Velocity distribution function and effective restitution coefficient for a granular gas of viscoelastic particles

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We perform large-scale event-driven molecular dynamics (MD) simulations for granular gases of particles interacting with the impact-velocity-dependent restitution coefficient $\varepsilon(v_{\rm imp})$. We use $\varepsilon(v_{\rm imp})$ as it follows from the simplest first-principles collision model of viscoelastic spheres. Both cases of force-free and uniformly heated gases are studied. We formulate a simplified model of an effective constant restitution coefficient $\varepsilon_{\rm eff}$, which depends on a current granular temperature, and we compute $\varepsilon_{\rm eff}$ using the kinetic theory. We develop a theory of the velocity distribution function for driven gases of viscoelastic particles and analyze the evolution of granular temperature and of the Sonine coefficients, which characterize the form of the velocity distribution function. We observe that for not large dissipation the simulation results are in an excellent agreement with the theory for both the homogeneous cooling state and uniformly heated gases. At the same time, a noticeable discrepancy between the theory and MD results for the Sonine coefficients is detected for large dissipation. We analyze the accuracy of the simplified model based on the effective restitution coefficient $\varepsilon_{\rm eff}$, and we conclude that this model can accurately describe granular temperature. It provides also an acceptable accuracy for the velocity distribution function for small dissipation, but it fails when dissipation is large.

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I. INTRODUCTION

Granular materials are abundant in nature; possible examples range from sands and powders on Earth [1–5] to more dilute systems, termed as granular gases [6–9], in space. Astrophysical objects, such as planetary rings or dust clouds, are typical examples of granular gases [6,10]. Depending on the average kinetic energy of individual grains, granular systems can be in a solid, liquid, or gaseous state. In the present study, we address a low-density granular gas—a system comprised of a macroscopically large number of macroscopic grains that move ballistically between inelastic pairwise collisions. The inelastic nature of interparticle collisions is the main feature which distinguishes a granular gas from a molecular gas and gives rise to many astonishing properties of dissipative gases. If no external forces act on the system, it evolves freely and permanently cools down. During the first stage of a granular gas evolution, its density remains uniform and no macroscopic fluxes are present. This state, which is called a homogeneous cooling state (HCS) [8,11], is, however, unstable with respect to fluctuations and in later times clusters and vortices develop. The system then continues to lose its energy in the inhomogeneous cooling state (ICS) [12–19].

If energy is injected into a granular gas to compensate for its losses in dissipative collisions, the system settles into a nonequilibrium steady state [20,21]. There are different ways to put energy into the system: via shearing [22], vibration [23,24] or rotating [25,26] of the walls of a container, applying electrostatic [27] or magnetic [28] forces, etc. In planetary rings, the energy losses due to particle collisions are replenished by gravitational interactions [29,30]. To describe theoretically the injection of energy into a system, a few types of thermostat, which mimic an action of external forces, have been proposed [20]. Here we exploit a white-noise thermostat [31], where all particles are heated uniformly and independently (see the next sections for more details).

Energy dissipation in a pairwise collision is quantified by the restitution coefficient ε,

$$\varepsilon = \left| \frac{(\mathbf{v}_{12}' \cdot \mathbf{e})}{(\mathbf{v}_{12} \cdot \mathbf{e})} \right|, \tag{1}$$

where $\mathbf{v}_{12}' = \mathbf{v}_2' - \mathbf{v}_1'$ and $\mathbf{v}_{12} = \mathbf{v}_2 - \mathbf{v}_1$ are the relative velocities of two particles after and before a collision, and $\mathbf{e}$ is a unit vector connecting their centers at the collision instant. The postcollision velocities are related to the precollision velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ as follows [8]:

$$\mathbf{v}_{1/2}' = \mathbf{v}_{1/2} \mp \frac{1+\varepsilon}{2}(\mathbf{v}_{12} \cdot \mathbf{e})\mathbf{e}. \tag{2}$$

To date, most studies of the HCS and ICS have focused on the case of a constant restitution coefficient [11–19]. This assumption contradicts, however, experimental observations [32–34] along with basic mechanical laws [35,36], which indicate that ε does depend on the impact velocity [34,36–39]. This dependence may be obtained by solving the equations of motion for colliding particles with explicit account of the dissipative forces acting between the grains. The simplest first-principles model of inelastic collisions assumes viscoelastic properties of particle material, which results in viscoelastic interparticle force [37] and finally in the restitution coefficient [36,39]:

$$\varepsilon = 1 - C_1 A \kappa^{2/5} |\mathbf{v}_{12} \cdot \mathbf{e}|^{1/5} + C_2 A^2 \kappa^{4/5} |\mathbf{v}_{12} \cdot \mathbf{e}|^{2/5} + \cdots . \tag{3}$$

Here, the numerical coefficients are $C_1 \simeq 1.15$ and $C_2 \simeq 0.798$. The elastic constant

$$\kappa = \left(\frac{3}{2}\right)^{3/2} \frac{Y\sqrt{\sigma}}{m(1-\nu^2)} \tag{4}$$

is a function of Young's modulus Y, Poisson ratio ν, mass m, and diameter σ of the particles; the constant A quantifies the viscous properties of the particle material. While the last equation is valid for not very large inelasticity, in the recent work of Schwager and Poschel [40] an expression for ε, which is valid for high inelasticity and accounts for the delayed recovery of the shape of colliding particles, has been derived:

$$\varepsilon = 1 + \sum_{k=1}^{N_\varepsilon} h_k \delta^{k/2} \left[2u\left(t\right)\right]^{k/20} \left|\left(\mathbf{c}_{12}\cdot\mathbf{e}\right)\right|^{k/10}, \tag{5}$$

where $h_1 = 0$, $h_2 = -C_1$, $h_3 = 0$, $h_4 = C_2$, and other numerical coefficients up to $N_\varepsilon = 20$ are given in Ref. [40]. In the latter equation, $u(t) = T(t)/T_0$ $[T_0 = T(0)]$, with the usual definition of temperature $T(t)$,

$$\frac{3}{2} n T(t) = \int d\mathbf{v} \frac{m\mathbf{v}^2}{2} f(\mathbf{v},t). \tag{6}$$

Here n is the number density of the gas, $\mathbf{v}$ is a particle velocity, $f(\mathbf{v},t)$ is the time-dependent velocity distribution function, and $\mathbf{c}_{12}$ is the dimensionless relative velocity, defined as $\mathbf{c}_{12} = \mathbf{v}_{12}/v_T$, where $v_T(t) = \sqrt{2T(t)/m}$ is the thermal velocity. The dissipation constant δ in Eq. (5) reads

$$\delta = A\kappa^{2/5}\left(\frac{T_0}{m}\right)^{1/10}, \tag{7}$$

so that the first few terms in Eq. (5) are identical to those in Eq. (3). Note that $\delta = 0$ corresponds to an elastic system, $\varepsilon = 1$, and dissipation in the system increases with increasing δ. Furthermore, the restitution coefficient for all collisions increases and tends to unity, $\varepsilon \to 1$, when the (reduced) temperature decreases, that is, when a granular gas cools down.

Note that although we address here dry granular particles, they still can stick in collisions with a very small impact velocity. Namely, if the normal component of the relative velocity $|\mathbf{v}_{12}\cdot\mathbf{e}|$ is smaller than the sticking threshold,

$$g_{\rm st} = \sqrt{\frac{4q_0(\pi^5\gamma^5 D^2)^{1/3}}{m\varepsilon^2}\left(\frac{\sigma}{4}\right)^{4/3}},$$

where $D = 3(1-\nu^2)/2Y$, $q_0 \simeq 1.45$, and γ is the adhesion coefficient, the colliding particles cling together [42,43]; in this case, the restitution coefficient drops to zero. In the present study, we neglect these processes.

The velocity distribution function in granular gases usually deviates from a Maxwellian distribution [18,19,21,44,45] and may be described by the *Sonine-polynomial* expansion [8]:

$$\begin{aligned} f(\mathbf{v},t) &= \frac{n}{v_T^3}\tilde{f}(\mathbf{c},t), \\ \tilde{f}(\mathbf{c},t) &= \pi^{-3/2}\exp(-c^2)\left[1+\sum_{p=1}^{\infty} a_p(t) S_p(c^2)\right], \end{aligned} \tag{8}$$

where $\mathbf{c} = \mathbf{v}/v_T$ and the Sonine polynomials read for $d = 3$

$$S_p(c^2) = \sum_{k=0}^{p} \frac{(-1)^k (1/2+p)!}{(1/2+k)!(p-k)!k!} c^{2k}. \tag{9}$$

Hence the evolution of $f(\mathbf{v},t)$ is completely determined by the time dependence of the Sonine coefficients $a_p(t)$. The first Sonine coefficient is trivial, $a_1 = 0$, as it follows from the definition of temperature, e.g., [8]; in the elastic limit, that is, for a Maxwellian distribution, all Sonine coefficients are zero, $a_p = 0$. The Sonine coefficients characterize the successive moments $\langle c^{2k}\rangle$ of the velocity distribution function (8), so that the first few moments read

$$\begin{aligned} \langle c^2\rangle &= \frac{3}{2}, \\ \langle c^4\rangle &= \frac{15}{4}(1+a_2), \\ \langle c^6\rangle &= \frac{105}{8}(1+3a_2-a_3). \end{aligned} \tag{10}$$

It has been shown recently that for a granular gas with a constant ε, the expansion (8) converges for small and moderate dissipation, up to $\varepsilon \gtrsim 0.6$, but breaks down otherwise [46]. For a moderate dissipation, it is sufficient to consider only the first two nontrivial coefficients a_2 and a_3, which have been studied both analytically [21,44,46,47] and by means of computer simulations [15,18,19,46–49].

The theory developed for a granular gas with a constant ε in a homogeneous cooling state predicts a rapid relaxation of a_p to steady-state values, which depend on ε [21,46]. After a certain time, spatial structures, such as vortices and clusters, spontaneously arise in such a granular gas [12–19]. The velocity distribution function becomes effectively Gaussian, which happens due to an averaging over a large number of independent clusters, where particle velocities are approximately parallel; a_p, however, shows very strong fluctuations around zero mean values [18,19]. In a granular gas, comprised of viscoelastic particles, clusters and vortices appear only as transient phenomena [50]; moreover, cluster formation is completely suppressed in a granular gas, orbiting a massive body in its gravitational potential—a typical example of such systems is a Kepler disk in astrophysics [29].

In contrast to the case of a constant ε, the evolution of the velocity distribution function in a gas of viscoelastic particles demonstrates a complicated nonmonotonous time dependence [51,52]. Typically, the magnitudes of the Sonine coefficients first increase with time, reach maximal values, and finally tend to zero, so that asymptotically the Maxwellian distribution is recovered [51,52]. This simply follows from the fact that the collisions become more and more elastic as temperature decreases. Although the time dependence of $a_2(t)$ and $a_3(t)$ in a force-free granular gas of viscoelastic particles has been studied theoretically in [52], the respective analysis for heated gases is still lacking. Moreover, the evolution of the velocity distribution function and its properties in such gases have been never studied numerically by means of MD either for force-free or for driven systems. Hence, it is still not known how accurate the theory is for the velocity distribution function in this case and what is the range of convergence of the Sonine series with respect to the dissipation parameter δ.

Another interesting question refers to the possibility of a simplified description of a granular gas of viscoelastic particles using an effective constant restitution coefficient: While it is known that the model of a constant ε fails qualitatively to describe a complicated evolution of such gas in a homogeneous cooling state [50], it is still not clear how accurate could be the respective description for the case of a driven system. Also, it would be interesting to check whether a model of a "quasiconstant" restitution coefficient, where ε does not depend on an impact velocity but depends on current temperature, may be used for a force-free gas.

In the present study, we perform thorough large-scale event-driven MD simulations of the velocity distribution function in a gas of viscoelastic particles, both in a homogeneous cooling state and under a uniform heating. We develop a theory for $f(\mathbf{v},t)$ for driven gases and observe that our MD results for $a_2(t)$ and $a_3(t)$ are in excellent agreement with theoretical predictions of [52] for the HCS and of the novel theory for driven systems. We report simulation results for the high-order Sonine coefficients a_4 and a_5, which can barely be obtained from the kinetic theory. The analysis of these coefficients allows us to assess the convergence of the Sonine series. We also analyze the model of a "quasiconstant" restitution coefficient $\varepsilon_{\rm eff}$ and show that it can be used for an accurate description of a granular gas if dissipation is not large.

The paper is organized as follows. In Sec. II we develop a theory for the velocity distribution function for uniformly heated gases, that is, we compute time-dependent Sonine coefficients $a_2(t)$ and $a_3(t)$. In Sec. III we derive an effective "quasiconstant" restitution coefficient $\varepsilon_{\rm eff}$. In Sec. IV we report our MD results for a force-free and heated gas of viscoelastic particles. In Sec. V the theoretical predictions are compared with simulation results and the accuracy of the theory, based on $\varepsilon_{\rm eff}$, is scrutinized. Finally, in Sec. VI we summarize our findings.

II. VELOCITY DISTRIBUTION FUNCTION FOR UNIFORMLY HEATED GAS

Assuming the molecular chaos, which is an adequate hypothesis for dilute granular gases addressed here, e.g., [8], we can write the Boltzmann-Enskog (BE) equation, supplemented by the diffusive Fokker-Planck term, which mimics a uniform heating [21],

$$\frac{\partial f(\mathbf{v},t)}{\partial t} = g_2(\sigma) I(f,f) + \frac{\xi_0^2}{2}\frac{\partial^2}{\partial \mathbf{v}^2} f(\mathbf{v},t). \tag{11}$$

Here $I(f,f)$ is the collision integral and $g_2(\sigma)$ is the contact value of the pair correlation function that takes into account the excluded volume effects [8]. For the scaling distribution function $\tilde{f}(\mathbf{c},t)$ [cf. Eq. (8)], the above equation reads

$$\frac{\partial \tilde{f}}{\partial t} - \frac{1}{v_T}\frac{dv_T}{dt}\left(3\tilde{f} + c\frac{\partial \tilde{f}}{\partial c}\right) = g_2\sigma^2 n v_T \tilde{I} + \frac{1}{v_T^2}\frac{\xi_0^2}{2}\frac{\partial^2}{\partial \mathbf{c}^2}\tilde{f}, \tag{12}$$

where $\tilde{I}(\tilde{f},\tilde{f}) = \sigma^{-2}n^{-2}v_T^2 I(f,f)$ is the dimensionless collision integral, defined as

$$\tilde{I}(\tilde{f},\tilde{f}) = \int d\mathbf{c}_2 \int d\mathbf{e}\,\Theta(-\mathbf{c}_{12}\cdot\mathbf{e})|-\mathbf{c}_{12}\cdot\mathbf{e}| \times [\chi \tilde{f}(\mathbf{c}_1'',t)\tilde{f}(\mathbf{c}_2'',t) - \tilde{f}(\mathbf{c}_1,t)\tilde{f}(\mathbf{c}_2,t)], \tag{13}$$

with $\mathbf{c}_1''$ and $\mathbf{c}_2''$ being the (reduced) precollision velocities in the so-called inverse collision, resulting in the postcollision velocities $\mathbf{c}_1$ and $\mathbf{c}_2$ [8]. The Heaviside function $\Theta(-\mathbf{c}_{12}\cdot\mathbf{e})$ selects the approaching particles and the factor χ equals the product of the Jacobian of the transformation $(\mathbf{c}_1'',\mathbf{c}_2'') \to (\mathbf{c}_1,\mathbf{c}_2)$ and the ratio of the lengths of the collision cylinders of the inverse and the direct collisions [8]:

$$\chi \equiv \frac{|\mathbf{c}_{12}''|}{|\mathbf{c}_{12}|}\frac{\mathcal{D}(\mathbf{c}_1'',\mathbf{c}_2'')}{\mathcal{D}(\mathbf{c}_1,\mathbf{c}_2)}. \tag{14}$$

Now we substitute the Sonine-polynomial expansion for $\tilde{f}(\mathbf{c},t)$ [Eq. (8)] into Eq. (12) and neglect all terms with $p > 3$. Multiplying the resulting equation with c_1^2, c_1^4, and c_1^6, and integrating over $\mathbf{c}_1$, we obtain

$$\begin{aligned}
\frac{\partial\langle c^2\rangle}{\partial t} + \frac{1}{T}\frac{dT}{dt}\langle c^2\rangle &= -\sqrt{\frac{2T}{m}}g_2(\sigma)\sigma^2 n\mu_2 + 3\frac{m\xi_0^2}{2T},\\
\frac{\partial\langle c^4\rangle}{\partial t} + \frac{2}{T}\frac{dT}{dt}\langle c^4\rangle &= -\sqrt{\frac{2T}{m}}g_2(\sigma)\sigma^2 n\mu_4 + 10\langle c^2\rangle\frac{m\xi_0^2}{2T},\\
\frac{\partial\langle c^6\rangle}{\partial t} + \frac{3}{T}\frac{dT}{dt}\langle c^6\rangle &= -\sqrt{\frac{2T}{m}}g_2(\sigma)\sigma^2 n\mu_6 + 24\langle c^4\rangle\frac{m\xi_0^2}{2T}\\
&.
\end{aligned} \tag{15}$$

Here,

$$\mu_p = -\int d\mathbf{c}\, c^p \tilde{I}(\tilde{f},\tilde{f}) \tag{16}$$

is the pth moment of the dimensionless collision integral. The moments μ_p can be calculated analytically up to $O(\delta^{10})$ using a formula manipulation program as explained in detail in Ref. [8]. They can be written as follows:

$$\mu_p = \sum_{k=0}^{20}\big(M_k^{(p,0)} + M_k^{(p,2)}a_2 + M_k^{(p,3)}a_3 + M_k^{(p,22)}a_2^2 + M_k^{(p,33)}a_3^2 + M_k^{(p,23)}a_2a_3\big)\delta^{k/2}(2u)^{k/20}. \tag{17}$$

The numerical values of the coefficients $M_k^{(p,i)}$ for μ_p with $p = 2,4,6$ are given in the Appendix, Tables II–IV.

Let us define the dimensionless time $\tau = t/\tau_c(0)$, where

$$\tau_c^{-1}(t) = 4\sqrt{\pi}g_2(\sigma)\sigma^2 n\sqrt{\frac{T(t)}{m}}. \tag{18}$$

Using Eqs. (10), which express the moments of the reduced velocity $\langle c^{2k}\rangle$ in terms of the Sonine coefficients a_k, we recast

Eqs. (15) into the following form:

$$\begin{aligned}
\frac{du}{d\tau} &= -\frac{\sqrt{2}\mu_2}{6\sqrt{\pi}}u^{\frac{3}{2}} + \frac{K}{4},\\
\frac{da_2}{d\tau} &= \frac{\sqrt{2}\sqrt{u}}{3\sqrt{\pi}}\mu_2(1+a_2) - \frac{\sqrt{2}}{15\sqrt{\pi}}\mu_4\sqrt{u} - \frac{Ka_2}{2u}, \qquad (19)\\
\frac{da_3}{d\tau} &= \frac{\sqrt{u}}{\sqrt{2\pi}}\mu_2(1-a_2+a_3)\\
&\quad - \frac{\sqrt{2}}{5\sqrt{\pi}}\mu_4\sqrt{u} + \frac{2\sqrt{2u}}{105\sqrt{\pi}}\mu_6 - \frac{3a_3K}{4u},
\end{aligned}$$

where the moments μ_p, $p = 2,4,6$ are the functions of the Sonine coefficients a_2,a_3 and known numerical constants $M_k^{(p,i)}$, Eq. (17). The constant K reads

$$K = \frac{m^{3/2}\xi_0^2}{\sqrt{\pi}g_2(\sigma)\sigma^2 n T_0^{3/2}}. \qquad (20)$$

Equations (19) describe the evolution of temperature and the velocity distribution function for a heated gas of viscoelastic particles. For $K = 0$, that is, for $\xi_0^2 = 0$, Eqs. (19) reduce to corresponding equations of Ref. [52], which describe evolution of a_2 and a_3 in a force-free gas.

When a heated gas settles in a steady state, the granular temperature and the velocity distribution function become time-independent. Then Eqs. (19) yield the algebraic equations for the steady-state temperature and moments μ_p, $p = 2,4,6$:

$$T_{\text{s.s.}} = \left[\frac{3}{2\sqrt{2}}\frac{\xi_0^2 m^{3/2}}{\mu_2 g_2(\sigma)\sigma^2 n}\right]^{2/3}, \qquad (21)$$

$$\mu_4 = 5\mu_2, \qquad (22)$$

$$\mu_6 = \frac{105}{4}\mu_2(1+a_2). \qquad (23)$$

Solving this system in the linear approximation with respect to the dissipation parameter δ, we obtain the steady-state Sonine coefficients,

$$a_2^{\text{s.s.}} = -A_2\delta, \quad A_2 = 2^{1/5}\frac{157}{500}\Gamma\left(\frac{21}{10}\right)C_1 \simeq 0.435, \qquad (24)$$

$$a_3^{\text{s.s.}} = -A_3\delta, \quad A_3 = 2^{1/5}\frac{28}{500}\Gamma\left(\frac{21}{10}\right)C_1 \simeq 0.078. \qquad (25)$$

The above expressions for a_2 and a_3, together with Eq. (17) for μ_2, yield the corresponding analytical expression for the steady-state temperature $T_{\text{s.s.}}$.

III. EFFECTIVE RESTITUTION COEFFICIENT

The expression (5) for the restitution coefficient gives this quantity for a collision with a particular impact velocity. A natural question arises: Is it possible to define an average value of ε, which characterizes the whole ensemble and may be exploited as an effective constant restitution coefficient ε_{eff} for all impacts? This, however, should depend on the current temperature of a granular gas; see Eq. (5). Since ε_{eff} describes the collisions, it is natural to define this as a "collision average," e.g. [41],

$$\varepsilon_{\text{eff}} = \frac{\int d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{e} f_1 f_2 |\mathbf{v}_{12}\cdot\mathbf{e}|\,\Theta(-\mathbf{v}_{12}\cdot\mathbf{e})\,\varepsilon(|\mathbf{v}_{12}\cdot\mathbf{e}|)}{\int d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{e} f_1 f_2 |\mathbf{v}_{12}\cdot\mathbf{e}|\,\Theta(-\mathbf{v}_{12}\cdot\mathbf{e})},$$

where the velocity distribution functions $f_1 = f(\mathbf{v}_1,t)$ and $f_2 = f(\mathbf{v}_2,t)$ are given by Eq. (8). The integral in the above equation can be calculated up to $O(\delta^{10})$ using a formula manipulation program as explained in detail in Ref. [8]. Neglecting in the Sonine expansion (8) for f_1 and f_2 the high-order coefficients a_p with $p \geqslant 4$, we obtain

$$\varepsilon_{\text{eff}}(t) = 1 + \sum_{k=1}^{N_\varepsilon}\left[\delta^{k/2}(2u)^{k/20}\frac{B_k + B_k^{(2)}a_2 + B_k^{(3)}a_3 + B_k^{(22)}a_2^2 + B_k^{(33)}a_3^2 + B_k^{(23)}a_2a_3}{1 + B_0^{(2)}a_2 + B_0^{(3)}a_3 + B_0^{(22)}a_2^2 + B_0^{(33)}a_3^2 + B_0^{(23)}a_2a_3}\right] \qquad (26)$$

with the numerical coefficients B_k and $B_k^{(i)}$ given in the Appendix, Table I. The above expression, although general, is rather involved. If we neglect the dependence of ε_{eff} on the Sonine coefficients (that is, perform the collision averaging of ε with a Maxwellian distribution), Eq. (26) reduces to

$$\varepsilon_{\text{eff}}(t) = 1 + \sum_{k=1}^{N_\varepsilon} B_k\,\delta^{k/2}\,[2T(t)/T_0]^{k/20}, \qquad (27)$$

where the above coefficients B_k may also be written in a compact analytical form,

$$B_k = h_k\frac{2^{1+\frac{k}{20}}}{2+\frac{k}{10}}\Gamma\left[2+\frac{k}{20}\right]. \qquad (28)$$

Here $\Gamma(x)$ is the Gamma function. We wish to stress again that in contrast to the impact-velocity-dependent coefficient ε, Eq. (5), the above restitution coefficient ε_{eff} is the same for all collisions. Its dependence on the dissipation parameter δ is shown in Fig. 1 for different N_ε. As it follows from this figure, the series (27) demonstrates an excellent convergence for $\delta \lesssim 0.5$. In what follows, we will use the number of terms in the expansion (27) equal to $N_\varepsilon = 20$, which guarantees an accurate description of the effective restitution coefficient ε_{eff} for the dissipative parameter δ up to $\delta \lesssim 0.5$, that is, for a rather large dissipation.

The system of equations, Eqs. (19), may also be used for a constant restitution coefficient, or, equally for the quasiconstant coefficient ε_{eff}, Eq. (26), provided the respective moments μ_p are expressed in terms of a_2, a_3, and ε_{eff}, as it follows for the case of a constant ε [46]. We write these in the following form:

$$\mu_2 = \frac{1}{16384}\sqrt{2\pi}\left(1-\varepsilon_{\rm eff}^2\right)\left(35a_3^2+144a_2^2 + 120a_2a_3+3072a_2+256a_3+16\,384\right),$$

$$\mu_4 = \frac{1}{32768}\sqrt{2\pi}(1+\varepsilon_{\rm eff})\sum_{k=0}^{3}\left(N_k^{(4,0)}+N_k^{(4,2)}a_2 + N_k^{(4,3)}a_3+N_k^{(4,22)}a_2^2+N_k^{(4,33)}a_3^2+N_k^{(4,23)}a_2a_3\right)\varepsilon_{\rm eff}^k,$$

$$\mu_6 = -\frac{3}{262\,144}\sqrt{2\pi}(1+\varepsilon_{\rm eff})\sum_{k=0}^{5}\left(N_k^{(6,0)}+N_k^{(6,2)}a_2 + N_k^{(6,3)}a_3+N_k^{(6,22)}a_2^2+N_k^{(6,33)}a_3^2+N_k^{(6,23)}a_2a_3\right)\varepsilon_{\rm eff}^k \tag{29}$$

with the coefficients $N_k^{(p,i)}$ for $p = 4{,}6$ listed in the Appendix, Table V.

Using Eqs. (19) and Eqs. (29) together with Eq. (26), one can find the temperature of a gas and the Sonine coefficients a_2 and a_3. Ultimately, $\varepsilon_{\rm eff}$ as a function of current temperature, as it follows from Eq. (26), may be found. This, however, is very close to the simplified form (27), so that the difference between the results, obtained with the use of the complete expression (26) and the simplified one, (27), can hardly be distinguished in the figures (see Fig. 2). Hence, for practical purposes one can safely use the simple expression (27) for the effective quasiconstant restitution coefficient.

IV. EVENT-DRIVEN SIMULATIONS

A. Force-free gas

We perform event-driven MD simulations [53] of a granular gas of smooth viscoelastic particles with the restitution coefficient (5). The grains are modeled as identical hard spheres of mass $m = 1$ and diameter $\sigma = 1$. In event-driven simulations, particles move freely between pairwise collisions, where the velocities of the grains are updated according to Eq. (2). We use the system of $N = 2.048 \times 10^6$ particles placed in a three-dimensional cubic box with the edge $L = 418$, which corresponds to the number density of $n = 0.028$.

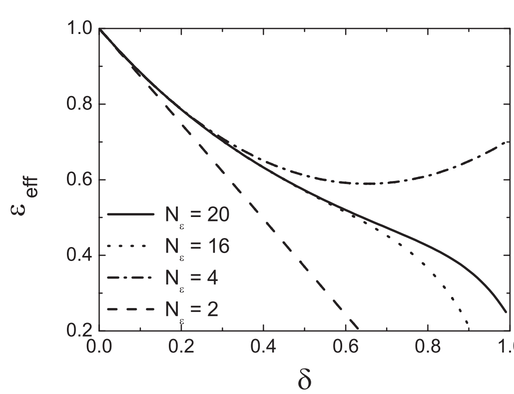

FIG. 1. An effective normal restitution coefficient $\varepsilon_{\rm eff}$, obtained as a "collision average" for $T = T_0$ and different values of N_ε [Eq. (27)].

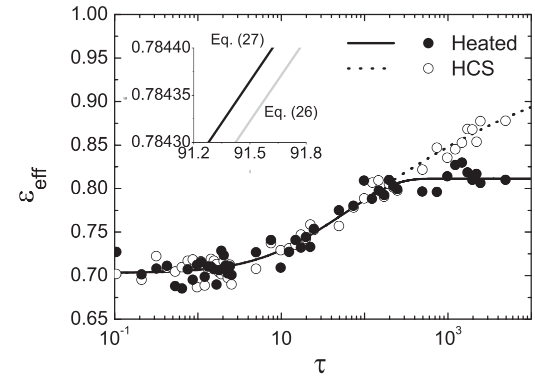

FIG. 2. Evolution of the average normal restitution coefficient, obtained in the MD simulations (symbols), and of the effective "quasiconstant" restitution coefficient $\varepsilon_{\rm eff}$, given by Eq. (26) (gray lines) and Eq. (27) (black lines). The difference between solutions of Eqs. (26) and (27) is shown in the inset (for heated case) and is not visible in the main plot. The dissipation constant $\delta = 0.3$. The dimensionless time $\tau = t/\tau_c(0)$ is expressed in terms of the initial mean collision time $\tau_c(0)$, Eq. (18). Note that the restitution coefficient tends to unity in the homogeneous cooling state and to a steady-state value for the heated gas.

This number density is rather small, which safely allows us to approximate the contact value of the correlation function by unity, $g_2(\sigma) \simeq 1$. Periodic boundary conditions are applied in all three directions.

The system was initialized by randomly placing particles in the box. For initial particle velocities, randomly directed vectors of the same length have been assigned. The initial total momentum $\sum_i m\mathbf{v}_i(0)$ was in this case with a high accuracy zero. We start with $\varepsilon = 1$, which corresponds to elastic collisions, and allow the system to relax to a Maxwellian velocity distribution, which used to happen in a few collisions per particle. Hence, a homogeneous system with a Maxwellian distribution was an initial condition for each MD run; then we evolve the system with a velocity-dependent restitution coefficient (5). In simulations we used the initial granular temperature $T(0) = 400/3$ with the corresponding initial mean collision time $\tau_c(0) = 0.436$. All statistical quantities presented here have been obtained as averages over 25 independent runs. We have confirmed that the system remains approximately homogeneous during our simulations and that the molecular chaos hypothesis holds true with a high degree of accuracy, as it used to be in dilute gases; see, e.g., [54].

In Fig. 2, we plot the time dependence of the current average of the restitution coefficient; this was obtained averaging ε over $(N/20)$ successive particles collisions. As it follows from the figure, the average restitution coefficient is in good agreement with the theoretical prediction for the effective "quasiconstant" coefficient $\varepsilon_{\rm eff}$, Eq. (27), for both freely evolving and heated granular gases.

In Fig. 3, we present simulation results for the time dependence of the reduced granular temperature $u(\tau)$ on the dimensionless time $\tau = t/\tau_c(0)$, expressed in terms of the initial mean collision time $\tau_c(0)$, Eq. (18). In a HCS, temperature decays asymptotically as the power law $u(\tau) \sim \tau^{-5/3}$ [8,51,52], while in a heated gas it relaxes to a steady-state value.

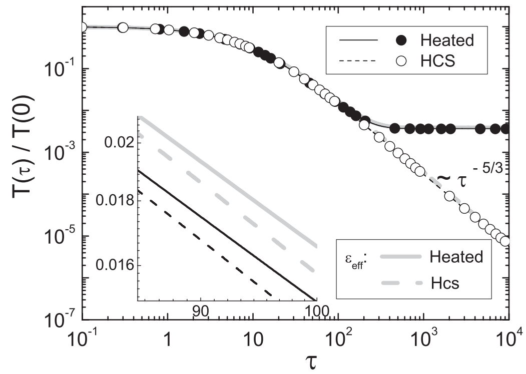

FIG. 3. Evolution of the reduced granular temperature $u(\tau) = T(\tau)/T(0)$ for the force-free and heated granular gases. The dissipation parameter $\delta = 0.3$. The symbols denote simulation results and thin black lines denote theory [numerical solution of Eqs. (19), where μ_p are given by Eqs. (17)]. For $\tau \gg 1$, $T/T(0) \sim \tau^{-5/3}$ for a force-free gas and $T \to T_{\rm s.s.}$ for a heated gas. Thick gray lines depict the theoretical predictions for temperature of a granular gas with a "quasiconstant" restitution coefficient $\varepsilon_{\rm eff}(t)$ [numerical solution of Eqs. (19), where μ_p are given by Eqs. (29) and $\varepsilon_{\rm eff}(t)$ by Eq. (27)].

Next, we analyze the evolution of the velocity distribution function, characterized by the time-dependent Sonine coefficients [cf. Eq. (8)]. To calculate a_2 and a_3 numerically, we compute the moments of the velocity distribution function $\langle c^{2k}\rangle$ and use Eqs. (10). In Fig. 4, we show the evolution of a_2 and a_3 for a force-free gas for $\delta = 0.15$, while in Fig. 5 the respective simulation data for the Sonine coefficients a_2, a_3, a_4, and a_5 for $\delta = 0.1$ and 0.3 are given.

B. Uniformly heated gas

The equation of motion for the ith particle in a white-noise thermostat, that is, for the gas under a uniform heating, reads

$$m\frac{d\mathbf{v}_i}{dt} = \mathbf{F}_i^{\rm coll} + \mathbf{F}_i^{\rm ext}. \tag{30}$$

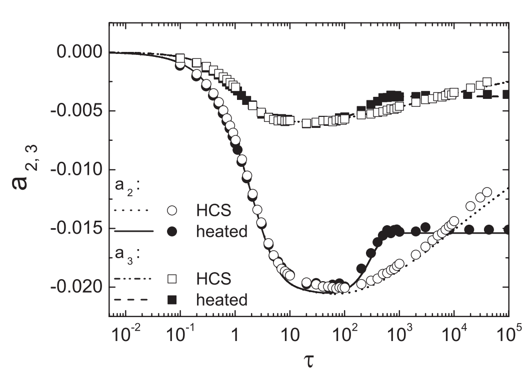

FIG. 4. Evolution of a_2 and a_3 for the force-free and heated granular gases with $\delta = 0.15$. The symbols correspond to simulation results, the lines to our theory [numerical solution of Eqs. (19), where μ_p are given by Eqs. (17)].

Here $\mathbf{F}_i^{\rm coll}$ is the force acting on the ith particle only in a pairwise collision with another gas particle, while $\mathbf{F}_i^{\rm ext}$ is the external force acting from the thermostat. Since we use the event-driven simulations [53], we do not need to solve explicitly the equation of motion for colliding particles with the forces $\mathbf{F}^{\rm coll}$. Instead, we directly apply the collision rules (2) and write the after-collision velocities, expressed in terms of the precollision ones with the restitution coefficient ε, Eq. (5) [8,53]. For the external force $\mathbf{F}^{\rm ext}$ we use the model of a Gaussian white noise:

$$\langle \mathbf{F}_i^{\rm ext}(t)\rangle = 0, \quad \langle F_{i,\alpha}^{\rm ext}(t) F_{j,\beta}^{\rm ext}(t')\rangle = m^2\xi_0^2\delta_{ij}\delta_{\alpha\beta}\delta(t-t'), \tag{31}$$

where $\alpha,\beta = x,y,z$, and ξ_0^2 characterizes the magnitude of the stochastic force.

We apply the algorithm suggested in Ref. [31] (see also [20,55]): During the event-driven simulations, we heat the system after a time step dt by adding to the velocity of each ith particle a random increment, which mimics the heating by noise,

$$v_{i,\alpha}(t+dt) = v_{i,\alpha}(t) + \sqrt{r}\sqrt{dt}\varphi, \tag{32}$$

where $\alpha = x,y,z$. The random number φ is uniformely distributed within the interval $[-0.5, 0.5]$, and r is the amplitude of the noise, $r = 12\xi_0^2$. After the change in velocities, the system is transferred to the center-of-mass frame,

$$\mathbf{v}_i = \mathbf{v}_i - \frac{1}{N}\sum_{i=1}^{N}\mathbf{v}_i,$$

to ensure the conservation of the momentum. In simulations we used $r = 0.1$ and $dt = 0.1$, which implies that the random kicking interval is small as compared to the mean free time. The number of particles and the system size were the same as in the force-free simulations; the initial conditions have also been prepared as in the force-free case. All statistical quantities were obtained as averages over 10 independent runs. The results are presented in Figs. 2–5, in which we show evolution of the average restitution coefficient (Fig. 2), of temperature (Fig. 3), and of the Sonine coefficients (Figs. 4 and 5).

V. RESULTS AND DISCUSSION

Let us compare the results of the MD simulations with the predictions of our theory. As we already mentioned, the current average of the restitution coefficient, obtained by a straightforward averaging of ε in successive collisions, is in good agreement with the theoretical value of the effective quasiconstant restitution coefficient $\varepsilon_{\rm eff}$, found as a collision average from the kinetic theory, Fig. 2. In Fig. 3, the evolution of granular temperature is shown. As it follows from the figure, during the early stages of cooling, the temperature of a heated granular gas evolves in the same way as the respective force-free gas with the same δ. However, while the temperature of the heated gas settles at a steady-state value, the force-free gas continues to cool down. The results of our theory [numerical solution of a system of differential equations

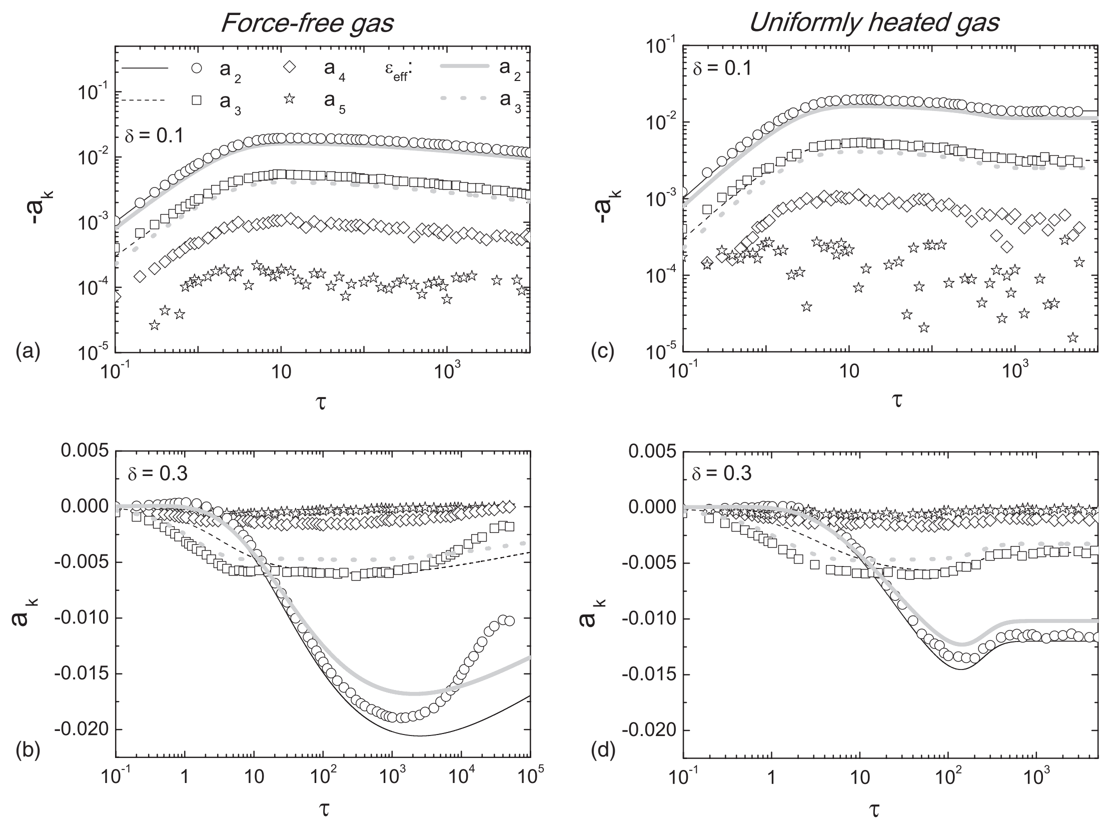

FIG. 5. Evolution of the Sonine coefficients a_k ($k = 2,3,4,5$) for a force-free (a,b) and uniformly heated (c, d) granular gas with $\delta = 0.1$ (a,c) and $\delta = 0.3$ (b,d). The symbols correspond to the MD simulation results and the thin black lines to the theory [numerical solution of Eqs. (19), where μ_p are given by Eqs. (17)]. Thick gray lines depict theoretical predictions [numerical solution of Eqs. (19), where μ_p are given by Eqs. (29)] for a granular gas with a "quasiconstant" restitution coefficient $\varepsilon_{\rm eff}(t)$, Eq. (27).

(19), where moments of collisional integral μ_p are given by Eqs. (17)] are in excellent agreement with the simulation data for both the homogeneous cooling state and uniformly heated gas. Moreover, the kinetic theory based on the effective quasiconstant restitution coefficient $\varepsilon_{\rm eff}$ [numerical solution of Eqs. (19), where μ_p are given by Eqs. (29)] yields almost the same accuracy for the description of granular temperature as the complete theory.

In Fig. 4, the evolution of the Sonine coefficients a_2 and a_3 is presented for the force-free and heated gas with $\delta = 0.15$. At the initial stage of evolution, for $\tau \leqslant 100$, the values of a_2 and a_3 in a heated and force-free gas are practically indistinguishable. At later times, the Sonine coefficients for the heated gas settle at the steady state [cf. Eqs. (24) and (25)]. For the freely evolving gas they continue to decay and evolve as follows: Starting from zero, $a_k(0) = 0$ ($k = 1,2,\ldots$), which corresponds to the initial Maxwellian distribution, a_2 and a_3 decrease, reach a minimum, and eventually return back to zero; see Fig. 4. As the thermal velocity decreases during the gas cooling, the behavior of the system tends to that of a gas of elastic particles, that is, the velocity distribution tends asymptotically to a Maxwellian distribution. For both the force-free and heated gases, our theory agrees perfectly with the computer simulations.

To investigate the convergence of the Sonine-polynomial expansion (8), we analyze the high-order Sonine coefficients, a_4 and a_5. In Fig. 5(a), the simulation data for the time dependence of a_2, a_3, a_4, a_5 along with the analytical predictions for a_2 and a_3 are shown. As it may be seen from Fig. 5(a), a_4 and a_5 evolve in a similar fashion as a_2 and a_3, that is, their magnitudes first increase, reach a maximum, and relax back to zero. Certainly this is not surprising, since it is a simple manifestation of the fact that the system starts from a Maxwellian velocity distribution due to initial conditions and returns to it later, when the collisions become elastic. Furthermore, a_4 is an order of magnitude smaller than a_3, while a_5 is $\sim 10^2$ times smaller than a_2. This confirms the convergence of the Sonine-polynomial expansion for the velocity distribution, at least for these values of δ. In this figure, we also show, in addition to the predictions of the complete theory, theoretical results for the case of an effective quasiconstant restitution coefficient $\varepsilon_{\rm eff}(t)$. Again we see that the accuracy of the simplified theory, based on the effective $\varepsilon_{\rm eff}(t)$, is surprisingly good.

In Fig. 5(b), the evolution of the Sonine coefficients for the larger dissipation, $\delta = 0.3$, is illustrated. The behavior of the coefficients is qualitatively the same as for the smaller δ. Moreover, the subsequent coefficients a_k decrease by an order of magnitude with increasing k, which again indicates the convergence of the Sonine-polynomial expansion for this dissipation. At the same time, a rather noticeable discrepancy between the kinetic theory and the MD results is obvious. In the earlier work [52], two of us have reported that the expansion (5) for ε with $N_\varepsilon = 20$ does not yield an acceptable accuracy for the Sonine coefficient a_2 and a_3 for $\delta \geqslant 0.3$ (when the complete theory is used). The discrepancy between the theoretical curves and the simulation data, seen in Fig. 5(b), seemingly supports this theoretical conclusion.

Figure 5(c) demonstrates the convergence of the Sonine-polynomial expansion for heated granular gases. As in Fig. 5(a), we plot here the dependence of a_2, a_3, a_4, and a_5 on τ for $\delta = 0.1$. It may be seen that the successive Sonine coefficients decrease by an order of magnitude, which clearly indicates the convergence of this series. As has been already mentioned in the discussion of Fig. 4, our complete theory is in perfect agreement with the simulation data for the heated gas, but here we also notice that the simplified model with a quasiconstant restitution coefficient $\varepsilon_{\rm eff}$ gives a rather accurate description of the evolution of a_2 and a_3 too.

Finally, in Fig. 5(d), we plot the time dependence of a_2, a_3, a_4, and a_5 for a heated gas with relatively large dissipation, $\delta = 0.3$. In this case, one can again see a noticeable discrepancy between our theory and the numerical results [cf. Fig. 5(b) for the freely evolving gas]. Nevertheless, the smallness of the high-order coefficients a_4 and a_5 seemingly manifests the convergence of the Sonine series. Interestingly, although the simulation data deviate from the theoretical curves during the relaxation to the steady state, in the steady state itself the numerical and theoretical values of the complete theory for the Sonine coefficients a_2 and a_3 are very close; see Fig. 5(d) for $\tau > 10^3$. At the same time, the results of the simplified model of the quasiconstant restitution coefficient $\varepsilon_{\rm eff}$ do not agree well with the MD results at the steady state for this dissipation.

VI. CONCLUSION

To date, most studies of granular gases have been focused on systems with a constant restitution coefficient, although the basic physics requires impact-velocity dependence of ε. Here we studied numerically, by means of molecular dynamics, and theoretically a granular gas of particles with the impact-velocity-dependent restitution coefficient, as it follows from the simplest first-principles model of viscoelastic spheres. Since the impact-velocity dependence of ε significantly complicates the description of a granular gas dynamics, we tried to develop a simplified collision model. Namely, we considered a model in which ε is the same for all collisions but depends on the current value of the thermal velocity of the gas, or, equally, on granular temperature. Using the collision average of the kinetic theory, we obtained an explicit expression for this quasiconstant restitution coefficient $\varepsilon_{\rm eff}[T(t)]$.

We explored the evolution of granular temperature and the velocity distribution function in large-scale MD simulations of force-free and uniformly heated gases, and we developed a theory of the velocity distribution function $f(\mathbf{v},t)$ for the case of driven gases. In the MD simulations, we found a few first Sonine coefficients a_2, a_3, a_4, and a_5, which quantify deviations of $f(\mathbf{v},t)$ from a Maxwellian distribution, and we noticed that the successive-order Sonine coefficients decrease by an order of magnitude with respect to the preceding ones. In other words, we observed that a_3 is approximately 10 times smaller than a_2, a_4 is 10 times smaller than a_3, and a_5 is 10 times smaller than a_4. This implies the convergence of the Sonine series for both heated and force-free gases in the range of dissipation studied.

We obtained theoretical predictions for a_2 and a_3 using the complete theory and the one based on an effective quasiconstant restitution coefficient $\varepsilon_{\rm eff}$. We found that the results of our complete theory for granular temperature are in very good agreement with the simulation data for both force-free and heated gases. Furthermore, if dissipation is not large, $\delta = 0.1$ or 0.15, the dependence on time of the Sonine coefficient, obtained in simulations, is also in excellent agreement with the theoretical predictions of the complete theory for a homogeneous cooling state (the previously existing theory) as well as for a uniform heating (the novel theory). At the same time for large dissipation, $\delta = 0.3$, a noticeable discrepancy between the theoretical and numerical curves is observed, especially for the homogeneous cooling; deviations for the uniform heating are also noticeable but less pronounced. Moreover, the steady-state values of a_2 and a_3 are given rather accurately by the new theory for heated gases.

We also analyzed the accuracy of the simplified model based on the effective quasiconstant restitution coefficient $\varepsilon_{\rm eff}(t)$. We have observed that this model can accurately describe the evolution of temperature for all studied dissipations and of the velocity distribution function, provided dissipation is not large, for both, force-free and heated gases. However, for large dissipation the simplified model allows only a qualitative description of $f(\mathbf{v},t)$ and fails to provide a quantitative one. Still we wish to stress that the simple model, where $\varepsilon_{\rm eff}(t)$ is determined by the granular temperature and material properties of the grains, may be important for applications. Indeed, the description of a granular gas dynamics for the case of a constant restitution coefficient is significantly simpler than that of an impact-velocity-dependent ε.

ACKNOWLEDGMENTS

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APPENDIX: NUMERICAL COEFFICIENTS

The effective restitution coefficient $\varepsilon_{\rm eff}$ and moments of the dimensionless collision integral $\mu_p = -\int d\mathbf{c}\, c^p \tilde{I}(\tilde{f},\tilde{f})$, where $p = 2,4,6$, can be calculated using a formula manipulation program as explained in detail in Ref. [8]. In Table I, we present coefficients $B_k^{(i)}$ which define $\varepsilon_{\rm eff}$ [Eqs. (27) and (26)]. In Tables II–IV, we give numerical values of the coefficients $M_k^{(p,i)}$ for $\mu_p (p = 2,4,6)$ [Eqs. (17)] in the case of a granular gas of viscoelastic particles and in Table V–the coefficients $N_k^{(p,i)}$ for $\mu_p (p = 4,6)$ [Eqs. (29)] for a granular gas with a constant restitution coefficient.

TABLE I. Numerical coefficients B_k and $B_k^{(i)}$ for calculation of the effective quasiconstant restitution coefficient $\varepsilon_{\rm eff}$ [Eqs. (26) and (27)].

k	B_k	$B_k^{(2)}$	$B_k^{(3)}$	$B_k^{(22)}$	$B_k^{(33)}$	$B_k^{(23)}$
0		−0.0625	−0.0156	−0.0146	−0.0064	−0.017
1	0	0	0	0	0	0
2	−1.176	0.07	0.016	0.0148	0.00616	0.0168
3	0	0	0	0	0	0
4	0.84	−0.044	−0.0096	−0.00826	−0.00326	−0.009
5	0.287	−0.0135	−0.0028	−0.00237	−0.00091	−0.0026
6	−0.578	0.0231	0.0046	0.0038	0.00142	0.0041
7	−0.524	0.0167	0.0032	0.00258	0.000937	0.0027
8	0.408	−0.00919	−0.00168	−0.00133	−0.00047	−0.0014
9	0.547	−0.0065	−0.0011	−0.00087	−0.0003	−0.00089
10	−0.184	0	0	0	0	0
11	−0.479	−0.0063	−0.00099	−0.000727	−0.000235	−0.00072
12	−0.0586	−0.00161	−0.00024	−0.00017	−0.000054	−0.00017
13	0.398	0.017	0.00243	0.00169	0.00051	0.0016
14	0.218	0.013	0.00175	0.00118	0.00035	0.0011
15	−0.277	−0.022	−0.0027	−0.00178	−0.00051	−0.0016
16	−0.289	−0.028	−0.00329	−0.002	−0.00058	−0.0019
17	0.112	0.0133	0.0014	0.00089	0.00024	0.00079
18	0.308	0.043	0.0043	0.00258	0.00067	0.0022
19	0.0478	0.0078	0.00071	0.00042	0.0001	0.00035
20	−0.276	−0.0517	−0.0043	−0.0024	−0.00059	−0.002

TABLE II. Numerical coefficients for μ_2 for a gas of viscoelastic particles [Eqs. (17)].

k	$M_k^{(2,0)}$	$M_k^{(2,2)}$	$M_k^{(2,3)}$	$M_k^{(2,22)}$	$M_k^{(2,33)}$	$M_k^{(2,23)}$
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	6.49	1.56	0.10	0.054	0.012	0.044
3	0	0	0	0	0	0
4	−9.29	−2.76	−0.14	−0.067	−0.014	−0.052
5	−1.80	−0.59	−0.025	−0.012	−0.0023	−0.0087
6	10.40	3.74	0.12	0.056	0.011	0.041
7	5.91	2.32	0.058	0.025	0.0047	0.018
8	−10.44	−4.46	−0.074	−0.031	−0.0055	−0.021
9	−10.53	−4.88	−0.04	−0.016	−0.0028	−0.011
10	8.77	4.39	-6.28×10^{-8}	1.57×10^{-7}	-6.91×10^{-7}	-1.10×10^{-6}
11	14.13	7.60	−0.063	−0.023	−0.0036	−0.015
12	−4.54	−2.62	0.044	0.015	0.0023	0.0093
13	−16.37	−10.12	0.25	0.080	0.012	0.050
14	−1.58	−1.04	0.035	0.010	0.0015	0.0063
15	16.74	11.77	−0.49	−0.14	−0.018	−0.08
16	8.09	6.05	−0.30	−0.079	−0.01	−0.045
17	−14.29	−11.33	0.66	0.16	0.02	0.089
18	−13.91	−11.69	0.779	0.175	0.02	0.0935
19	8.82	7.83	−0.59	−0.12	−0.013	−0.063
20	18.13	16.99	−1.42	−0.27	−0.028	−0.13

TABLE III. Numerical coefficients for μ_4 for a gas of viscoelastic particles [Eqs. (17)].

k	$M_k^{(4,0)}$	$M_k^{(4,2)}$	$M_k^{(4,3)}$	$M_k^{(4,22)}$	$M_k^{(4,33)}$	$M_k^{(4,23)}$
0	0	10.03	−2.51	0.31	0.049	0.16
1	0	0	0	0	0	0
2	36.32	46.85	−6.50	−0.29	−0.015	−0.14
3	0	0	0	0	0	0
4	−71.50	−100.66	16.09	0.59	0.029	0.25
5	−10.36	−15.39	2.66	0.077	0.0034	0.032
6	116.21	170.73	−29.24	−0.98	−0.054	−0.39
7	45.69	73.09	−13.997	−0.275	−0.012	−0.105
8	−169.5	−260.38	47.12	1.01	0.055	0.37
9	−117.70	−196.09	39.54	0.34	0.016	0.12
10	219.56	354.27	−67.36	9.42×10^{-7}	-9.59×10^{-5}	2.20×10^{-6}
11	234.61	406.45	−85.03	0.94	0.042	0.30
12	−240.62	−406.38	79.92	−2.49	−0.11	−0.75
13	−398.72	−724.58	157.86	−6.21	−0.248	−1.76
14	198.27	345.6	−67.8	5.33	0.22	1.42
15	597.98	1147.26	−261.54	19.24	0.66	4.81
16	−57.49	−83.78	7.19	−3.89	−0.149	−0.91
17	−797.17	−1618.27	386.31	−43.42	−1.25	−9.41
18	−208.15	−471.79	132.09	−11.56	−0.26	−2.31
19	935.74	2010.21	−501.69	77.96	1.81	14.30
20	604.56	1387.68	−379.52	55.88	1.1	9.31

TABLE IV. Numerical coefficients for μ_6 for a gas of viscoelastic particles [Eqs. (17)].

k	$M_k^{(6,0)}$	$M_k^{(6,2)}$	$M_k^{(6,3)}$	$M_k^{(6,22)}$	$M_k^{(6,33)}$	$M_k^{(6,23)}$
0	0	112.80	−84.60	−4.93	0.022	−0.82
1	0	0	0	0	0	0
2	209.94	633.78	−245.02	16.75	0.26	2.23
3	0	0	0	0	0	0
4	−525.04	−1718.36	717.54	−48.53	−0.45	−4.85
5	−61.21	−203.84	86.30	−7.2	−0.061	−0.60
6	1097.057	3731.6	−1607.5	133.88	0.64	8.93
7	338.22	1206.7	−544.06	46.04	0.18	2.3
8	−2042.4	−7140.7	3126.2	−347.5	−0.75	−11.58
9	−1116.58	−4131.0	1913.43	−196.42	−0.21	−3.27
10	3412.0	12219.7	−5395.76	806.66	−0.0022	−0.0012
11	2856.1	10832.9	−5076.7	683.4	−0.647	−11.39
12	−5055.2	−18485.3	8165.6	−1634.9	3.29	54.5
13	−6187.9	−24012.3	11320	−2006.6	5.40	100.33
14	6435.9	23886	−10417.0	2818.3	−10.83	−187.9
15	11807	46936.9	−22224.5	5062.8	−21.37	−421.9
16	−6433.2	−23805.0	9879.0	−3867.0	21.14	386.7
17	−20203.4	−82418.7	39185.8	−11160.6	60.27	1302.07
18	3203.3	10084	−2516.6	3251.2	−23.5	−433.49
19	31188.6	130743.0	−62400.8	21721.9	−133.14	−3258.3
20	5784.3	29433.95	−17812.7	2307.9	−6.45	−384.65

TABLE V. Numerical coefficients for μ_4 and μ_6 for a gas with a constant restitution coefficient [Eqs. (29)].

k	$N_k^{(4,0)}$	$N_k^{(4,2)}$	$N_k^{(4,3)}$	$N_k^{(4,22)}$	$N_k^{(4,33)}$	$N_k^{(4,23)}$
0	147456	277504	−46336	2192	455	1096
1	−147456	−211968	29952	−144	−135	−72
2	32768	30720	−2560	−480	−50	−240
3	−32768	−30720	2560	480	50	240
k	$N_k^{(6,0)}$	$N_k^{(6,2)}$	$N_k^{(6,3)}$	$N_k^{(6,22)}$	$N_k^{(6,33)}$	$N_k^{(6,23)}$
0	−1884160	−8496128	4311296	26448	−3113	4408
1	1884160	7054336	−3229952	10416	1193	1736
2	−720896	−2920448	1401856	52032	2396	8672
3	720896	2396160	−1008640	−2880	−860	−480
4	−131072	−286720	71680	−13440	−280	−2240
5	131072	286720	−71680	13440	280	2240

[1] *Physics of Dry Granular Media*, edited by H. J. Herrmann, J.-P. Hovi, and S. Luding, NATO Advanced Study Institute Series (Kluwer, Dordrecht, 1998).
[2] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).
[3] *The Physics of Granular Media*, edited by H. Hinrichsen and D. E. Wolf (Wiley, Berlin, 2004).
[4] G. H. Ristow, *Pattern Formation in Granular Materials* (Springer-Verlag, Berlin, 2000).
[5] J. Duran, *Sands, Powders and Grains* (Springer-Verlag, Berlin, 2000).
[6] *Granular Gases*, edited by T. Poeschel and S. Luding, Lecture Notes in Physics, Vol. 564 (Springer, Berlin, 2001).
[7] *Granular Gas Dynamics*, edited by T. Poeschel and N. V. Brilliantov, Lecture Notes in Physics, Vol. 624 (Springer, Berlin, 2003).
[8] N. V. Brilliantov and T. Pöschel, *Kinetic Theory of Granular Gases* (Oxford University Press, Oxford, 2004).
[9] I. Goldhirsch, Annu. Rev. Fluid Mech. **35**, 267 (2003).
[10] *Planetary Rings*, edited by R. Greenberg and A. Brahic (Arizona University Press, Tucson, 1984).
[11] P. K. Haff, J. Fluid Mech. **134**, 401 (1983).
[12] I. Goldhirsch and G. Zanetti, Phys. Rev. Lett. **70**, 1619 (1993).
[13] S. K. Das and S. Puri, Europhys. Lett. **61**, 749 (2003).
[14] S. K. Das and S. Puri, Phys. Rev. E **68**, 011302 (2003).
[15] H. Nakanishi, Phys. Rev. E **67**, 010301 (2003).
[16] T. P. C. van Noije, M. H. Ernst, R. Brito, and J. A. G. Orza, Phys. Rev. Lett. **79**, 411 (1997).
[17] T. P. C. van Noije, M. H. Ernst, and R. Brito, Phys. Rev. E **57**, R4891 (1998).
[18] S. R. Ahmad and S. Puri, Europhys. Lett. **75**, 56 (2006).
[19] S. R. Ahmad and S. Puri, Phys. Rev. E **75**, 031302 (2007).
[20] J. M. Montanero and A. Santos, Granular Matter **2**, 53 (2000).
[21] T. P. C. van Noije and M. H. Ernst, Granular Matter **1**, 57 (1998).
[22] L. Bocquet, W. Losert, D. Schalk, T. C. Lubensky, and J. P. Gollub, Phys. Rev. E **65**, 011307 (2001).
[23] R. D. Wildman and D. J. Parker, Phys. Rev. Lett **88**, 064301 (2002).
[24] K. Feitosa and N. Menon, Phys. Rev. Lett **88**, 198301 (2002).
[25] O. Zik, D. Levine, S. G. Lipson, S. Shtrikman, and J. Stavans, Phys. Rev. Lett. **73**, 644 (1994).
[26] S. Puri and H. Hayakawa, Physica A **270**, 115 (1999); **290**, 218 (2001).
[27] I. S. Aranson and J. S. Olafsen, Phys. Rev. E **66**, 061302 (2002).
[28] A. Snezhko, I. S. Aranson, and W.-K. Kwok, Phys. Rev. Lett. **94**, 108002 (2005).
[29] F. Spahn, U. Schwarz, and J. Kurths, Phys. Rev. Lett. **78**, 1596 (1997).
[30] J. Schmidt, K. Ohtsuki, N. Rappaport, H. Salo, and F. Spahn, in *Dynamics of Saturn's Dense Rings*, in *Saturn from Cassini-Huygens*, edited by M. K. Dougherty, L. W. Esposito, and S. M. Krimigis (Springer, Heidelberg, Germany, 2009), pp. 413–458.
[31] D. R. M. Williams and F. C. MacKintosh, Phys. Rev. E **54**, R9 (1996); D. R. M. Williams, Physica A **233**, 718 (1996).
[32] W. Goldsmit, *The Theory and Physical Behavior of Colliding Solids* (Arnold, London, 1960).
[33] F. C. Bridges, A. Hatzes, and D. N. C. Lin, Nature (London) **309**, 333 (1984).
[34] G. Kuwabara and K. Kono, J. Appl. Phys., Pt. 1 **26**, 1230 (1987).
[35] T. Tanaka, T. Ishida, and Y. Tsuji, Trans. Jpn. Soc. Mech. Eng. **57**, 456 (1991).
[36] R. Ramirez, T. Pöschel, N. V. Brilliantov, and T. Schwager, Phys. Rev. E **60**, 4465 (1999).
[37] N. V. Brilliantov, F. Spahn, J. M. Hertzsch, and T. Pöschel, Phys. Rev. E **53**, 5382 (1996).
[38] W. A. M. Morgado and I. Oppenheim, Phys. Rev. E **55**, 1940 (1997).
[39] T. Schwager and T. Pöschel, Phys. Rev. E **57**, 650 (1998).
[40] T. Schwager and T. Pöschel, Phys. Rev. E **78**, 051304 (2008).
[41] E. Trizac and P. L. Krapivsky, Phys. Rev. Lett. **91**, 218302 (2003).
[42] N. V. Brilliantov, N. Albers, F. Spahn, and T. Pöschel, Phys. Rev. E **76**, 051302 (2007).
[43] N. V. Brilliantov and F. Spahn, Math. Comput. Simul. **72**, 93 (2006).
[44] A. Goldshtein and M. Shapiro, J. Fluid Mech. **282**, 75 (1995).
[45] S. E. Esipov and T. Pöschel, J. Stat. Phys. **86**, 1385 (1997).
[46] N. V. Brilliantov and T. Pöschel, Europhys. Lett. **74**, 424 (2006).
[47] A. Santos and J. M. Montanero, Granular Matter **11**, 157 (2009).
[48] M. Huthmann, J. A. G. Orza, and R. Brito, Granular Matter **2**, 189 (2000).

[49] J. J. Brey, M. J. Ruiz-Montero, and D. Cubero, Phys. Rev. E **54**, 3664 (1996).
[50] N. Brilliantov, C. Saluena, T. Schwager, and T. Poschel, Phys. Rev. Lett. **93**, 134301 (2004).
[51] N. V. Brilliantov and T. Pöschel, Phys. Rev. E **61**, 5573 (2000).
[52] A. S. Bodrova and N. V. Brilliantov, Physica A **388**, 3315 (2009).
[53] T. Pöschel and T. Schwager, *Computational Granular Dynamics* (Springer, Berlin, 2005).
[54] J. J. Brey and M. J. Ruiz-Montero, Phys. Rev. E **69**, 011305 (2004).
[55] T. P. C. van Noije, M. H. Ernst, E. Trizac, and I. Pagonabarraga, Phys. Rev. E **59**, 4326 (1999).