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Optimal back-to-front airplane boarding

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The problem of finding an optimal back-to-front airplane boarding policy is explored, using a mathematical model that is related to the $1 + 1$ polynuclear growth model with concave boundary conditions and to causal sets in gravity. We study all airplane configurations and boarding group sizes. Optimal boarding policies for various airplane configurations are presented. Detailed calculations are provided along with simulations that support the main conclusions of the theory. We show that the effectiveness of back-to-front policies undergoes a phase transition when passing from lightly congested airplanes to heavily congested airplanes. The phase transition also affects the nature of the optimal or near-optimal policies. Under what we consider to be realistic conditions, optimal back-to-front policies lead to a modest 8–12 % improvement in boarding time over random (no policy) boarding, using two boarding groups. Having more than two groups is not effective.

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I. INTRODUCTION

Airlines and their passengers alike have a mutual interest in minimizing the time spent at the gate while the passengers are boarding the airplane. For the airlines, airport infrastructure, and passengers, reducing the boarding time means decreased operational costs, increased passenger throughput, and better passenger experience [1–4].

A common airline strategy aimed at decreasing the boarding time is to employ a back-to-front boarding policy. The boarding is performed in stages, by means of announcements such as, "passengers from row number 40 and above are now welcome to board the airplane; all other passengers, please remain seated." Such policies try to board passengers from the back of the airplane first, by announcing groups of rows whose passengers are allowed to board at the same time.

Researchers have mostly been studying airline boarding through discrete simulations [1,3,5–7]. The studies have concluded that back-to-front boarding policies with multiple groups are ineffective; however, these studies assumed that the sizes of the group are equal. The case of general group sizes has not been explored in these studies. More recently [4], the boarding procedure at SWISS airlines, where the back third of the airplane is boarded first, with the rest of the passengers boarding after them, was examined. The conclusion was that random boarding was slightly better than this particular strategy as well. In another direction [8], researchers also looked at the process of walking from the gate to the airplane and within the airplane. While the study is of interest, it is orthogonal to the study of back-to-front policies, or more generally to the comparison between policies that are seat-location based.

Airplane boarding was also recently considered [9] by Frette and Hemmer in the context of a particle system with distinguishable particles on a substrate. The model considered an airplane with a single passenger (particle) per row. The authors provided a detailed study of the random boarding policy for a very small number (up to 16) of passengers and via numerical extrapolation suggested a power law for average boarding time with more passengers. They argued that their findings on the random policy suggest that back-to-front boarding with two equal-size groups would not be beneficial, in accordance with previous studies. The authors were not able to extend their analysis to more than one passenger per row.

In a Comment by Bernstein on [9] (see [10]), a different power law, for the average boarding time of the random boarding policy, was found using simulations with more passengers. The author also suggested a model for back-to-front airplane boarding that differed from that of [9] and led to the opposite conclusion that boarding with two groups of equal size is beneficial. It is important to note that the study was also restricted to a single passenger per row.

In this paper we analyze back-to-front boarding with any number of groups, for all group sizes, whether equal or not, and for any number of passengers per row. We provide near-optimal back-to-front policies and in addition estimates on the amount of boarding time improvement with respect to random boarding. We verify our analysis via simulations. The analysis is based on a mathematical model of the boarding process that was developed in [11]. The model is strongly related to the $(1 + 1)$-dimensional polynuclear growth model [12,13] with a concave boundary. The concave boundary case differs from the well studied convex boundary case even in terms of some basic exponents (see the Appendix for more details). The model is also strongly related to the causal set program for gravity as outlined in [14].

The models that were used in [9,10] are very special cases of our model. Our analysis resolves the apparent contradiction between the results of the two papers while showing that there are major differences between the realistic case of several passengers per row and the case that was studied of a single passenger per row.

Mathematically, back-to-front boarding policies are parametrized by an infinite-dimensional simplex, although for any finite number of passengers the parametrization is finite. The analysis of the model proceeds by assigning to each back-to-front policy a Lorentzian metric on the unit square. After normalization, the boarding time is estimated by the proper time of the maximal curve in the resulting space-time domain. Consequently, our task can be viewed as minimizing the diameter of a space-time domain within an

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infinite-dimensional family. While the problem does not have an absolute minimum, we can find near-optimal solutions that, coupled with practical considerations, essentially solve the problem.

We show that having two passenger groups is sufficient for a near-optimal back-to-front boarding policy. The optimal policy provides mild boarding time improvement over random boarding (having no policy), but is easier to implement than other policies that can provide more benefit.

II. MODELING THE BOARDING PROCESS

A back-to-front boarding policy F with m groups will be represented by a partition of the unit interval

$$1 = \rho_0 > \rho_1 > \cdots > \rho_m = 0.$$

Let V be the number of rows in the airplane. We place h passengers in each row. Thus, if the number of passengers is n, the number of rows is $V = n/h$. To implement a policy, an airline would first announce the boarding of the passengers with row numbers $V\rho_1$ and above, then rows $V\rho_2$ to $V\rho_1$, etc.

As a very special case of back-to-front boarding we have the unique policy with $m = 1$, which corresponds to having no policy, i.e., there is no airline control on the boarding queue order. This (lack of) policy will be called random boarding.

The following model of the boarding process was introduced in [11,15]. We assume that passengers queue to board the airplane according to the above rules of a back-to-front policy, but in random order within each boarding group. Once the queueing order is fixed, each passenger proceeds to their designated row, advancing as much as possible along the aisle. When a passenger reaches his or her designated row, they block the aisle for one time unit while getting organized before sitting and clearing the aisle. While standing in the aisle, the passenger blocks other passengers who queue behind, waiting for the aisle to be cleared so they can advance further towards their designated row. We assume that each passenger occupies a portion l of the aisle between successive rows. For example, suppose a passenger who is seated in row 20 has reached his or her row and is getting organized in the aisle. Further assume that there are 10 passengers queued behind him or her, waiting to get to row numbers beyond 20 and that each such passenger occupies as personal space half the distance between rows ($l = 1/2$). Then these 10 passengers will occupy the aisle space from rows 15 to 19 and thus a passenger behind them who is trying to reach row 16 will have to wait for the passenger in row 20 to clear the aisle. The process proceeds in rounds, each taking a single time unit, in which passengers who have reached their row are seated. The number of rounds (time units) until all passengers have sat is the boarding time. Note that a passenger who was seated in round i must have been blocked from reaching their row by a passenger who was seated in round $i-1$; otherwise they could have sat earlier. A chain is a sequence of passengers where each passenger is blocked by its predecessor in the chain. The boarding time is thus the longest chain of passengers blocking each other. We refer to such a chain as a maximal chain. The above model, with the special parameters $l = h = 1$, is also the model used in [9] for random boarding and in [10] for both random and back-to-front boarding.

We may also consider the more realistic model in which the amount of time that a passenger blocks the aisle before being seated is given by a random variable X rather than just a single time unit. Variability in blocking time exists since different passengers have different amounts of luggage, different reaction times, and passengers already seated may have to get up to let them pass to their seats.

III. ESTIMATING THE AVERAGE BOARDING TIME

A. Basic quantities

To get an estimate for the average boarding time of a back-to-front policy in the airplane boarding model, we follow the method that is outlined in [11,15]. The passengers will be represented as points (q,r) in the unit square $[0,1]^2$. The q coordinate is given by the position of a given passenger in the boarding queue divided by the number of passengers n, while the r coordinate is given by the assigned row number divided by the number of rows V.

The choice of a policy F leads to a joint probability distribution density $p_F(q,r)dqdr$ on the row and queue location coordinates of passengers. For a policy F given by the partition $1 = \rho_0 > \rho_1 > \cdots > \rho_m = 0$ we define the square S_i, given by $\rho_{i-1} \geqslant r \geqslant \rho_i$ and $1 - \rho_{i-1} \leqslant q \leqslant 1 - \rho_i$. The density function associated with F is given by $p_F(q,r) = \frac{1}{\rho_{i-1}-\rho_i}$ when $(q,r) \in S_i$ and zero otherwise.

When restricted to a given boarding group, the joint density induces a uniform distribution since the ordering of passengers within a group is uniformly random. While the boarding policy is responsible for producing a joint density distribution in the queue and row coordinates, the airplane configuration, namely, the inter-row distance (leg room), the number of aisles (single or double), which we denote by b, and the number of passengers per row, affects the boarding time through the congestion parameter

$$k = lh/b.$$

The congestion parameter measures the total aisle length (per aisle), in units of distance between successive rows, occupied by all the passengers in a single row. Intuitively, $1/k$ measures the fraction of passengers that can stand along the aisles simultaneously during the boarding process. An estimate for k can be obtained when passengers are preparing to exit the airplane. When passengers exit, they tend to crowd the aisle so that they can take their luggage and prepare to leave the airplane. These rather crowded aisle conditions that tend to produce minimal personal space produce a lower bound for k. From personal observations, we estimate that at most half the passengers can stand in the aisles in these very crowded conditions showing that $k \geqslant 2$ in realistic settings.

Let

$$\alpha(q,r) = \alpha_F(q,r) = \int_r^1 p_F(q,z)dz$$

and consider the Lorentzian metric

$$ds^2 = p_F(q,r)[dqdr + k\alpha(q,r)dq^2] \quad (1)$$

on the unit square. We note that at some points we have $F(q,r) = 0$ and the Lorentzian metric becomes degenerate. At such points, the metric should be understood as determining the condition for causal curves, namely, $dqdr + k\alpha(q,r)dq^2 \geqslant 0$ has to hold, however, there is no contribution to the proper time computation.

Let $T(F,k)$ be the maximal proper time of a causal curve in the space-time domain consisting of the unit square equipped with the Lorentzian metric above and let $R(F,k,n)$ denote the average boarding time (measured in rounds) of n passengers, with congestion parameter k and boarding policy F in our model. It can be shown that as the number of passengers goes to infinity we have

$$2T(F,k)\sqrt{n}/R(F,k,n) \to 1.$$

Since the factor $2\sqrt{n}$ is policy independent, we will use $T(F,k)$ as a normalized estimate for the average boarding time, given a boarding policy and a congestion parameter. More generally, if instead of allowing passengers a single time unit to block the aisle before being seated we assume that the blocking time is given by a random variable X, then it can be shown (see discussion in [15]) that as the number of passengers goes to infinity we have

$$C_X T(F,k)\sqrt{n}/B(F,k,n) \to 1, \tag{2}$$

where C_X is a constant that depends on X but not on the boarding policy. Consequently, we can also use $T(F,k)$, as a normalized estimate for comparing boarding policies, in the presence of X. The idea behind these results is that when the number of passengers is large, the blocking relation essentially coincides with the past-future relation of the space-time domain, the maximal chain of passengers blocking each other will closely follow the maximal curve, and the number of passengers in the maximal chain (the boarding time) will be proportional to the proper time of the maximal curve (with a proportionality factor of the order of $\sqrt{n}$).

B. Comparison with other models

The models of [9,10] for random boarding are special cases of our model that are obtained by setting $l = h = b = 1$. This leads to a congestion factor $k = 1$. The model of [10] for back-to-front boarding with equal size groups is also a special case of our model with $l = h = b = 1$. The back-to-front model of [9] is somewhat inconsistent. It assumes that the boarding times of the two groups are mutually exclusive, an assumption that corresponds to the case of $k \to \infty$, while within a given boarding group $k = 1$ is assumed, as in the random boarding case. While these assumptions are conflicting, the $k \to \infty$ assumption on the interactions between different boarding groups turns out to be dominant.

The description of the simulations in [1,3,5–7] is not always sufficiently clear, however, it seems that they use the parameters $l = b = 1$ and $h = 6$, which leads to $k = 6$. They also include additional details that mostly affect the aisle blocking distribution X, but as seen from Eq. (2), this essentially scales the boarding time of any given policy rather than affecting the comparison between different boarding policies. Consequently, we will provide the analysis and simulations for the fixed one-time-unit delay model.

C. Equal-size groups with low-*k* and high-*k* limits

As we change the value of k from $k = 0$ to $k \to \infty$ the behavior of back-to-front strategies changes dramatically. As explained in [15], when $k = 0$, a situation that corresponds to cardboard thin passengers with no carry-on luggage, the best back-to-front boarding strategy with m groups is to divide into groups of equal size. Moreover, the boarding time steadily decreases with the number of groups and is proportional to $m^{-1/2}$. It is easy to show that these conclusions, essentially extend to the range $k \leqslant 1$. These observations, which are based on analytical results, were reproduced via simulations in [10].

A particular feature of the case $k \leqslant 1$ is that the improvement in boarding time that can be attained via back-to-front boarding is limited only by the number of passengers, which in turn limits the number of groups. As the number of passengers grows to infinity, so can the number of groups and we can get unbounded improvements in boarding time over the random policy.

At the value $k = 1$ a phase transition in the behavior of back-to-front policies occurs. For $k > 1$, the amount of boarding time improvement relative to random boarding is bounded regardless of the number of groups and their sizes. Specifically, let $T(k)$ denote the normalized, estimated average boarding time of the random policy. It is shown in [16] that for a given value of $k > 1$ we have

$$\frac{T(F,k)}{T(k)} \geqslant \frac{\sqrt{k-1}}{\sqrt{k} + \frac{1-\ln(2)}{\sqrt{k}}} \tag{3}$$

for any back-to-front policy F, regardless of the number of groups and their sizes. In Table I we plot the values of the bound of Eq. (3) for $k = 2, \ldots, 6$. Asymptotically, as $k \to \infty$, the bound is approximated by $1 - \frac{3/2-\ln(2)}{k} \sim 1 - \frac{0.8}{k}$, which gives a reasonable approximation of the bound for $k > 6$.

In addition, for $k > 1$, equal-size groups are no longer optimal among back-to-front boarding strategies. Let F_m denote the back-to-front boarding policy with m equal size groups. We note that F_1 is random boarding. The values of $T(F_m,k)$ are computed in [15]. For random boarding we have

$$T(F_1,k) = \sqrt{\frac{e^k - 1}{k}} \tag{4}$$

for $k \leqslant \ln 2$ and

$$T(F_1,k) = \sqrt{k} + \frac{1-\ln 2}{\sqrt{k}}, \quad k > \ln 2. \tag{5}$$

TABLE I. Lower bound on the relative reduction in boarding time, according to Eq. (3), of a back-to-front policy in comparison with random boarding for various congestion factors k.

k	Bound
2	0.613
3	0.741
4	0.804
5	0.842
6	0.868

For back-to-front boarding with two equal-size groups we have

$$T(F_2,k) = \sqrt{2k} + \frac{3/4 - 2\ln 2}{\sqrt{2k}} \quad (6)$$

when $k \geqslant 2\ln 2$ and for $1 \leqslant k \leqslant 2\ln 2$ we have

$$T(F_2,k) = \sqrt{\frac{1}{2k}}\left(k + \frac{e^k - 1}{4}\right). \quad (7)$$

For the back-to-front policy with $m > 2$ equal-size groups we have for $k \geqslant 3/4 + \ln(2)$

$$T(F_m,k) = \sqrt{mk} - \frac{m-2}{\sqrt{mk}}(\ln 2 + 1/4) - \frac{2\ln 2 - 3/4}{\sqrt{mk}}. \quad (8)$$

As can be seen from the formulas above, random boarding is better than back-to-front boarding with two (or more) equal-size groups whenever $k > \ln(2) + \frac{\sqrt{2}-3/4}{2-\sqrt{2}} \sim 1.83$. A similar computation shows that random boarding is better than back-to-front boarding with three equal-size groups whenever $k > 1.67$.

Let $D_m(k) = T(F_m,k)/T(F_1,k)$ denote the ratio between average boarding times of a back-to-front policy with m equal-size groups and that of random boarding, as $n \to \infty$. As noted previously, when $k \leqslant 1$, $D_m(k) \sim m^{-1/2}$. In contrast, it was proven in [15] that as $k \to \infty$, we have $D_m(k) \sim m^{1/2}$ and in fact equal-size groups become asymptotically the worst choice for a back-to-front policy.

To see why there is such a vast difference between the cases of small and large values of k and why $k = 1$ is the phase transition point, we note that when $k \leqslant 1$, all passengers in a given boarding group can stand in the airplane aisle next to the seats that they are assigned to. This means that they do not create significant backlogs that spill over to the seats that are assigned to adjacent groups and interfere with their boarding, or stated otherwise, passenger interactions are local. When $k = 0$ no interference can occur and thus the different groups may board in parallel (or, in practice, in relatively short succession). Consequently, the boarding time is determined by the boarding time of the largest group and choosing equal-size groups minimizes the size of the largest group. Since boarding time is proportional to $\sqrt{n}$, we obtain a savings ratio on the order of $\sqrt{\frac{1}{m}}$. In contrast, when $k > 1$, the passengers cannot all stand next to the rows to which they were assigned and they spill over to the aisle space next to other rows creating backlogs that block passengers from other groups; stated otherwise, they create long-range interactions. These backlogs create a cascading effect and therefore the groups cannot all board in parallel, but rather serially. Examining the explicit calculations that are given in the Appendix, it can be observed that the effect of groups interfering with each other is especially pronounced when the groups are of roughly equal size, thus choosing all the groups to be of equal size creates the worst cascading effects. In good policies, the group sizes will roughly form a geometric sequence, so that any two adjacent groups have relatively different sizes. This means that the number of groups of substantial size in such policies will tend to be very small and thus few boarding groups are sufficient to obtain near-optimal results.

The transition between the regimes of small values of k and large values takes place mostly in the range $k \in [1,6]$, which matches the range that is most relevant to airplane boarding. In this regime it is hard to predict what level of improvement over random boarding back-to-front policies can offer. We are thus forced to make detailed computations.

IV. BACK-TO-FRONT POLICIES AT INTERMEDIATE-*k* VALUES

To compute $T(F,k)$, for some policy F, we need to find the curve with maximal proper time in the associated space-time domain. Such curves are composed of geodesic segments and boundary segments. We first consider the case $m = 2$ of two boarding groups.

A. Calculations for two-group policies

We will use the notation of Sec. III. Let us set $x = 1 - \rho_1$. The two boarding groups consist of passengers whose row numbers are $[(1-x)V] + 1, \ldots, V$ followed by passengers from rows $1, \ldots, [(1-x)V]$, where $[a]$ denotes integer value. For example, when $x = 0.2$, we first board the passengers from the last 20% of the rows and then the remaining 80%. The (q,r) coordinates of these passengers will belong to the squares S_1 and S_2, respectively. The maximal curve with respect to the metric in (1) may be contained entirely within S_1, entirely within S_2, or may span both squares. We let $T_1(x,k)$, $T_2(x,k)$, and $T_{1,2}(x,k)$ denote the maximal proper times of causal curves contained entirely in S_1, in S_2, and spanning both S_1 and S_2, respectively. Obviously, the proper time of the maximal curve will be given by

$$T(x,k) = \max[T_1(x,k), T_2(x,k), T_{1,2}(x,k)].$$

Since the metric (1), restricted to S_1 or S_2, is a scaled version of the metric corresponding to the random policy on the unit square, a simple calculation shows that

$$T_1(x,k) = \sqrt{x}\,T(0,k) \quad (9)$$

and

$$T_2(x,k) = \sqrt{1-x}\,T(0,k), \quad (10)$$

so it remains to compute $T_{1,2}(x,k)$. The computation is given in the Appendix. We state the end result. For given values of k and x we let

$$\delta^* = \delta^*(k) = (1 - 2e^{-k})^2, \quad (11)$$

$$\delta_{\mathrm{crit}} = \delta_{\mathrm{crit}}(x) = \frac{4 - 3x - 4\sqrt{x - x^2}}{4(1-x)}, \quad (12)$$

and

$$\delta_{\min} = \delta_{\min}(x) = \max\left(\frac{1-(k+1)x}{1-x}, 0\right). \quad (13)$$

The value of $T_{1,2}(x,k)$ depends on the relation between these three quantities.

(i) If $\delta_{\min} \geqslant \delta^*$ then

$$T_{1,2}(x,k) = \frac{\delta_{\min}(1-x) + (k+1)x - 1}{\sqrt{kx}} \quad (14)$$

$$+ \sqrt{\frac{1-x}{k}}\sqrt{(1-\delta_{\min})(e^k - 1)}. \quad (15)$$

The maximal curve spanning both S_1 and S_2 will not have a boundary component in S_2.

(ii) If $\delta^* \geqslant \delta_{\rm crit} \geqslant \delta_{\rm min}$ then

$$T_{1,2} = \frac{\delta_{\rm crit}(1-x)+(k+1)x-1}{\sqrt{kx}} \quad (16)$$

$$+\sqrt{\frac{1-x}{k}}[\sqrt{\delta}_{\rm crit}+\ln(1-\sqrt{\delta}_{\rm crit})+k+1-\ln 2]. \quad (17)$$

The maximal curve in this case will have a boundary component in S_2.

(iii) If $\delta^* \geqslant \delta_{\rm min} \geqslant \delta_{\rm crit}$ then

$$T_{1,2} = \frac{\delta_{\rm min}(1-x)+(k+1)x-1}{\sqrt{kx}} \quad (18)$$

$$+\sqrt{\frac{1-x}{k}}[\sqrt{\delta}_{\rm min}+\ln(1-\sqrt{\delta}_{\rm min})+k+1-\ln 2]. \quad (19)$$

In this case, the maximal curve has a boundary component $\delta_{\rm min}$ representing the lowest point on the left edge of S_2, where a causal curve meeting S_1 can enter S_2.

(iv) If $\delta_{\rm crit} \geqslant \delta^* \geqslant \delta_{\rm min}$ then

$$T_{1,2} = \frac{\delta^*(1-x)+(k+1)x-1}{\sqrt{kx}} \quad (20)$$

$$+\sqrt{\frac{1-x}{k}}[\sqrt{\delta}^*+\ln(1-\sqrt{\delta}^*)+k+1-\ln 2]. \quad (21)$$

In this case the maximal curve is tangent to the bottom edge of S_2.

As we explained before, the graph of $T(x,k)$ for a fixed value of k differs between the cases of $k \leqslant 1$ and $k \geqslant 1$. The bottom curve in Fig. 1 is $T(x,1)$. It has essentially a single local minimum (also global) near $x = 0.5$. All the other curves in Fig. 1 display $T(x,k)$ for increasing values of $k > 1$. In these graphs there are two local minima. The first local minimum, which as k increases approaches $x = 0$, is obtained where $T_{1,2}(x,k) = T_2(x,k)$. At this meeting point $T_{1,2}(x,k)$ is increasing while $T_2(x,k)$ is decreasing, hence $T(x,k)$, which is the maximum of the two near this value of x, is not differentiable at the minimum but has directional derivatives. The second local minimum is similarly obtained near $x = 1$ when $T_{1,2}(x,k) = T_1(k,x)$; the function $T(x,k)$, is again nondifferentiable at this point.

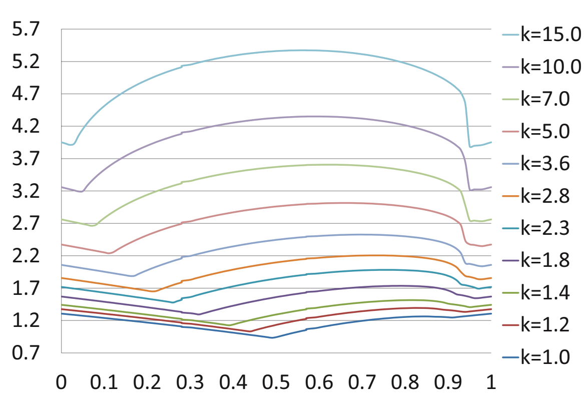

FIG. 1. (Color online) Function $T(x,k)$ for various values of the congestion parameter k. As the congestion k increases, so does the value of $T(x,k)$, hence the curves are sorted from bottom to top in increasing order of k. The estimated average boarding time with n passengers is $2T(x,k)\sqrt{n}$. As k increases, the minimal value of $T(x,k)$ is obtained at values of x that drift from $x = 1/2$ towards $x = 0$. The maximal value is attained at values of x that drift from $x = 1$ towards $x = 1/2$. The graphs also show that it is always better to have a small first boarding group, i.e., $T(x,k) < T(1-x,k)$ for $x < 1/2$.

B. Results for two-group policies

Using the equations above, we can calculate the estimated average boarding time for any values of k and x for a two-group policy. As we explained in the previous section, the most interesting and relevant range for the parameter k is $k = 1,\ldots,6$. We simulated the boarding process with $l = 1$ (as in most simulation-based studies) and with $h = 1,\ldots,6$ passengers per row (per aisle). The simulations assumed 200 passengers (per aisle) and a single-time-unit delay, hence the boarding time is given as the number of rounds. Note that with 200 passengers and $k = 1$, we have 200 rows in the airplane and x can only take values that are multiples of 0.005. A similar discretization occurs for the other values of k; for example, for $k = 4$ we have 50 rows and x increases in multiples of 0.02. We simulated all such possible values of x. We also computed $T(x,k)$ for all values of x and k corresponding to our simulations. The simulations were run on each fixed value of k and x, 100 000 times.

The main results are presented in Table II. For each value of $k = 1,\ldots,6$ we give the value of x that minimized the boarding time in the simulations. This value is denote by x_o (x optimal) and is presented in the second column. The average of the boarding time with the optimal policy, measured in rounds $R(x_o)$ is presented in the third column. The fourth column provides the value of x that should be

TABLE II. Boarding times in rounds for the optimal two-group policy and the recommended two-group policy. The value of x_o, the optimal partitioning point according to the simulation, is given by the fraction of rows whose passengers form the first boarding group out of the total number of rows; $R(x_o) = R(x_o,k,200)$ is the average boarding time, measured in rounds, for the optimal policy; x_r is the recommended value of the partitioning point according to the analytical calculations; and $R(x_r)$ is the average boarding time, measured in rounds, of the recommended policy according to the simulations. The last column provides the analytically predicted average boarding time (in rounds) for the recommended policy. As can be seen, even though the analytical predictions for the average number of rounds are not accurate, the conclusions of the analytical model in terms of policy comparisons are highly accurate and lead to near-optimal recommendations.

k	Simulation x_o	Simulation $R(x_o)$	Analytical x_r	Simulation $R(x_r)$	Analytical $R_e(x_r)$
1	90/200	21.747	98/200	22.128	26.398
2	26/100	30.080	29/100	30.324	38.875
3	12/66	36.054	13/66	36.190	48.422
4	6/50	41.247	7/50	41.380	56.525
5	4/40	45.454	4/40	45.454	63.726
6	2/33	48.997	3/33	49.096	69.470

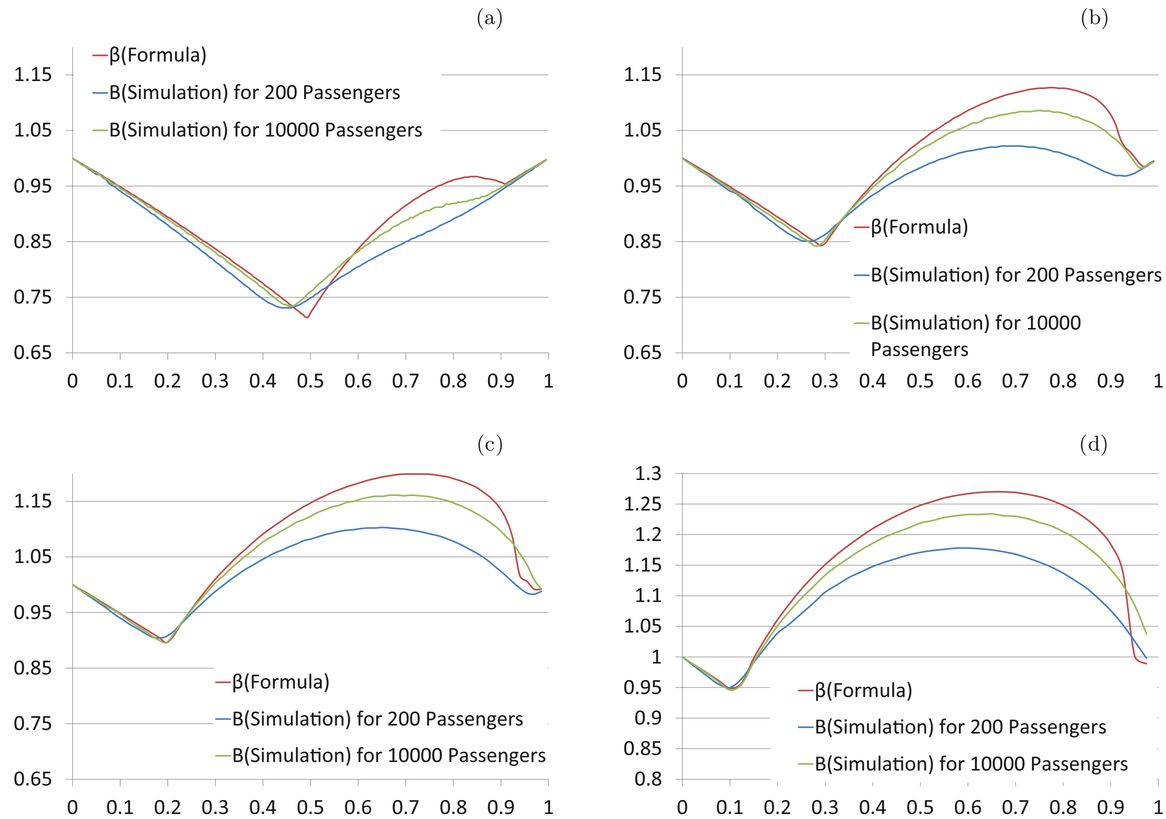

FIG. 2. (Color online) Ratio $B(x,k)$ between the average boarding time with two groups and the avrage boarding time without groups (random boarding) for (a) $k=1$, (b) $k=2$, (c) $k=3$, and (d) $k=5$. The ratio is shown in the bottom curve (at $x=0.7$) of each figure. For $k=1$ there is always a reduction in boarding time with groups, with the largest reduction around $x=0.5$. For $k>1$ low-x values (the first group relatively small) give an improvement, while high-x values lead to boarding times that are higher than for random boarding. As k increases, the interval in x that gives an improvement, the optimal x value, and the improvement both decrease. These facts are consistent with Table II. The maximum value of the curves (corresponding to the worst choice of the x value) increases with k and is attained near $x=0.5$ when k is large. Also shown are the analytically computed ratios $\tilde{B}(x,k)$ (top curve at $x=0.7$) that correspond to infinitely many passengers and the results for 10 000 passengers (middle curve at $x=0.7$), which are closer to the analytical predictions.

optimal according to our estimate $T(x,k)$. We call it x_r (x recommended) since this is the value of x that is recommended by our analytical calculations. The fifth column $R(x_r)$ provides the average boarding time, in rounds, with the recommended policy. The last column (rounds expected for x recommended) provides the theoretically expected average boarding time for the recommended policy. This theoretically expected value is based on the limit as the number of passengers tends to infinity.

We see that the performance of the recommended policy is very close to that of the optimal policy for all values of k. In fact, for all values $k \geqslant 2$ the difference is less than 1% and for $k=1$ it is less than 1.5%. This may seem surprising since the estimates on which the recommended policy are based are clearly not accurate (compare columns 3 and 6).

Considering the issue of accuracy, we note that when comparing different airplane boarding policies and finding the optimal ones we are less interested in the absolute boarding time of each policy than we are in the ratio of boarding times. We chose to compare all the different back-to-front policies to the random boarding policy. Consequently, we define the boarding time ratio $B=B(k,x,n)=R(x,k,n)/R(0,k,n)$ to be the ratio of the average boarding time of the two-group policy corresponding to x, i.e., $R(x,k,n)$, and the average boarding time with random boarding $R(0,x,n)$ for n passengers. The ratio $\beta(x,k)=T(x,k)/T(0,k)$ is the analytical estimate for the value of B. This estimate assumes an infinite number of passengers, i.e., as $n\to\infty$ we have $B(x,k,n)\to\beta(x,k)$. One of the reasons for the accuracy of the results is that the convergence of the ratio is much faster than the convergence of either the numerator $R(x,k,n)/(2\sqrt{n})\to T(x,k)$ or the denominator $R(0,k,n)/(2\sqrt{n})\to T(0,k)$ alone. Figure 2 displays the functions $\beta(x,k)$, $B(x,k,200)$, and $B(x,k,10\,000)$ for $k=1,2,3,5$. We see that these graphs are not so far from each other already for 200 passengers. Unlike the absolute errors, which can be very large, the ratio of errors between any given policy and the random policy that serves as a measure is substantially smaller. This partially explains why the predictions regarding the optimal policy are rather accurate. We have also included in the graphs of Fig. 2 the values of B for simulations with 10 000 passengers and it can be seen that the difference from β becomes even smaller and the predictions even more accurate (not shown).

A second observation from Table II that also adds to the accuracy of the predictions is that as a function of x, $R(x,k,200)$ behaves as if it has a vanishing derivative at x_o, i.e., a small error in the recommended value of x has a negligible effect on the average boarding time. This should be contrasted with the behavior of $T(x,k)$ at its minimum x_r, where there are nonzero directional derivatives. We also notice that in all cases $x_o \leqslant x_r$, that is, the first boarding group in the optimal policy is somewhat smaller than in the recommended policy.

To gain a better understanding of these observed phenomena, we consider the error term that measures the difference between the analytical estimates and the observed results. Comparing the estimates to the simulation results shows the error due to a finite population size of 200 passengers. Let

$$\Delta(x,k,n) = E(R(F,k,n)) - 2T(F,k)\sqrt{n}$$

denote the error, which is the difference between the actual average boarding time of n passengers and the estimated average boarding time, coming from the calculation of the proper time of a maximal curve in the associated Lorentzian domain.

The error for the random boarding policy was analyzed in [11]. Following the analysis, for $k > \ln(2) \sim 0.69$, the error Δ, which can be shown to be always negative, is expected to asymptotically ($n \to \infty$) obey a power law of the form

$$a(x,k)n^b,$$

where $1/6 < b < 1/4$ is independent of x and k and $a(x,k) < 0$. Thus the behavior of boarding time for random boarding is given by a sum of two power laws, one providing the main term estimate $2T(0,k)\sqrt{n}$ and a secondary law governing the error. This explains the discrepancy between the power laws of [9,10]. In [9], when estimating boarding time with a small number of passengers, the error is very influential. Since the authors considered only a single power law, they obtained one that fits only very small values of n, but is asymptotically incorrect. In [10], the main term was rediscovered numerically by considering thousands of passengers, but the secondary term was not considered.

When carried over to back-to-front boarding, the analysis of [11] suggests that the absolute value of the relative error

$$E = |\Delta(x,k,n)/R(x,k,n)| = -\Delta(x,k,n)/R(x,k,n)$$

is in general an increasing function of k. For 200 passengers the relative error is expected to be very large.

Table III presents the range of values of the relative error as percentage points, i.e., $100E$, for $k = 1,\ldots,6$ with 200 passengers. As can be seen, the relative error for different values of x is large and grows with k as expected. The table also presents the value of x where the relative error is minimized (see column 3). The relative error generally has two local minima (except for $k = 1$, where they converge) and they are near the two local minima of the expected normalized boarding time $T(x,k)$. They are also near the local minima of the observed boarding time since these are near the local minima of $T(x,k)$, however, the correlation with the minima of $T(x,k)$ is stronger. As with $T(x,k)$, for $k = 2,3,4$, the first local minimum is also the global one. For $k = 5,6$, unlike $T(x,k)$, the second local minimum happens to be the global one.

TABLE III. Range of the relative error (in percentage points) of the boarding time estimate and the value of x, denoted by x_e, with a minimal relative error for $k = 1,\ldots,6$ and 200 passengers. The minimal relative error is obtained near one of the local minima of $T(x,k)$. The size of the relative error increases with k.

k	E range	x_e
1	19–23	99/200
2	28–47	30/100
3	33–50	13/66
4	36–52	8/50
5	39–55	38/40
6	41–57	32/33

When computing the estimates for the normalized average boarding time $T(x,k)$, we are considering the case of $n \to \infty$. In that regime there is no discretization and the recommended value for x_r is obtained when the values of $T_2(k,x)$ and $T_{1,2}(k,x)$ coincide. This means that there are two distinct maximal curves, one residing only in S_2 and the other spanning both S_1 and S_2, which partially coincide after meeting at a boundary point in S_2. When $N \to \infty$, the maximal chain in the blocking partial order will be contained in the neighborhood of one of the two curves, depending on whether it consists entirely of passengers in the second boarding group or not. Using the notions of Sec. II, let $L_1(x)$ denote the longest chain consisting of passengers in the first group, $L_2(x)$ the longest chain in the second group, and $L_{1,2}(x)$ the longest chain spanning both groups. Let $L(x) = \max[L_2(x),L_1(x),L_{1,2}(x)]$ denote the longest chain. Note that $L(0) = L_2(0)$. As explained above, the arguments in [11] suggest that the expectation $E(L_2(x_r))$ is approximated to second order by an expression of the form $2T_2(k,x_r)\sqrt{n} - b_1[(1-x_r)n]^{c_1}$, with $b_1 > 0$ and $1/6 \leqslant c_1 \leqslant 1/4$, and similarly, since $T_2(k,x_r) = T_{1,2}(k,x_r)$, the quantity $E(L_{1,2}(x_r))$ is approximated by $2T_2(k,x_r)\sqrt{n} - b_{1,2}(x_r)n^{c_{1,2}}$. A similar occurrence happens at the second local minimum of the expected boarding time where $T_1 = T_{1,2}$.

For values of x that are close to one of the two local minima, the fact that $L(x)$ is the maximum of two random variables of roughly equal size [either $L_1(x)$ and $L_{1,2}(x)$ or $L_2(x)$ and $L_{1,2}(x)$] and the fact that the error in the estimate for each one separately is negative [$b_1,b_2,b_{1,2}(x) < 0$] suggest that the error will be somewhat less negative than usual, i.e., locally maximized, and as a result the relative error E (which has opposite sign) experiences a local (and sometimes global) minimum near the local minima of the expected boarding time function. Since the average boarding time B is the sum of the estimate $2T(x,k)\sqrt{n}$ that is minimized at x_r and the error Δ that is locally maximized at x_r, it tends to be flat around x_r and this provides further explanation to the unexpected accuracy of the analytical predictions.

The competition between $L_2(x)$ and $L_{1,2}(x)$ for values of x near x_r also explains why the optimal value in simulations x_o is always somewhat smaller than x_r. As an example, at the granularity level of 200 passengers, four per row ($k = 4$), the two values of x closest to the minimum of $T(x,4)$, which is roughly 0.1485, are $x = 0.14$ and 0.16. On a sample of size 1000, with $x = 0.14$ (last seven rows board first), we had $L(x) = L_2(x)$, 737 times, while for $x = 0.16$, it happened 379 times. Since at x_r, $L_2(x)$ decreases (as a distribution) while $L_{1,2}(x)$ is increasing, a small change in x to a value where one of the two options is dominant yields a smaller maximum of the two, hence a smaller expected boarding time. Considering the derivatives of $T_2(k,x)$ and $T_{1,2}(k,x)$ at x_r suggests that taking a somewhat smaller value of x is more efficient, in line with the simulation results.

Considering the results, one can see that the optimal policy achieves visible, but not dramatic, savings over the random boarding policy with realistic congestion values. For $k = 3$ we save approximately 10% in boarding time with the optimal choice of x. In this case, the airline should board 1/5 of the passengers and then all others. For $k = 4$ the optimal savings is 7–8 %. Compared to other methods such as the reverse pyramid method of [5], it saves less boarding time but requires substantially fewer groups, hence it is somewhat easier to manage. The most efficient method seems to be back-to-front boarding by arithmetic sequences of rows [3,7]; however, this method requires very strong control over the boarding process, well beyond what is currently practiced.

C. More than two groups

Our methods allow us to iteratively compute $T(F,k)$ for any back-to-front boarding policy F. However, unlike the case of $m = 2$, the computation is numerical and proceeds by induction on m, via a dynamic programming procedure. More details of the computation are presented in the Appendix. As we noted in Sec. III, for values of $k \geqslant 2$ we do not expect major gains in response time, by adding more groups, since the groups will roughly form a geometric sequence and will therefore be restricted to a few groups of non-negligible size. Table IV summarizes the results of the computations that are based on finding the length of the maximal proper time curve in the appropriate Lorentzian domain. For $k = 2$, one still gets some mild improvement by switching from two groups to three. The smallest of the groups in the four-group optimal solution has a single row of two passengers when there are 200 passengers and therefore the four-group solution is essentially irrelevant for this finite population scenario. In the case of $k = 4$, already the three-group case involves a group consisting of a single row and the expected improvement is negligible.

TABLE IV. Boarding times under the optimal m-group policies, and the inner partitioning points $\rho_{m-1},\ldots,\rho_1$ of the recommended policy for $m = 2,3,4$ and congestion factors $k = 2,4$. The third column shows the expected (analytically computed) ratio of average boarding times between the recommended policy m-group policy and random boarding. The recommended policy becomes optimal when $n \to \infty$. Starting with the fourth column, the optimal back-to-front m-group policy, for $k = 2,4$ and $m = 2,3,4$, is given by the inner points $\rho_{m-1},\ldots,\rho_1$ of the corresponding partition. The table shows that for $k = 2$, a three-group policy may still offer a mild advantage over a two-group policy. For $k = 4$ the expected improvement is minimal and the optimal three-group policy is a perturbed version of the optimal two-group policy, which is obtained by adding a small first group.

k	m	T_m/T_1	F_m^*		
2	2	0.840	0.29436		
2	3	0.792	0.11030	0.37226	
2	4	0.774	0.04477	0.15009	0.40041
4	2	0.923	0.14853		
4	3	0.911	0.02526	0.17004	
4	4	0.909	0.00439	0.02953	0.17368

TABLE V. Optimal and recommended three-group policies and their boarding times for $k = 2,4$. The F_o row presents the optimal policy and its observed boarding time and the F_r row displays the recommended policy, after discretization of x for 200 passengers, and its boarding time.

k	Policy	x_1	x_2	R
2	F_o	0.09	0.32	28.591
2	F_r	0.11	0.37	28.985
4	F_o	0.12	0.98	40.893
4	F_r	0.02	0.16	41.218

We checked the calculations against simulations of all possible three-group policies for 200 passengers and $k = 2,4$. We ran each policy 10 000 times. Table V shows the results.

In comparison to the optimal two-group policy, we see for $k = 2$ a 5% improvement. For $k = 4$ the improvement is negligible. Both results are in line with the analytical predictions. In the case of $k = 4$ the optimal and recommended three-group policies seem rather different, but in fact they are closely related. The optimal three-group policy is a perturbation of the optimal two-group policy, with the first row of the airplane being split from the second group to form the third boarding group. The recommended three-group policy is a perturbation of the recommended two-group policy; in this case, the recommended two-group policy has a first boarding group consisting of the last seven rows. The recommended three-group policy first boards the last row and then the next seven rows from the back. The perturbations are different, one involving the first row and one involving the last, but in both cases they have a minor effect of a single row and the average boarding time difference between the optimal and recommended policy is negligible as in the two-group case.

Given the good accuracy of the theoretical calculations, we did not simulate four-group policies since no real improvements are expected. We conclude that two groups are sufficient for back-to-front boarding with realistic values of k.

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APPENDIX

1. Airplane boarding and (1 + 1)-dimensional polynuclear growth

The connection between airplane boarding and the $(1+1)$-dimensional polynuclear growth (PNG) process is provided through the notion of an increasing subsequence. We recall that a sequence of points in the plane (x_i,y_i) is said to be increasing if for all $i < j$ we have $x_i \leqslant x_j$ and $y_i \leqslant y_j$.

We recall the PNG process. Consider a crystal in contact with a supersaturated vapor. Occasionally, in a uniformly random time and location (Poisson process) a nucleus is formed on the existing surface and one assumes that the nucleus spreads evenly in all directions at a constant speed, forming a round new layer. When two such layers collide they coalesce. A PNG droplet is a PNG process that starts with a single nucleation event A on a flat one-dimensional substrate. Consider the layer (height) of a second (terminal) nucleation event B. This and several other initial conditions were considered in [13]. It was observed that by a clockwise rotation by $\pi/4$ (conversion to lightlike coordinates) a sequence of nucleation events, each one on a layer formed by the previous one, is converted to an increasing sequence in the plane. Consequently, the height of the nucleation event B is the maximal increasing sequence among uniformly distributed points (rotated nucleation events) in a square or rectangle.

As explained in [11], the boarding time for random boarding (no policy) and $k = 0$ is also given by the maximal increasing sequence among n uniformly distributed points in a square or rectangle. It was also shown that when $k > 0$ the boarding time is given by a maximal increasing subsequence among uniformly distributed points in a nonrectangular domain. For $k > \ln(2) \sim 0.69$, which is the case considered in the present paper, the domain is nonconvex. After some normalization, the domain consists of the points that satisfy $0 \leqslant x \leqslant b > 1$ and $\frac{-1}{x+1} + 1 \leqslant y \leqslant 1$. All four boundary curves of the domain are the orbits of points under a one-parameter subgroup of generalized Lorentz transformations, i.e., symmetries in Minkowski space that do not necessarily fix the origin. The statistical behavior of the PNG droplet in a nonconvex domain ($b > 1$) is rather different from that of the convex domain, which has been thoroughly explored in [13] and many other papers. In this sense, airplane boarding provides a natural setting for a different type of PNG droplet. The statistical behavior is different in terms of the exponents governing height fluctuations (larger than the convex case), error term (larger in absolute value than the convex case), and transversal fluctuations (smaller than the convex case).

When we move to back-to-front boarding, the boarding process restricted to each group is still modeled by a PNG droplet model, however, the interaction between the two groups is not modeled by the PNG process since the corresponding space-time is not flat, the curvature of the model being concentrated on the lower boundary of S_1 and the left boundary of S_2, where the model is not continuous. For more on the relations between airplane boarding and other physical and mathematical systems see the surveys in [17–19].

2. Computations for two-group policies

For the random policy F_1, the length (proper time) of a causal curve $r = r(q)$, given by (1), takes the form

$$L(r) = \int_{q_0}^{q_1} \sqrt{r' + k(1-r)}dq. \tag{A1}$$

The general solution for the corresponding Euler-Lagrange equation has the form

$$r(q) = c_1 e^{2kq} + c_2 e^{kq} + 1 \tag{A2}$$

and its length is

$$L(r) = (e^{kq_1} - e^{kq_0})\sqrt{\frac{c_1}{k}}. \tag{A3}$$

To compute the normalized expected boarding time $T(x,k)$, we first consider the case when the maximal proper time curve φ spans both square cells S_1 and S_2, i.e., $T(x,k) = T_{1,2}(x,k)$.

In this case, it was shown in [15] that the maximum length curve must consist of a horizontal line segment in S_1, between $(0,1-x)$ and $(q_0,1-x)$, for some $0 \leqslant q_0 \leqslant x$, then a straight-line segment sloping down to (x,r_1), where it enters S_2 with slope $-k$, and then the maximal curve in the lower-right square S_2, ending at $(1,1-x)$. By the description we have the relation

$$r_1 = 1 - x - k(x - q_0) = 1 - (k+1)x + kq_0. \tag{A4}$$

Each segment of the maximal curve φ must either be of the form in Eq. (A2) or lie on the bottom boundary of a square. The following are the three cases that we need to consider.

(i) Restricted to S_2, φ has no boundary component.

(ii) φ is tangent to the bottom of S_2.

(iii) φ contains a boundary component in S_2.

Let L be the length of the maximal curve and let $\tilde{L}_1$ and $\tilde{L}_2$ denote the lengths of the portions of the maximal curve in the corresponding squares S_1 and S_2. Since the density distribution is uniform in each square we can apply Eqs. (A1)–(A3) after scaling to a unit size square. Let us call L_1 and L_2 the lengths of the resulting scaled curves. A simple computation reveals that scaling the density distribution and coordinates introduces a square root factor, so $\tilde{L}_1 = \sqrt{x}L_1$ and similarly $\tilde{L}_2 = \sqrt{1-x}L_2$, leading to

$$L = \sqrt{x}L_1 + \sqrt{1-x}L_2. \tag{A5}$$

Since the maximal curve in S_1 is a horizontal line segment (not a geodesic), we have by direct computation from (A1)

$$L_1 = \sqrt{k}\frac{q_0}{x}. \tag{A6}$$

Consider the second square scaled to unit size. The maximal curve enters the scaled square at

$$\delta = \frac{r_1}{1-x}. \tag{A7}$$

By Eq. (A4), δ is constrained by the inequality $\delta \in (\frac{1-x-xk}{1-x}, 1) \cap (0,1)$, i.e.,

$$1 > \delta > \frac{1-x-xk}{1-x} \tag{A8}$$

whenever $0 < x < \frac{1}{k+1}$. Using Eqs. (A4) and (A7) we can express L_1 via x and δ,

$$L_1 = \frac{\delta(1-x) + (k+1)x - 1}{x\sqrt{k}}. \tag{A9}$$

We turn to the computation of $L_2(\delta)$, which is the contribution of the maximal curve in S_2, assuming that S_2 was scaled to unit size and that the starting point of the curve is $(0,\delta)$. The ending point of the curve has to be $(1,1)$. We first consider the case where the maximal curve has no boundary component. In this case, by Eqs. (A2) and (A3) we have

$$L_2(\delta) = \sqrt{\frac{(1-\delta)(e^k - 1)}{k}}. \tag{A10}$$

This formula holds as long as $\delta \geqslant \delta^*$, where δ^* is the value that corresponds to case (ii) of tangency. We compute δ^* by applying to the general solution (A2) the tangency condition $r = r' = 0$.

The condition is satisfied at $\widetilde{q}_1$, which satisfies

$$\widetilde{q}_1 = 1 - \frac{\ln 2}{k}. \quad \text{(A11)}$$

Plugging c_1, c_2, and $\widetilde{q}_1$ into $r(0) = \delta$ and $r(\widetilde{q}_1) = 0$, we obtain

$$\delta^* = (1 - 2e^{-k})^2. \quad \text{(A12)}$$

In the remaining case $\delta < \delta^*$, we have by [15]

$$L_2(\delta) = \sqrt{\frac{1}{k}}[\sqrt{\delta} + \ln(1 - \sqrt{\delta}) + k + 1 - \ln 2]. \quad \text{(A13)}$$

Combining (A5) and (A9) and rescaling, we get a formula for the length of the maximal curve φ conditioned to pass through the point $[x, \delta(1-x)]$,

$$L(\delta) = \sqrt{x}\,\frac{\delta(1-x) + (k+1)x - 1}{x\sqrt{k}} + \sqrt{1-x}\,L_2(\delta). \quad \text{(A14)}$$

Given $x \in (0,1)$, we seek to maximize L as a function of δ. For this, we need to explore the derivative. After some computations we obtain

$$\frac{dL}{d\delta} = \frac{1-x}{\sqrt{kx}} - \frac{1}{2}\sqrt{\frac{1-x}{k}}\,\frac{\sqrt{e^k - 1}}{\sqrt{1-\delta}} \quad \text{(A15)}$$

when $\delta \geqslant \delta^*$ and

$$\frac{dL}{d\delta} = \frac{1-x}{\sqrt{kx}} - \frac{1}{2}\sqrt{\frac{1-x}{k}}\,\frac{1}{1-\sqrt{\delta}} \quad \text{(A16)}$$

otherwise. Calculating with (A15) one can show that the derivative does not vanish in the range $\delta \geqslant \delta^*$ and that the maximal value in that range is at δ^*.

When $\delta < \delta^*$, let δ_{crit} denote the value for which $\frac{dL}{d\delta} = 0$. Calculating with (A16) we see that

$$\delta_{\text{crit}} = \frac{4 - 3x - 4\sqrt{x - x^2}}{4(1-x)} \quad \text{(A17)}$$

and that it represents a local maximum. Note, however, that this calculation holds only if the admissibility conditions for the critical point, namely, $\delta_{\text{crit}} < \delta^*$ and $\delta_{\text{crit}} > \max\{0, \frac{1-x-xk}{1-x}\}$, hold. After some calculations, the first constraint holds when x satisfies

$$1 - \frac{1}{5 - 8\sqrt{\delta^*} + 4\delta^*} < x < 1 - \frac{1}{5 + 8\sqrt{\delta^*} + 4\delta^*}. \quad \text{(A18)}$$

Regarding the other constraint $\delta_{\text{crit}} > \max\{0, \frac{1-x-xk}{1-x}\}$, it further remains to check (A8). For $x \geqslant \frac{1}{k+1}$ it reduces to the trivial $\delta_{\text{crit}} > 0$.

Now assume $x < \frac{1}{k+1}$. One needs to check that

$$\delta_{\text{crit}} > \frac{1 - x - xk}{1-x} \quad \text{(A19)}$$

and in case it holds, δ_{crit} is admissible; otherwise, the boundary value at $L(\frac{1-x-xk}{1-x})$ should be considered instead of $L(\delta_{\text{crit}})$. The above inequality holds if (and only if)

$$\frac{16}{17 + 8k + 16k^2} < x. \quad \text{(A20)}$$

Using all the formulas and constraints above yields the resulting computation of $T(x,k)$ as appears in the text.

3. Optimal *m*-group policy

We would like to compute the optimal m-group policy and to measure its efficiency relative to random boarding. To keep track of all possible policies we will first discretize the values of ρ_i. The problem is that the number of possible group sizes grows exponentially in m. We therefore add an auxiliary variable and employ a dynamic programming procedure that computes the optimal score inductively and whose complexity grows linearly with m.

We will need an auxiliary quantity, which we denote by $L_1^{(m)}(z)$, the length of the maximal curve in an m-group partition of the unit square constrained to end at the point $(z,0)$. We compute $L_1^{(m)}(z)$ inductively. To begin the induction, we know from (A6) that

$$L_1^{(1)}(z) = \sqrt{k}z. \quad \text{(A21)}$$

In order to proceed with the induction we need to compute the maximal proper time (length) of a curve in the unit square that begins at the point $(0,r)$ and ends at $(q,0)$. The length is computed with respect to the Lorentzian metric given in (A1), which corresponds to the random boarding policy. We call this the corner length and denote it by $L_C(r,q)$. The length can be computed using (A3). Let $q^*(r) = \frac{1}{k}\ln(\frac{1}{1-\sqrt{r}})$ and $q_* = q_*(r) = \frac{1}{k}\ln(\frac{1}{1-r})$. We note that $q_*(r)$ is the minimal value of q for which there is a causal curve from $(0,r)$ to $(q,0)$. Let $Q = Q(q) = e^{kq}$. We have

$$L_C(r,q) = \sqrt{\frac{(Q-1)[(1-r)Q - 1]}{kQ}} \quad \text{(A22)}$$

when $q_* \leqslant q < q^*$ and

$$\frac{kq + \sqrt{r} + \ln(1 - \sqrt{r})}{\sqrt{k}} \quad \text{(A23)}$$

when $q > q^*$.

We can now use the corner length to compute inductively $L_1^{(m)}(z)$. We shall mostly reuse the arguments for the two-group case, however, instead of L_1 we use a scaled version of $L_1^{(m-1)}(z)$ to account for the contribution of the first $m-1$ squares, conditioned on the end point of the maximal curve. We then maximize over the possible choices of z. For $z \geqslant \frac{1}{k+1}$, we are led to the following inductive computation. Let

$$M(\delta,m,x,z) = \sqrt{x}L_1^{(m-1)}\left(\frac{\delta(1-x) + (k+1)x - 1}{kx}\right) + \sqrt{1-x}\,L_C\left(\delta, \frac{z-x}{1-x}\right) \quad \text{(A24)}$$

and let

$$\tilde{M}(m,x,z) = \max_{\delta} M(\delta,m,x,z),$$

where $\max_\delta$ is taken over $\delta_{\min} < \delta < 1 - \exp(-k\frac{z-x}{1-x})$. For $z \geqslant \frac{1}{k+1}$ we have

$$L_1^{(m)}(z) = \min_{0<x<z} \max\left\{\sqrt{1-x}L_1^{(1)}\left(\frac{z-x}{1-x}\right), \tilde{M}(m,x,z)\right\}. \tag{A25}$$

For $z < \frac{1}{k+1}$ we also need to consider the possibility that $x > z$ and that only the first $m-1$ squares will contribute; thus, if $z < \frac{1}{k+1}$, then $L_1^{(m)}(z)$ is defined as the minimum of (A25) and

$$\min_{z<x<1-kz}\left\{\sqrt{x}L_1^{(m-1)}\left[\left(z - \frac{1-x}{k}\right)\Big/x\right]\right\}. \tag{A26}$$

Having computed $L_1^{(m)}(z)$ recursively, we can compute T_m, the optimal boarding time for an m group back-to-front policy, recursively using $L_1^{(m-1)}$ and $L_2(\delta)$. This is done using the procedure for two-group policies and taking the appropriate value of the variable z that matches δ and x. Let

$$z(\delta,x) = \frac{\delta(1-x)+(k+1)x-1}{kx}.$$

We obtain the recursive formula

$$L^{(m)}(\delta,x) = \sqrt{x}L_1^{(m-1)}[z(\delta,x)] + \sqrt{1-x}L_2(\delta). \tag{A27}$$

Optimizing over all possible values of δ we set

$$\tilde{L}^{(m)}(x) = \max_\delta L^{(m)}(\delta,x),$$

where the $\max_\delta$ is, by (A8), taken over the interval $\delta_{\min} < \delta < 1$ Combining with the cases where not all cells are used, we see that the optimal time is given by

$$T_m = \min_{0<x<1} \max\{\sqrt{x}T_{m-1}, \sqrt{1-x}T_1, \tilde{L}^{(m)}(x)\}.$$

This is the value of the boarding time that appears in Table IV.

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