Free energy of the anisotropic Ising lattice with Brascamp-Kunz boundary conditions

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The free energy of the finite and anisotropic Ising lattice with Brascamp-Kunz boundary conditions is calculated exactly as a series in the absence of an external magnetic field.

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I. INTRODUCTION

The free energy of conformally invariant two-dimensional systems has been studied for some time [1–3]. In the case where the system considered is an infinite strip, it is possible to find an exact expression for the free energy per unit length. This has been done in the isotropic case, with equal horizontal and vertical coupling constants [4]. This was done by considering a cylindrical Ising lattice of height $\mathcal{M}$ and circumference $2\mathcal{N}$ with so-called *Brascamp-Kunz* boundary conditions [5]. Other aspects of finite size corrections with Brascamp-Kunz boundary conditions have been considered in Ref. [6].

In this paper, the calculation corresponding to the calculation of the free energy in Ref. [4] will be done, but with different vertical and horizontal coupling constants.

Blöte *et al.* [1] wrote the limit of the free energy at criticality per unit length $\lim_{\mathcal{N}\to\infty} F/2\mathcal{N}$ as

$$\lim_{\mathcal{N}\to\infty} \frac{F}{2\mathcal{N}} = f\mathcal{M} + f^{\times} + \frac{\Delta}{\mathcal{M}} + \cdots, \tag{1}$$

where f is the bulk free energy per unit area and $f^{\times}/2$ is the surface free energy. In the isotropic case Δ is given by

$$\Delta = -\frac{\pi}{24}(c - 24h), \tag{2}$$

where $c - 24h$ is called the *effective* central charge. The effective central charge has been discussed elsewhere, for instance, by Izmailian *et al.* [7]. For the Ising lattice, the central charge is $c = 1/2$, and the allowed values of the conformal weight h are 0, 1/2, and 1/16. It can be seen from Eq. (17) of Ref. [4] that the partition function is a multiple of the Virasoro character $\chi_{1/16} = \sqrt{\vartheta_2/2\eta}$ [where the suppressed argument is $i\mathcal{N}/(\mathcal{M}+1)$]. Thus h must be 1/16. It will be shown later in this paper that in the isotropic case

$$\Delta = \frac{\pi}{24}. \tag{3}$$

Since $c = 1/2$, (2) and (3) confirm that $h = 1/16$.

Equation (3) is only valid if the classical system is rotationally invariant at large distances [1,2]. In terms of an Ising lattice, this means that the lattice has to be isotropic. If this is not the case, then (3) must be modified by dividing the right-hand side by v, the "speed of light." The speed of light should be obtained from a dispersion relation $\omega \sim vu$, where ω is the frequency and u is the momentum.

The Ising model of size $M \times N$ has exact solutions for various boundary conditions, and finite size corrections have also been studied by many authors. For toroidal boundary conditions, Izmailian and Hu [8] and Salas [9] calculated the free energy up to order N^{-5}. Lu and Wu [10] found expressions for the partition functions of a quadratic Ising lattice on a Möbius strip and on a Klein bottle and found finite size corrections of the free energy to order N^{-1}. Janke and Kenna [6] considered an Ising lattice of size $M \times 2N$ with Brascamp-Kunz boundary conditions. They calculated corrections of the specific heat to order M^{-3}.

Izmailian *et al.* [4] calculated the finite size correction of the free energy of an isotropic lattice with Brascamp-Kunz boundary conditions up to every order. To do this, they used the fact that the partition function of a lattice with these boundary conditions can be written in terms of a partition function of a lattice with "twisted" boundary conditions. They could then expand the free energy as a series and use the Kronecker double series. Before this, Ivashkevich *et al.* [11] had rewritten the partition function of the Ising lattice with toroidal boundary conditions in the same way. In this paper, we will apply this method to the anisotropic lattice with Brascamp-Kunz boundary conditions. For an introduction to Kronecker's double series, see Refs. [4,11,12].

II. THE BRASCAMP-KUNZ BOUNDARY CONDITIONS

The Brascamp-Kunz boundary conditions were introduced by Brascamp and Kunz [5] to study the zeros of the partition function of a finite Ising lattice. It had previously been conjectured by Fisher [13] that in the isotropic case, the zeros of the partition function $Z_{M,N}$ in the variable $x := e^{2\beta E}$ will approach the circles $|x \pm 1| = \sqrt{2}$ in the thermodynamic limit $M,N \to \infty$. Brascamp and Kunz used a result by McCoy and Wu [14] to show that with these boundary conditions, the zeros do not just approach these circles but lie on them even when the lattice is finite. To put it briefly, McCoy and Wu considered a cylindrical lattice with free boundary conditions and with a magnetic field $\mathfrak{H}$ on the lower boundary. Brascamp and Kunz found that in the limit $\beta\mathfrak{H} \to i\pi/2$, the expression for the corresponding partition function is much simplified. Moreover, in this limit the zeros approach the circles mentioned above, even when the lattice is finite. They then constructed a lattice whose dual lattice is the lattice of McCoy and Wu with $\beta\mathfrak{H} = i\pi/2$. (For a general discussion of duality relations, see, for example, Ref. [15].) This lattice is the Brascamp-Kunz lattice. Its partition function can be written in such a way that it is easy to calculate finite size corrections of the free energy. We consider the lattice $\Lambda = \{(m,n)|1 \leqslant m \leqslant \mathcal{M}, 1 \leqslant n \leqslant 2\mathcal{N}, (m,2\mathcal{N}+1) = (m,1)\}$. The boundary

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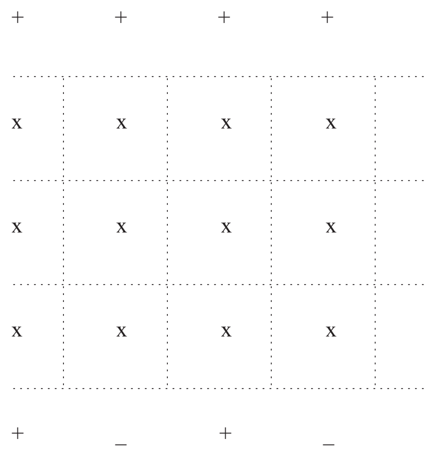

FIG. 1. The lattice Λ (marked by crosses; $\mathcal{M}=3$, $\mathcal{N}=2$) and the dual lattice Λ^* (at intersections of lines).

conditions are as follows: (i) The lattice interacts with a row of fixed, positive spins above it. (ii) The lattice interacts with a row of fixed, alternating spins below it. (See Fig. 1.)

III. THE PARTITION FUNCTION WITH BRASCAMP-KUNZ BOUNDARY CONDITIONS

The interaction between neighboring spins on Λ is E_1 in the horizontal direction and E_2 in the vertical direction. In what follows, the dimensionless parameters $K_l = \beta E_l (l = 1,2)$ will be used instead of E_1 and E_2. Apart from interactions between nearest neighbors, there is an external magnetic field. Thus the Hamiltonian is defined as

$$\begin{aligned}\mathcal{E}_\Lambda(\sigma,K_1,K_2,H) = &-E_1 \sum_{j=1}^{\mathcal{M}} \sum_{k=1}^{2\mathcal{N}} \sigma_{j,k}\sigma_{j,k+1} \\ &- E_2 \sum_{j=0}^{\mathcal{M}} \sum_{k=1}^{2\mathcal{N}} \sigma_{j,k}\sigma_{j+1,k} \\ &- \sum_{j=1}^{\mathcal{M}} \sum_{k=1}^{2\mathcal{N}} H(j,k)\sigma_{j,k},\end{aligned} \tag{4}$$

where $\sigma_{j,k} = \sigma_{j,k+2\mathcal{N}} = \pm 1$, $\sigma_{0,k} = 1$, and $\sigma_{\mathcal{M}+1,k} = (-1)^{k+1}$. The partition function

$$Z_\Lambda(K_1,K_2,H) = \sum_{\sigma\in\{-1,1\}^\Lambda} \exp -\beta\mathcal{E}_\Lambda(\sigma,K_1,K_2,H) \tag{5}$$

has been calculated for the constant external magnetic fields $H \equiv 0$ and $\beta H \equiv i\pi/2$, for which the problem is exactly solvable. Brascamp and Kunz calculated $Z_\Lambda(K,K,0)$. $Z_\Lambda(K_1,K_2,0)$ and $Z_\Lambda(K_1,K_2,i\pi/2)$ were calculated in Ref. [16]. $Z_\Lambda(K_1,K_2,0)$ is given by

$$\begin{aligned}Z_\Lambda(K_1,K_2,0) = 2^{2\mathcal{M}\mathcal{N}} \prod_{j=1}^{\mathcal{N}} \prod_{k=1}^{\mathcal{M}} \{&\cosh 2K_1 \cosh 2K_2 \\ &- \sinh 2K_1 \cos\theta_j - \sinh 2K_2 \cos\varphi_k\},\end{aligned} \tag{6}$$

where

$$\theta_j = (2j-1)\pi/2\mathcal{N}, \varphi_k = k\pi/(\mathcal{M}+1). \tag{7}$$

IV. THE FREE ENERGY

The partition function (6) can be written as

$$Z_\Lambda = 2^{\mathcal{M}\mathcal{N}} e^{2\mathcal{M}\mathcal{N}\mu(K_1,K_1)} \prod_{j=1}^{\mathcal{N}} \prod_{k=1}^{\mathcal{M}} F(j,k), \tag{8}$$

where

$$\begin{aligned}F(j,k) := 4\Bigg[&2\sinh^2 \left(\frac{\sinh 2K_2}{\sinh 2K_1}\right)^{1/2} \mu(K_1,K_2) \\ &+ \sin^2\theta_j/2 + \frac{\sinh 2K_2}{\sinh 2K_1} \sin^2\varphi_k/2\Bigg]\end{aligned} \tag{9}$$

and the mass $\mu(K_1,K_2)$ is defined as

$$\begin{aligned}\sinh^2 \mu(K_1,K_2) := &\frac{1}{4(\sinh 2K_1 \sinh 2K_2)^{1/2}} \\ &\times (\cosh 2K_1 \cosh 2K_2 - \sinh 2K_1 \\ &- \sinh 2K_2);\end{aligned} \tag{10}$$

in particular

$$\mu(K,K) = \tfrac{1}{2} \ln \sinh 2K. \tag{11}$$

Define $\omega(K_1,K_2;u)$ by the equation

$$\begin{aligned}\sinh^2 \omega(K_1,K_2;u) := &2\sinh^2 \left(\frac{\sinh 2K_2}{\sinh 2K_1}\right)^{1/2} \mu(K_1,K_2) \\ &+ \frac{\sinh 2K_2}{\sinh 2K_1} \sin^2 u,\end{aligned} \tag{12}$$

so that

$$F(j,k) = 4[\sinh^2 \omega(K_1,K_2;\varphi_k/2) + \sin^2\theta_j/2]. \tag{13}$$

Then, using the identity [17]

$$\prod_{j=0}^{2\mathcal{N}-1} 4\left(\sinh^2\omega + \sin^2\theta_j/2\right) = 4\cosh^2 2\mathcal{N}\omega, \tag{14}$$

one obtains

$$\begin{aligned}&\prod_{j=0}^{2\mathcal{N}-1} F(j+1,0)F(j+1,\mathcal{M}+1) \\ &\quad = [4\cosh 2\mathcal{N}\omega(K_1,K_2;0)\cosh 2\mathcal{N}\omega(K_1,K_2;\pi/2)]^2.\end{aligned} \tag{15}$$

Therefore, using the same argument as in Ref. [4], one finds

$$\begin{aligned}Z_\Lambda^2 = &\frac{2^{2\mathcal{M}\mathcal{N}} e^{4\mathcal{M}\mathcal{N}\mu(K_1,K_1)}}{4\cosh 2\mathcal{N}\omega(K_1,K_2;0)\cosh 2\mathcal{N}\omega(K_1,K_2;\pi/2)} \\ &\times \tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K_1,K_2)),\end{aligned} \tag{16}$$

where $\tilde{\Lambda}$ is a lattice of size $(2(\mathcal{M}+1),2\mathcal{N})$ and the partition function of $\tilde{\Lambda}$ with twisted boundary conditions is given by the equation

$$[\tilde{Z}_{\tilde{\Lambda}}(\alpha,\beta,\mu)]^2 = \prod_{j=0}^{2\mathcal{N}-1}\prod_{k=0}^{2\mathcal{M}+1} 4\left(2\sinh^2\mu + \sin^2\frac{\pi(j+\alpha)}{2\mathcal{N}} + \frac{\sinh 2K_2}{\sinh 2K_1}\sin^2\frac{\pi(k+\beta)}{2(\mathcal{M}+1)}\right). \tag{17}$$

$[\tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K_1,K_2))]^2$ is given by

$$\begin{aligned}[\tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K_1,K_2))]^2 &= \prod_{j=0}^{2\mathcal{N}-1}\prod_{k=0}^{2\mathcal{M}+1} 4\left[2\sinh^2\mu(K_1,K_2) + \sin^2\theta_{j+1}/2 + \frac{\sinh 2K_2}{\sinh 2K_1}\sin^2\varphi_k/2\right] = \prod_{j=0}^{2\mathcal{N}-1}\prod_{k=0}^{2\mathcal{M}+1} F(j+1,k)\\ &= \left(\frac{2}{\sinh 2K_1}\right)^{4\mathcal{N}(\mathcal{M}+1)}\prod_{j=0}^{2\mathcal{N}-1}\prod_{k=0}^{2\mathcal{M}+1}\left[\left(\frac{\sinh 2K_1}{\sinh 2K_2}\right)^{1/2}(\cosh 2K_1\cosh 2K_2 - \sinh 2K_1 - \sinh 2K_2)\right.\\ &\quad\left. + 2\sinh 2K_1\sin^2\theta_{j+1}/2 + 2\sinh 2K_2\sin^2\varphi_k/2\right]. \end{aligned}\tag{18}$$

Define $\tilde{\omega}(K_1,K_2;u)$ by the lattice dispersion relation

$$\begin{aligned}\sinh^2\tilde{\omega}(K_1,K_2;u) &= \frac{\sinh 2K_1}{\sinh 2K_2}[2\sinh^2\mu(K_1,K_2) + \sin^2 u]\\ &= \frac{(\sinh 2K_1)^{1/2}}{2(\sinh 2K_2)^{3/2}}(\cosh 2K_1\cosh 2K_2 - \sinh 2K_1 - \sinh 2K_2) + \frac{\sinh 2K_1}{\sinh 2K_2}\sin^2 u. \end{aligned}\tag{19}$$

At criticality it reads

$$\tilde{\omega}(K_1,K_2;u) \sim \left(\frac{\sinh 2K_1}{\sinh 2K_2}\right)^{1/2} u. \tag{20}$$

One would thus expect that

$$v = \left(\frac{\sinh 2K_1}{\sinh 2K_2}\right)^{1/2}. \tag{21}$$

It will be shown later that this is in fact the case. It follows from (18) and (19) that

$$[\tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K_1,K_2))]^2 = \left(\frac{\sinh 2K_2}{\sinh 2K_1}\right)^{4\mathcal{N}(\mathcal{M}+1)}\prod_{j=0}^{2\mathcal{N}-1}\prod_{k=0}^{2\mathcal{M}+1} 4[\sinh^2\tilde{\omega}(K_1,K_2;\theta_{j+1}/2) + \sin^2\varphi_k/2]. \tag{22}$$

Using the identity

$$\prod_{k=0}^{2\mathcal{M}+1} 4\left(\sinh^2\omega + \sin^2\varphi_k/2\right) = 4\sinh^2 2(\mathcal{M}+1)\omega, \tag{23}$$

one obtains from (22)

$$\tilde{Z}_{\tilde{\Lambda}}(1/2,0;\mu(K_1,K_2)) = \left(\frac{\sinh 2K_2}{\sinh 2K_1}\right)^{2\mathcal{N}(\mathcal{M}+1)}\prod_{j=0}^{2\mathcal{N}-1} 2\sinh 2(\mathcal{M}+1)\tilde{\omega}\left(K_1,K_2;\theta_j/2\right). \tag{24}$$

V. ASYMPTOTIC EXPANSION OF THE FREE ENERGY

A. The free energy at criticality

Let $(K_1,K_2) = (K^*,K)$ be the curve on which $\mu(K_1,K_2) = 0$, or, equivalently,

$$\sinh 2K_1\sinh 2K_2 = 1. \tag{25}$$

Let the Taylor expansion of $\tilde{\omega}(K^*,K;u)$ be

$$\tilde{\omega}(K^*,K;u) = \sum_{n=0}^{\infty}\frac{\lambda_{2n}}{(2n)!}u^{2n+1}. \tag{26}$$

In particular

$$\lambda_0 = \left(\frac{\sinh 2K^*}{\sinh 2K}\right)^{1/2} = \sinh 2K^*. \tag{27}$$

Then

$$\tilde{\omega}(K^*,K;u) = \sinh^{-1}\lambda_0\sin u = \sinh^{-1}\sinh 2K^*\sin u, \tag{28}$$

and similarly,

$$\omega(K^*,K;u) = \sinh^{-1}\lambda_0^{-1}\sin u = \sinh^{-1}\sinh 2K\sin u. \tag{29}$$

Thus, on the critical curve, the free energy as obtained from (16) is

$$F = -\ln Z_\Lambda = -\mathcal{M}\mathcal{N}\ln 2 - 2\mathcal{M}\mathcal{N}\mu(K^*,K^*) + \ln 2 + \frac{1}{2}\ln\cosh 2\mathcal{N}\sinh^{-1}\left(\frac{\sinh 2K}{\sinh 2K^*}\right)^{1/2} - \frac{1}{2}\ln\tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K^*,K))$$
$$= -\mathcal{M}\mathcal{N}\ln 2 - 2\mathcal{M}\mathcal{N}\mu(K^*,K^*) + \ln 2 + \frac{1}{2}\ln\cosh 4\mathcal{N}K - \frac{1}{2}\ln\tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K^*,K)). \quad (30)$$

Clearly

$$\ln\tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K^*,K)) = 2\mathcal{N}(\mathcal{M}+1)\ln\left(\frac{\sinh 2K}{\sinh 2K^*}\right) + 2(\mathcal{M}+1)\sum_{j=0}^{2\mathcal{N}-1}\tilde{\omega}(K^*,K,\theta_j/2)$$
$$+\sum_{j=0}^{2\mathcal{N}-1}\ln[1-\exp -4(\mathcal{M}+1)\tilde{\omega}(K^*,K,\theta_j/2)]. \quad (31)$$

The two sums in (31) can be calculated exactly up to an exponentially small correction $O(e^{-\mathcal{N}})$.

B. Calculation of (31)

The first sum in (31) can be written as a power series using the Euler-Maclaurin summation formula:

$$2(\mathcal{M}+1)\sum_{j=0}^{2\mathcal{N}+1}\tilde{\omega}(K^*,K,\theta_j/2) = \frac{S}{\pi}\int_0^\pi \tilde{\omega}(K^*,K,u)du - 2\pi\xi\sum_{n=0}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^n\frac{\lambda_{2n}}{(2n)!}\frac{\mathrm{B}_{2n+2}(1/2)}{2n+2}, \quad (32)$$

where $S = 4\mathcal{N}(\mathcal{M}+1)$, $\xi = (\mathcal{M}+1)/\mathcal{N}$, and $\mathrm{B}_p(\cdot)$ is the pth Bernoulli function. It is defined as

$$\mathrm{B}_p(x) := -\frac{p!}{(2\pi i)^p}\sum_{k\in\mathbf{Z}\setminus\{0\}} k^{-p}e^{2\pi i k x}. \quad (33)$$

It remains to calculate the second sum of (31).

The second sum in (31) can be written as

$$\sum_{j=0}^{2\mathcal{N}-1}\ln(1-e^{-4(\mathcal{M}+1)\tilde{\omega}(K^*,K,\theta_j/2)}) = -2\sum_{m=1}^{\infty}\frac{1}{m}\sum_{j=0}^{\mathcal{N}-1}e^{-2m2(\mathcal{M}+1)\tilde{\omega}(K^*,K,\theta_j/2)}. \quad (34)$$

Let $P(p) = \{\pi = (q_1,\ldots,q_\nu,r_1,\ldots,r_\nu)|q_j,r_j\in\mathbf{N}, 1\leqslant\nu\leqslant p, q_j\neq q_k \text{ if } j\neq k, \sum_{j=1}^{\nu}q_jr_j = p\}$. The exponential on the right-hand side of (34) can be written as

$$e^{-2m2(\mathcal{M}+1)\tilde{\omega}(K^*,K,\theta_j/2)} = \exp\left[-2\pi m\lambda_0\xi(j+1/2) - 2\pi m\xi\sum_{p=1}^{\infty}\frac{\lambda_{2p}}{(2p)!}\left(\frac{\pi^2\xi}{S}\right)^p(j+1/2)^{2p+1}\right]$$
$$= \left[1-2\pi m\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{(j+1/2)^{2p+1}}{(2p)!}\Lambda_{2p}\right]e^{-2\pi m\lambda_0\xi(j+1/2)}, \quad (35)$$

where

$$\Lambda_{2p} = (2p)!\sum_{\pi\in P(p)}\left(\prod_{l=1}^{\nu(\pi)}\frac{1}{r_l!}\left(\frac{\lambda_{2q_l}}{(2q_l)!}\right)^{r_l}\right)[-2\pi m\xi(j+1/2)]^{r_1+\cdots+r_{\nu(\pi)}-1}. \quad (36)$$

Together, (34), (35), and (36) imply that

$$\sum_{j=0}^{2\mathcal{N}-1}\ln(1-e^{-4(\mathcal{M}+1)\tilde{\omega}(K^*,K,\theta_j/2)})$$
$$= -2\sum_{m=1}^{\infty}\frac{1}{m}\left(\sum_{j=0}^{\mathcal{N}-1}e^{-2\pi m\lambda_0\xi(j+1/2)}\right) + 4\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{1}{(2p)!}\Lambda_{2p}\left(\sum_{m=1}^{\infty}\sum_{j=0}^{\mathcal{N}-1}(j+1/2)^{2p+1}e^{-2\pi m\lambda_0\xi(j+1/2)}\right). \quad (37)$$

As in Ref. [11], if, for large $\mathcal{N}$, the finite sum $\sum_{j=0}^{\mathcal{N}-1}$ is replaced by the infinite sum $\sum_{j=0}^{\infty}$ in (37), then equality still holds up to an exponentially small correction $O(e^{-\mathcal{N}})$. Thus

$$\sum_{j=0}^{2\mathcal{N}-1} \ln(1 - e^{-4(\mathcal{M}+1)\tilde{\omega}(K^*,K,\theta_j/2)}) = -2\sum_{m=1}^{\infty}\frac{1}{m}\left(\sum_{j=0}^{\infty} e^{-2\pi m\lambda_0\xi(j+1/2)}\right) + 4\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{1}{(2p)!}\Lambda_{2p}\left(\sum_{m=1}^{\infty}\sum_{j=0}^{\infty}(j+1/2)^{2p+1}e^{-2\pi m\lambda_0\xi(j+1/2)}\right) + O(e^{-\mathcal{N}}). \quad (38)$$

Combining (32), (38), and (31), one obtains

$$\ln \tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K^*,K)) = 2\mathcal{N}(\mathcal{M}+1)\ln\left(\frac{\sinh 2K}{\sinh 2K^*}\right) + \frac{S}{\pi}\int_0^{\pi}\tilde{\omega}(K^*,K,u)du - 2\pi\xi\sum_{n=0}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^n\frac{\lambda_{2n}}{(2n)!}\frac{\mathrm{B}_{2n+2}(1/2)}{2n+2} - 2\sum_{m=1}^{\infty}\frac{1}{m}\sum_{j=0}^{\infty}e^{-2\pi m\lambda_0\xi(j+1/2)} + 4\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{1}{(2p)!}\Lambda_{2p}\sum_{m=1}^{\infty}\sum_{j=0}^{\infty}(j+1/2)^{2p+1}e^{-2\pi m\lambda_0\xi(j+1/2)} + O(e^{-\mathcal{N}}). \quad (39)$$

This expression may be further simplified in terms of elliptic θ functions. However, it is simplest to consider the limit $\mathcal{N}\to\infty$.

C. The free energy in the limit $\mathcal{N}\to\infty$

In the limit $\mathcal{N}\to\infty$ while $\mathcal{M}$ is fixed, an exact result can be obtained.

Combining (30) and (39), one obtains

$$\lim_{\mathcal{N}\to\infty}\frac{F}{2\mathcal{N}} = -\frac{1}{2}\mathcal{M}\ln 2 - \mathcal{M}\mu(K^*,K^*) + K - \lim_{\mathcal{N}\to\infty}\frac{1}{4\mathcal{N}}\ln\tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K^*,K))$$
$$= -\frac{1}{2}\mathcal{M}\ln 2 - (\mathcal{M}+2)\mu(K,K) + K - \lim_{\mathcal{N}\to\infty}\frac{\mathcal{M}+1}{\pi}\int_0^{\pi}\tilde{\omega}(K^*,K,u)du + \frac{1}{2}\lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\sum_{m=1}^{\infty}\frac{1}{m}\sum_{j=0}^{\infty}e^{-2\pi m\lambda_0\xi(j+1/2)} - \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{1}{(2p)!}\Lambda_{2p}\sum_{m=1}^{\infty}\sum_{j=0}^{\infty}(j+1/2)^{2p+1}e^{-2\pi m\lambda_0\xi(j+1/2)}. \quad (40)$$

The limit of the double sum can be calculated to be

$$\frac{1}{2}\lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\sum_{m=1}^{\infty}\frac{1}{m}\sum_{j=0}^{\infty}e^{-2\pi m\lambda_0\xi(j+1/2)} = \frac{\pi}{24(\mathcal{M}+1)\lambda_0}. \quad (41)$$

Further, by Ref. [11]

$$\sum_{m=1}^{\infty}\sum_{j=0}^{\infty}(j+1/2)^{2p+1}e^{-2\pi m\lambda_0\xi(j+1/2)} = \frac{1}{4(p+1)}\left[\mathrm{B}_{2p+2}(1/2) - \mathrm{K}_{2p+2}^{1/2,0}(i\lambda_0\xi)\right], \quad (42)$$

where $\mathrm{B}_p(\cdot)$ is the pth Bernoulli function and

$$\mathrm{K}_p^{\alpha,\beta}(\tau) := -\frac{p!}{(-2\pi i)^p}\sum_{\substack{m,n\,\in\,\mathbf{Z}\\(m,n)\neq(0,0)}}\frac{e^{-2\pi i(n\alpha+m\beta)}}{(n+\tau m)^p} \quad (43)$$

is Kronecker's double series.

It can be shown [4] that $\mathrm{K}_{2p}^{1/2,0}(i\xi)$ can be expressed in terms of the elliptic θ functions ϑ_2, ϑ_3, and ϑ_4. It can therefore be shown [4] that for small ξ

$$\mathrm{K}_{2p+2}^{1/2,0}(i\lambda_0\xi) = \mathrm{B}_{2p+2}(\lambda_0\xi)^{-2p-2} + O(\xi^{-2p-1}), \quad (44)$$

where $\mathrm{B}_n := \mathrm{B}_n(1)$ is the nth Bernoulli number. Since

$$\Lambda_{2p} = \lambda_{2p} + O(\xi) \quad (45)$$

for small ξ, it follows that

$$\lim_{\mathcal{N}\to\infty}\frac{F}{2\mathcal{N}} = -\frac{1}{2}\mathcal{M}\ln 2 - (\mathcal{M}+2)\mu(K,K) - \frac{\mathcal{M}+1}{4\pi}\int_0^{\pi}\tilde{\omega}(K^*,K,u)du + K + \frac{\pi}{24(\mathcal{M}+1)\lambda_0} + \sum_{p=1}^{\infty}\left(\frac{\pi}{2(\mathcal{M}+1)}\right)^{2p+1}\frac{\mathrm{B}_{2p+2}}{(2p)!(2p+2)}\frac{\lambda_{2p}}{\lambda_0^{2p+2}}. \quad (46)$$

In particular, one sees that the expected value of v given in (21) is correct.

D. The free energy for large $\mathcal{N}$

We now consider the case of large $\mathcal{N}$, where the partition function is given by (39). According to Ref. [11],

$$-2\sum_{m=1}^{\infty}\frac{1}{m}\sum_{j=0}^{\infty}e^{-2\pi m\lambda_0\xi(j+1/2)} = 2\sum_{j=0}^{\infty}\ln\left(1-e^{-2\pi\lambda_0\xi(j+1/2)}\right) = \ln\frac{\vartheta_4(i\lambda_0\xi)}{\eta(i\lambda_0\xi)} + \pi\lambda_0\xi\mathrm{B}_2(1/2), \quad (47)$$

where $\eta(\tau) := [\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau)/2]^{1/3} = e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1-e^{i2\pi\tau n})$. From (39), (42), and (47) one thus obtains

$$\ln\tilde{Z}_{\tilde{\Lambda}}(1/2,0,\mu(K^*,K)) = 2\mathcal{N}(\mathcal{M}+1)\ln\left(\frac{\sinh 2K}{\sinh 2K^*}\right) + \frac{S}{\pi}\int_0^{\pi}\tilde{\omega}(K^*,K,u)du - 2\pi\xi\sum_{n=0}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^n\frac{\lambda_{2n}}{(2n)!}\frac{\mathrm{B}_{2n+2}(1/2)}{2n+2} + \ln\frac{\vartheta_4(i\lambda_0\xi)}{\eta(i\lambda_0\xi)} + \pi\lambda_0\xi\mathrm{B}_2(1/2) + \pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{1}{(2p)!}\Lambda_{2p}\frac{1}{p+1}\left[\mathrm{B}_{2p+2}(1/2) - \mathrm{K}_{2p+2}^{1/2,0}(i\lambda_0\xi)\right] + O(e^{-\mathcal{N}}). \quad (48)$$

It now follows from (30) and (48) that

$$F = -\mathcal{M}\mathcal{N}\ln 2 - 2\mathcal{M}\mathcal{N}\mu(K^*,K^*) + \frac{1}{2}\ln 2 + 2\mathcal{N}K - \mathcal{N}(\mathcal{M}+1)\ln\left(\frac{\sinh 2K}{\sinh 2K^*}\right) - \frac{S}{2\pi}\int_0^{\pi}\tilde{\omega}(K^*,K,u)du + \pi\xi\sum_{n=0}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^n\frac{\lambda_{2n}}{(2n)!}\frac{\mathrm{B}_{2n+2}(1/2)}{2n+2} - \frac{1}{2}\ln\frac{\vartheta_4(i\lambda_0\xi)}{\eta(i\lambda_0\xi)} - \frac{1}{2}\pi\lambda_0\xi\mathrm{B}_2(1/2) - \frac{1}{2}\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{1}{(2p)!}\Lambda_{2p}\frac{1}{p+1}\left[\mathrm{B}_{2p+2}(1/2) - \mathrm{K}_{2p+2}^{1/2,0}(i\lambda_0\xi)\right] + O(e^{-\mathcal{N}}). \quad (49)$$

VI. CONCLUSION

In this paper we have generalized the result found in Ref. [4]. The calculation in Ref. [4] depended on the expression found by Brascamp and Kunz for the partition function of an isotropic Ising lattice Λ without external magnetic field and with Brascamp-Kunz boundary conditions, $Z_\Lambda(K,K,0)$. The form of that expression allows the partition function to be written in terms of the partition function of an Ising lattice with twisted boundary conditions. The twisted boundary conditions allow the calculation of the correction of the free energy to every order, with the use of the Kronecker double series.

The results obtained in this paper depend on the calculation of the partition function of the anisotropic lattice, $Z_\Lambda(K_1,K_2,0)$, done in Ref. [16]. We have found that this partition function can also be written as the partition function of a lattice with twisted boundary conditions, just like in the isotropic case. Hence we could expand the partition function and use the Kronecker double series to obtain our result.

We have calculated the free energy density $F/2\mathcal{N}$ of an Ising lattice with Brascamp-Kunz boundary conditions and obtained exact results up to exponentially small corrections. In particular, we have considered the free energy density of an infinite strip with Brascamp-Kunz boundary conditions, $\lim_{\mathcal{N}\to\infty}F/2\mathcal{N}$. In this case, the result is exact as a series expansion. Further, we have modified the effective central charge of Eq. (1) to an anisotropic lattice. This was done by calculating the speed of light corresponding to the anisotropy.

The twisted boundary conditions have been used for toroidal [11] and Brascamp-Kunz boundary conditions ([4] and the present paper). It has not yet been fully explored to which cases the twisted boundary conditions can be applied. This may be an interesting topic to investigate.

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