Higher-order expansions for the entropy of a dimer or a monomer-dimer system on d-dimensional lattices

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Recently, an expansion as a power series in $1/d$ has been presented for the specific entropy of a complete dimer covering of a d-dimensional hypercubic lattice. This paper extends from 3 to 10 the number of terms known in the series. Likewise, an expansion for the entropy, dependent on the dimer density p, of a monomer-dimer system, involving a sum $\sum_k a_k(d)p^k$, has been offered recently. We herein extend the number of known expansion coefficients from 6 to 20 for the hypercubic lattices of general dimensionality d and from 6 to 24 for the hypercubic lattices of dimensionalities $d < 5$. We show that these extensions can lead to accurate numerical estimates of the p-dependent entropy for lattices with dimension $d > 2$. The computations of this paper have led us to make the following marvelous conjecture: In the case of the hypercubic lattices, all the expansion coefficients $a_k(d)$ are positive. This paper results from a simple melding of two disparate research programs: one computing to high orders the Mayer series coefficients of a dimer gas and the other studying the development of entropy from these coefficients. An effort is made to make this paper self-contained by including a review of the earlier works.

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I. INTRODUCTION AND RESULTS

The dimer problem arose in a thermodynamic study of diatomic molecules and was abstracted into one of the most basic and natural problems in both statistical mechanics [1–3] and combinatorial mathematics [4]. In more recent years, dimers found interesting applications also in information [5] and string theories [6,7].

Given a hyper-simple-cubic (hsc) lattice with number of sites N in d dimensions, the dimer problem, loosely speaking, is to count the number of different ways dimers (dominoes) may be laid down in the lattice (without overlapping) to completely cover it. Each dimer covers two nearest-neighbor vertices. It is known [8] that the number of such coverings is roughly $\exp(\lambda_d N)$ for some constant λ_d as N goes to infinity. Minc [9] gave a proof of the asymptotic relation (asymptotic as $d \to \infty$)

$$\lambda_d \sim \tfrac{1}{2}\ln(2d) - \tfrac{1}{2}. \tag{1}$$

In a series of papers [10–13], one of the present authors found a mathematical argument for a full asymptotic expansion

$$\lambda_d \sim \frac{1}{2}\ln(2d) - \frac{1}{2} + \frac{c_1}{d} + \frac{c_2}{d^2} + \cdots \tag{2}$$

and computed the first three terms in Table I, also making the conjecture that no further terms would be computed. He was very wrong. One of the results of the present paper is the set of coefficients from c_4 to c_{10} reported in Table I. Viewing the sequence of c_i, we are certainly led to expect the sum in Eq. (2) to be asymptotic and not convergent.

If we consider covering by dimers of a fraction of the vertices denoted here by $p = 2\rho$ [where ρ is the dimer density per site and the vertices not covered by dimers are considered covered by monomers (checkers)] and as above study the number of such coverings, we arrive similarly at a function $\lambda_d(p)$ where

$$\lambda_d(1) = \lambda_d. \tag{3}$$

Another common notation for λ_d is $\tilde{h}_d$. One also studies

$$h_d = \max_{0 \leqslant p \leqslant 1} \lambda_d(p). \tag{4}$$

For $\lambda_d(p)$ Friedland *et al.* [5,14] proved the asymptotic relation (asymptotic as $d \to \infty$)

$$\lambda_d(p) \sim \tfrac{1}{2}[p\ln(2d) - p\ln(p) - 2(1-p)\ln(1-p) - p]. \tag{5}$$

Both this equation and Eq. (1) may be viewed as the mean field approximations for the respective quantities. This was mentioned in Ref. [13] and is briefly discussed at the end of Sec. III. By a development similar to that in Ref. [13], Federbush and Friedland [15] argued for an expansion

$$\lambda_d(p) = \frac{1}{2}[p\ln(2d) - p\ln(p) - 2(1-p)\ln(1-p) - p] + \sum_{k=2}^{\infty} a_k(d)p^k \tag{6}$$

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where, setting $x(d) = \frac{1}{2d}$, those authors computed the five coefficients

$$a_2(d) = \tfrac{1}{4}x, \quad a_3(d) = \tfrac{1}{12}x^2, \quad a_4(d) = \tfrac{1}{24}x^2(-5x+3),$$
$$a_5(d) = \tfrac{1}{40}x^3(-39x+20),$$
$$a_6(d) = \tfrac{1}{60}x^3(-19x^2-30x+20).$$

The main result of this paper is the extension of known values

$$a_7(d) = \tfrac{1}{84}x^4(1093x^2-1008x+231), \quad a_8(d) = \tfrac{1}{112}x^4(967x^3-35x^2-602x+189),$$
$$a_9(d) = \tfrac{1}{144}x^5(-66047x^3+68712x^2-23556x+2856),$$
$$a_{10}(d) = \tfrac{1}{180}x^5(-67721x^4+18495x^3+29565x^2-15405x+2232),$$
$$a_{11}(d) = \tfrac{1}{220}x^6(5456221x^4-6452710x^3+2752860x^2-524700x+39710),$$
$$a_{12}(d) = \tfrac{1}{264}x^6(887437x^5+2477970x^4-3847316x^3+1824724x^2-378004x+31130),$$
$$a_{13}(d) = \tfrac{1}{312}x^7(-614279535x^5+794742624x^4-392705664x^3+95702984x^2-11868441x+621504),$$
$$a_{14}(d) = \tfrac{1}{364}x^7(678357525x^6-1192936836x^5+869146005x^4-339116960x^3+75444460x^2-9220393x+497016),$$
$$a_{15}(d) = \tfrac{1}{420}x^8(89365899701x^6-124219633888x^5+68478916835x^4-19687487260x^3+3185117250x^2 - 281248772x+10870055),$$
$$a_{16}(d) = \tfrac{1}{480}x^8(-206929670185x^7+330409603725x^6-221634792330x^5+83075676915x^4-19146441210x^3 + 2751382878x^2-231206020x+8907885),$$
$$a_{17}(d) = \tfrac{1}{544}x^9(-16388790941183x^7+24197151077904x^6-14547689415128x^5+4724677127184x^4-911997832372x^3 + 106422324240x^2-7073226040x+210678416),$$
$$a_{18}(d) = \tfrac{1}{612}x^9(55311212276891x^8-89669360611981x^7+61471303146642x^6-23833002227449x^5+5824219780656x^4 - 933123781978x^3+97025317251x^2-6063514389x+176829104),$$
$$a_{19}(d) = \tfrac{1}{684}x^{10}(3770925296332945x^8-5844092886538362x^7+3760855236979965x^6-1340101438257204x^5 + 293876531465913x^4-41181769780866x^3+3649368222699x^2-189574974180x+4489042410),$$
$$a_{20}(d) = \tfrac{1}{760}x^{10}(-16045042327489089x^9+26850617367263509x^8-19173445082939896x^7+7825625528101485x^6 - 2044727194575071x^5+359651992720132x^4-43125672212794x^3+3440152700645x^2-167626520550x + 3849436062).$$

For the hsc lattices of dimensionality $d < 5$, four more coefficients $a_k(d)$ are available. They are listed in Table II. In Ref. [15] it was conjectured that the series in Eq. (6) is convergent for $0 \leqslant p \leqslant 1$. Federbush in fact proved [16] that this series converges for small enough p. Using also (i) the result by Heilmann and Lieb [17] that $\lambda_d(p)$ is analytic for $0 < p < 1$, (ii) the conjecture that the $a_k(d)$ are all positive for integer values of d in the case of the hsc lattices (which we have checked for integer values of d and $k \leqslant 20$; see also the Appendix), and (iii) the theorem that for an analytic function represented in the vicinity of the origin by a power series with positive coefficients, one of the singularities nearest the origin lies on the positive real axis, we can extend the analyticity domain of $\sum_k a_k(d)p^k$ to a disk of radius $R < 1$. The convergence of this series also at $p = 1$ is then a trivial consequence of the positivity conjecture for the coefficients $a_k(d)$ and of the upper bound [9] $\lambda_d(1) < \frac{\ln(2d)!}{4d}$.

TABLE I. Expansion coefficients c_n of the dimer entropy $\lambda_d \sim \frac{1}{2}\ln(2d) - \frac{1}{2} + \sum_n \frac{c_n}{d^n}$ in the case of the hyper-simple-cubic lattices.

$c_1 = 1/8$	$c_6 = 20815/21504$
$c_2 = 5/96$	$c_7 = 9151/6144$
$c_3 = 5/64$	$c_8 = 39593/73728$
$c_4 = 237/1280$	$c_9 = -645691/61440$
$c_5 = 349/768$	$c_{10} = -107753037/901120$

In this paper we assume the validity of the positivity conjecture, from which the convergence of the series $\sum_k a_k(d)p^k$ for $0 \leqslant p \leqslant 1$ follows. Since for any r the partial sums $\sum_k^r a_k(d)p^k$ are positive for integer values of d, the expansion (6) gives good approximations of $\lambda_d(1)$ also in low dimensions, unlike the expansion (2), which is numerically useful only for sufficiently large d. In the Appendix we shall further discuss the positivity conjecture, while Sec. IV is devoted to the numerical approximations.

It is interesting to point out some results of historic importance for the dimer problem. The exact value of λ_2

TABLE II. Higher-order expansion coefficients $a_k(d)$ of the dimer entropy $\lambda_d(p)$ on the hsc lattices of dimension $d < 5$.

$d = 2$	$d = 3$	$d = 4$
$a_{21}(2) = 2556400845 61/92358976733 1840$	$a_{21}(3) = 662234724918 67/102372436321 7633280$	$a_{21}(4) = 1529954754778 4641/9684540638 6975145 9840$
$a_{22}(2) = 5027313191 9/1935140464 88576$	$a_{22}(3) = 1171503630290 797/2026974239 1709138944$	$a_{22}(4) = 1174316299551 8717 5/8522395762 0538128 46592$
$a_{23}(2) = 4312434281365/1780329227 6948992$	$a_{23}(3) = 690335743881 9689/1332011642 8837434 1632$	$a_{23}(4) = 9027160349821 0873 3/7467242001 0376264 941568$
$a_{24}(2) = 5789230773063/2589569785 7380352$	$a_{24}(3) = 406623703567 2469 7/8718621662 5111775 08864$	$a_{24}(4) = 6949151047 6070 6161 3/6516865746 3601103 9490048$

calculated by Fisher [1] and Kasteleyn [2] is given by the closed-form expression

$$\lambda_2 \equiv \tilde{h}_2 = \frac{1}{\pi}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right) = G/\pi = 0.291\,560\,904\,0\ldots, \tag{7}$$

with G Catalan's constant. The technique used in the proof of this relation had great influence in the field of exactly soluble models.

The one-dimensional problem has an even more complete solution [14]

$$\lambda_1(p) = \frac{p}{2}\ln(2) - \frac{p}{2}\ln(p) - (1-p)\ln(1-p) - \frac{p}{2} + \sum_{k=2}^{\infty}\frac{(p/2)^k}{(k-1)k}, \tag{8}$$

so that $\lambda_1(1) \equiv \tilde{h}_1 = 0$ and $h_1 = \ln\frac{1+\sqrt{5}}{2}$. Notice that in this simple case, all the $a_k(1)$ are positive and rapidly vanishing as $k \to \infty$, so the series converges for $0 \leqslant p \leqslant 1$.

Let us turn for the moment to consideration of a dimer gas on our d-dimensional lattice. The gas of dimers is taken as a hard-body system. Between each pair of dimers there is a potential energy 0 if the dimers are disjoint and $+\infty$ if they overlap. For this gas we are interested into the coefficients of the Mayer series [18] $b_1(d), b_2(d), \ldots$.

Both the formalism in Ref. [13] used to derive Eq. (2) and the formalism in Ref. [15] used to derive Eq. (6) take as inputs the $b_i(d)$ and have as outputs the c_i of Table I and the $a_k(d)$. Federbush did not have as good an algorithm for computing to high orders the $b_i(d)$ as in Refs. [19–21] and was not aware of the already existing lower-order expansions [22–24] for small lattice dimensionalities. This explains the many additional terms computed in Eqs. (2) and (6) when the computations of Ref. [21] were used as inputs. In Sec. II the technique used in Refs. [20,21] to compute the $b_i(d)$ is discussed. For the computations of the $a_i(d)$, with $i = 1,2,\ldots,20$, one needed exactly the $b_i(d)$ for $1 \leqslant i \leqslant 20$ and $1 \leqslant d \leqslant 10$. [Interestingly, these values in fact determine, for all d, the $b_i(d)$ with $1 \leqslant i \leqslant 20$. This will be shown in Sec. II and in an independent way in Sec. III.]

In Sec. III the machines in Refs. [13,15] to calculate the c_i and $a_k(d)$, respectively, are discussed; however, they are too technical to delve deeply into all of the theory. Recently, Federbush found another route from the $b_i(d)$ to expansions for $\lambda_d(p)$ that is simple enough for us to completely describe it in this paper [25]. We close this section by specializing [26] the expansion in Eq. (6) to $d = 2$ to see what such an expansion looks like:

$$\lambda_2(p) = \frac{1}{2}[p\ln(4) - p\ln(p) - 2(1-p)\ln(1-p) - p] + 2\left[\frac{1}{2\times 1}\left(\frac{p}{4}\right)^2 + \frac{1}{3\times 2}\left(\frac{p}{4}\right)^3 + \frac{7}{4\times 3}\left(\frac{p}{4}\right)^4 + \frac{41}{5\times 4}\left(\frac{p}{4}\right)^5 + \cdots\right]; \tag{9}$$

this equation is determined by an infinite sequence of integers

$$1,1,7,41,181,757,3291,14\,689,64\,771,276\,101,1\,132\,693, 4\,490\,513,17\,337\,685,\ldots, \tag{10}$$

of which the first 23 integers are known from the calculations of this paper. It is very natural to try to find a pattern in the successive terms of this sequence so that a closed-form expression for $\lambda_2(p)$ be realized, recalling that it exists for $\lambda_2 \equiv \lambda_2(1)$. Recently, we came across an early paper by Rushbrooke *et al.* [27] computing by a somewhat different method the first six coefficients in Eq. (6) for the square and the diamond lattices and the first five for other three-dimensional lattices.

The rest of the paper is organized as follows. In Sec. II we recall how the Mayer expansion for the dimer problem is related to the high-temperature (HT) low-field expansion of an Ising system. Section III sketches how the expansion of Eq. (6) is derived from the Mayer series. In Sec. IV we show how simply the expansion (6) can lead to accurate estimates of the p-dependent entropy $\lambda_d(p)$. The Appendix contains additional comments on the positivity conjecture and lists the coefficients appearing in some generalizations of Eq. (6) to lattices other than the hsc one. We have included in the Appendix a section on the graphical expansion procedure for the Ising model, which completes the exposition of Sec. II.

II. DIMERS AND THE ISING MODEL

It has long been known [1,2,22,23] that the number of ways to place s hard dimers onto a lattice can be evaluated by computing, to the same order s and on the same lattice, the HT and low-field series expansion of the free energy of a spin-1/2 Ising model in the presence of a uniform magnetic field. The dimer combinatorial problem can be simply formulated in the language of statistical mechanics. A set of dimers on an N-site lattice (N even) is described as a lattice gas of molecules occupying nearest-neighbor sites, subject to a nonoverlap constraint, in terms of a macrocanonical partition function

$$\Xi_N(z) = 1 + \sum_{s=1}^{N/2} Z_s z^s = 1 + \sum_{s=1}^{N/2} g_N(s) z^s. \tag{11}$$

Due to the nonoverlap constraint Z_s, the canonical partition function for a fixed number s of dimers simply counts the allowed dimer configurations so that $g_N(s)$ is precisely the number of ways of placing s dimers over the links of the lattice and $z = \exp(\beta\mu)$ is the dimer activity. The chemical potential μ, namely, the energy cost of adding one more dimer to the system, is zero whenever there is room on the lattice for adding one more dimer and infinite otherwise. Therefore, the value of $\beta = 1/k_B T$, with T the temperature and k_B the Boltzmann constant, is irrelevant and can be set to unity. Thus $z = 1$ is the value of the activity describing the combinatorics of a monomer-dimer system, i.e., of a dimer system that does not cover completely the lattice, while $z = \infty$ describes the complete coverings.

In the $N \to \infty$ (thermodynamical) limit one gets

$$\Xi(z) = \lim_{N\to\infty} [\Xi_N(z)]^{1/N} = 1 + \sum_{s=1}^{\infty} g(s) z^s, \tag{12}$$

from which a pressure (or macrocanonical potential) can be defined in the usual way

$$P(z) = \ln[\Xi(z)] = \sum_{s=1}^{\infty} b_s z^s \tag{13}$$

since $\beta = 1$. The dimer density per site ρ is expressed in terms of the pressure by

$$\rho(z) = z\frac{dP}{dz} = \sum_{s=1}^{\infty} s b_s z^s. \tag{14}$$

The series for $\rho(z)$ can be inverted to get z as a power series in the density and by substituting $z = z(\rho)$ in Eq. (13), P can be expressed as a power series in the density ρ, thus obtaining the virial expansion. Equations (13) and (14) are called the Mayer expansions of the dimer lattice gas.

The specific entropy $s_d(p)$ of a dimer system of density ρ in d dimensions is

$$\begin{aligned} s_d(p)/k_B \equiv \lambda_d(p) &= -\rho(z)\ln z + P(z) \\ &= \tfrac{1}{2}[p\ln(2d) - p\ln p] + O(p), \end{aligned} \tag{15}$$

which arises by setting $z = z(p)$ and $\rho = p/2$ and observing that on the hsc lattices $z = \frac{p}{2d} + O(p^2)$. Notice that one has

$$\frac{d\lambda_d}{dp} = \frac{-\ln(z)}{2}. \tag{16}$$

This structure was further specified in Refs. [5,14], as indicated in Eq. (6). One can also easily check that by changing the variable from z to p, the point $z = 1$ corresponds to a stationary point of the entropy with respect to p, thus linking the definition given above of h_d in terms of $\lambda_d(p)$ with the definition used in Ref. [19] as $P(z)|_{z=1}$. We now couple the relation (15) with the expansions above for $P(p)$ and $z(p)$. We write

$$z = \frac{p}{2b_1}[1 + F(p)] \tag{17}$$

and then get from Eqs. (15) and (17)

$$\lambda_d(p) = P(p) - \frac{p}{2}\ln\left(\frac{p}{2b_1}\right) - \frac{p}{2}\ln[1 + F(p)] \tag{18}$$

or

$$\lambda_d(p) = P(p) - \frac{p}{2}\ln(p) + \frac{p}{2}\ln(2d) - \frac{p}{2}\ln[1 + F(p)] \tag{19}$$

using $b_1 = d$. Referring to Eq. (6), we may put Eq. (19) in the form

$$\begin{aligned} \lambda_d(p) = {} & \frac{1}{2}[p\ln(2d) - p\ln p - 2(1-p)\ln(1-p) - p] \\ & + \sum_{k=2}^{\infty} a_k(d) p^k, \end{aligned} \tag{20}$$

where the $a_k(d)$ are suitably determined from the Mayer series coefficients in a straightforward manner.

The Mayer coefficients for the dimer system $b_s(d)$ on a d-dimensional lattice are simply obtained from the HT expansion of the free energy for the Ising model. To illustrate the relationship between the Ising and the dimer problems, recall the primitive [28] method of HT and low-field

graphical expansion for the partition function $Z_N(\beta,h) = \sum_{m\geqslant 0}\sum_{l=m}^{L_{\max}} \gamma_N(2m,l)\tanh(h)^{2m}\tanh(\beta)^l$ of a spin-1/2 Ising model on a lattice of N sites. Here $\beta = 1/k_BT$ denotes the inverse temperature and $h = \beta H$ with H the uniform external magnetic field. The expansion coefficient $\gamma_N(2s,s)$ counts all possible lattice configurations of graphs represented by precisely s disconnected edges placed onto disjoint links of the lattice and therefore coincides with the quantity $g_N(s)$ in Eq. (12). The procedure of forming the specific free energy $f_N(\beta,h) = \frac{1}{N}\ln Z_N$ and then taking the thermodynamical limit exactly parallels [23] the procedure leading to Eq. (13), so one concludes that from the expansion $f(\beta,h) = \sum_{m\geqslant 0}\sum_{l=m}^{L_{\max}} f_{2m,l}\tanh(h)^{2m}\tanh(\beta)^l$, the Mayer expansion coefficients can be read as $b_s(d) = f_{2s,s}(d)$.

Let us now recall that recently a significant extension of the HT series for several models in the Ising universality class, including the conventional spin-1/2 model, has been obtained for a sequence of bipartite lattices, in particular the hsc lattices of spatial dimensionality $1 \leqslant d \leqslant 10$ and the hyper-body-centered-cubic (hbcc) lattices of any dimensionality. In the case of the hbcc lattice, this is true at least in principle because the lattice dimensionality enters only in the power of the embedding number (see below) and thus the computation time increases very slowly with the dimensionality; so far we have only performed the computations for $d \leqslant 7$. It is also convenient at this point to give some simple details of these calculations. It is most convenient to refer [20,21] to the classical linked-cluster method [29] of graphical expansion. At each order l of the HT expansion, the series coefficients are expressed as the sum of an appropriate class of l-edge graphs. Each graph contributes a ratio of two integers, the free embedding number and the symmetry number of the graph, times a product of bare vertex functions associated with the vertices of the graph and depending on the magnetic field. The embedding number counts the number of distinct ways (per site of the underlying lattice) in which the graph can be placed onto the lattice, with each vertex assigned to a site and each edge to a link. This number depends on the topology of the graph and on the dimensionality d of the lattice. The important property is that, in the case of the hsc lattices (but not for the hbcc lattices), for a generic graph with l edges, the embedding number is a polynomial in d of degree l at most. The symmetry number counts the automorphisms of the graph and depends only on the topology of the graph. The great advantage of the linked-cluster method comes from the recognition that the huge variety of graphs that contribute at relatively high orders of expansion to the computation of a physical quantity, e.g., the magnetization, can be obtained by combining simpler graphs in a smaller class [29], thus making it possible to trade the computational complexity for algebraic complexity.

From the field-dependent free energy one can compute all its field derivatives usually called (higher) susceptibilities. It is clear at this point that on the hsc lattices, the computation of these quantities through tenth order can be extended to a generic d. It is sufficient to perform a simple interpolation of the series coefficients using the computation on a sequence of hsc lattices of dimensionalities $1 \leqslant d \leqslant 10$ and based on the fact that the lth-order expansion coefficient is a simple polynomial [30] of degree l in d (with zero constant term). Actually much more than this can be done. One can observe [21] that the knowledge of the free energy gives access to the HT expansions of the successive derivatives of the magnetic field with respect to the magnetization $\partial^{2p+1}h/\partial\mathcal{M}^{2p+1}$ for $p = 0,1,\ldots$ and that these quantities are expressed only in terms of connected graphs having no articulation vertex, i.e., no vertex whose deletion would disconnect the graph. What is decisive for our aims is the fact that the embedding number onto a hsc lattice of an l-edge graph in this particular class is a polynomial in d of degree $\lfloor l/2\rfloor$ at most [30]. Here $\lfloor l/2\rfloor$ denotes the integer part of $l/2$. Therefore, in spite of the fact that the HT expansion coefficients of the (higher) susceptibilities at order l are polynomials in d of degree l, the susceptibilities can be simply expressed in terms of the successive derivatives of the magnetic field with respect to the magnetization that, at the same expansion order, are polynomials in d of degree $\lfloor l/2\rfloor$ only. Thus one can conclude that the exact dependence on d of the HT coefficients of the higher susceptibilities can actually be determined up to order 20, using only data for a sequence of hsc lattices of dimensionalities $1 \leqslant d \leqslant 10$, by an interpolation in d of the series coefficients.

Let us finally stress that the elements of the coefficients matrix $f_{2m,l}(d)$ of the HT and low-field expansion for the free energy of the spin-1/2 Ising model can be linearly expressed in terms of the expansion coefficients of the susceptibilities and therefore they also are polynomials in d of degree l. This property holds in particular for the Mayer coefficients $b_s(d) = f_{2s,s}(d)$ of the dimer gas. From Eq. (16) it follows that $\frac{d\lambda_d}{dp}$ can also be determined to order 20 for all d. More details concerning the graphical expansion procedure can be found in Sec. 3 of the Appendix.

III. DERIVATION OF EXPANSIONS

As mentioned in the Introduction, we have a second route for deriving $\lambda_d(p)$ and λ_d expansions. The key initial step is the computation of the quantity $\tilde{J}_i(d)$ from the quantities $b_i(d)$. The $\tilde{J}_i(d)$ depend on the set of $b_n(d)$ with $n \leqslant i$. The computations are given in Ref. [31] as

$$\tilde{J}_1 = 0.$$

We first find $\tilde{J}_r^L$, with $\tilde{J}_1^L = 0$, and from $r = 2$ on, inductively defined by

$$\tilde{J}_r^L = \frac{1}{L}\left\{S_r - \left[\exp\left(L\sum_{i=1}^{r-1}\tilde{J}_i^L x^i\right)\right]\Bigg|_r\right\}, \tag{21}$$

where

$$S_r = \sum_{p=0}^{r}\left(\left\{\exp\left[L\sum_i b_i\left(\frac{x}{2d}\right)^i\right]\right\}\Bigg|_p \times \frac{1}{(r-p)!}\left(\frac{-1}{2(L-1)}\right)^{r-p}\frac{(L-2p)!}{(L-2r)!}\right). \tag{22}$$

The symbol $|$ with the subscript j indicates the jth coefficient in the formal power series in x. The $\tilde{J}_r$ are determined from the $\tilde{J}_r^L$ by taking L to infinity. We may also inductively go from the $\tilde{J}_i$ to the b_i by the same formulas.

This set of relations was implicitly used in Ref. [13], but not explicitly written down there. Just as the $b_i(d)$ are the cluster expansion coefficients of a dimer gas, the $\tilde{J}_i(d)$ are the cluster expansion coefficients of a certain polymer gas [13] and these coefficients of the two gases are related by the development surrounding Eqs. (21) and (22). This is a clean calculation that requires no hard proof. The $\tilde{J}_i(d)$ can be proved [11] to be of the form

$$\tilde{J}_s(d) = \frac{c_{s,r}}{d^r} + \frac{c_{s,r+1}}{d^{r+1}} + \cdots + \frac{c_{s,s-1}}{d^{s-1}} \tag{23}$$

with $r \geqslant s/2$.

Whereas our first development was basically for each d individually, we will see as with this last equation that the dependence on d is in the fundamentals of this second development. The present treatment allows us to get results relating the series for different d. As an example, suppose we know the $\tilde{J}_i(d)$ for $1 \leqslant d \leqslant 10$ and $i \leqslant 20$. Then one may derive $\tilde{J}_i(d)$ for $i \leqslant 20$ and all d. (One has enough information to compute all the $c_{s,r}$ for $i \leqslant 20$.) The same statement holds for the $b_i(d)$ since one may go between the set of $b_i(d)$ with $i < n$ and the set of $\tilde{J}_i(d)$ with $i < n$, as mentioned above.

So far all the results dealt with in this section have been true and rigorously proven. We now turn to the further development, certainly true, but for which we do not yet have a rigorous proof. We work for a given d and take as known the $\tilde{J}_i(d)$ [which as above could be calculated from the $b_i(d)$]. We then compute $\alpha_i(d)$ by iterations, from $\alpha_i = 0$, of

$$\alpha_k = (\tilde{J}_k p^k) \frac{1}{\left(1 - 2\sum_{i=2} i\alpha_i\right)^{2k}} \left(1 - 2\sum_{i=2} i\alpha_i \Big/ p\right)^k . \tag{24}$$

In iterating we take the mapping from the right-hand side of the equation to the left-hand side of the equation to be a mapping of formal power series in p. It is proven in Ref. [16] that there is an $m > 0$ such that each of the sequences of formal power series converges to a convergent power series of radius of convergence $\geqslant m$. (Even if the power series in $\lambda_d(p)$ [see Eq. (6)] has a radius of convergence $\geqslant 1$, as we assume, we do not know if m can be chosen as 1.) Then $\lambda_d(p)$ is given by

$$\lambda_d(p) = Q_1 + Q_2, \tag{25}$$

$$Q_1 = \tfrac{1}{2}[p\ln(2d) - p\ln p - 2(1-p)\ln(1-p) - p], \tag{26}$$

$$Q_2 = \sum_{i=2} \alpha_i - \sum_{k=2} \frac{1}{k} \left(2\sum_{i=2} i\alpha_i\right)^k + \frac{1}{2} p \sum_{k=2} \frac{1}{k} \left(2\sum_{i=2} i\alpha_i \Big/ p\right)^k . \tag{27}$$

Here Q_2 may be developed as a power series in p

$$Q_2 = \sum_{2}^{\infty} a_k(d) p^k, \tag{28}$$

where $a_k(d)$ is a polynomial in powers of $\frac{1}{d}$ with powers satisfying $k/2 \leqslant r < k$ [as the powers in Eq. (23)] (see Refs. [15,16]). So, for example, if we know $a_k(d)$ for $k \leqslant 20$ and $d \leqslant 10$, then we can deduce $a_k(d)$ for $k \leqslant 20$ and all d. To determine a_k only the values of $a_k(d)$, $d = 1, \ldots, \lfloor k/2 \rfloor$, are needed; the remaining $10 - \lfloor k/2 \rfloor$ values were used to give consistency checks for each $k < 20$. This is a consistency check on both the computation of the $b_i(d)$ and of the theory since as we mentioned above the development of Eqs. (24)–(28) has not been yet made rigorous.

We can deduce the series for λ_d [Eq. (2) above] basically by setting $p = 1$ in Eq. (25). It is important to note for this that each power of $\frac{1}{d}$ gets a contribution from only a finite number of $a_k(d)$. Specifically the $1/d^s$ get contributions from those $a_k(d)$ for which $k/2 \leqslant s < k$. For example, if we know $a_k(d)$ for $k \leqslant 20$, then we can deduce the terms in λ_d up to $1/d^{10}$.

To get at the theory [of the formal argument leading to Eqs. (25)–(28), our second development of the $\lambda_d(p)$ and λ_d expansions], we recommend to the reader starting by reading Ref. [13] or Sec. 5 of Ref. [15]. We now give a slightly hand-waving summary of the introductory portion of this theory up to the derivation of the mean field formulas (1) and (5) above.

We work on a periodic d-dimensional lattice with a number of sites N. A difunction is a translation-invariant periodic function on pairs of distinct vertices. We associate with dimers the difunction f, which is 1 if the two vertices are nearest neighbors and 0 otherwise. We call a sequence $X_1, X_2, \ldots$ of pN distinct vertices a p-sequence. We let $\sum$ denote the sum over all p-sequences. We note that the number of distinct dimer coverings that cover a fraction p of the vertices can be represented as

$$\frac{1}{2^{(pN/2)}} \frac{1}{(pN/2)!} \sum \prod_{\substack{i=1 \\ i \text{ odd}}}^{pN} f(X_i, X_{i+1}). \tag{29}$$

The numerical factors before the sum are divided by the number of different p-sequences that correspond to the same choice of dimers. The sum is over $\frac{N!}{[(1-p)N]!}$ p-sequences.

We let f_0 be the difunction of constant value $\frac{2d}{N-1}$. Here f and f_0 have the same normalization in the sense that if one fixes its first component and sums over the second, one gets the same answer for both functions. Replacing f in Eq. (29) by f_0 and using the Stirling formula gives the mean field answer

$$\exp(\lambda_{mf} N)$$

for the number of our dimer covers, where λ_{mf} is as in Eq. (5). We write

$$f = f_0 + \mathcal{V} \tag{30}$$

with

$$\mathcal{V} = f - f_0. \tag{31}$$

Expansions in powers of $\mathcal{V}$ may be converted into the expansions of this paper.

IV. NUMERICAL ESTIMATES

It is interesting at this point to get some feeling about the accuracy of the estimates of h_d and $\tilde{h}_d \equiv \lambda_d(1)$ that can be obtained from the expression (6) for $\lambda_d(p)$ when a sufficiently large number of coefficients $a_k(d)$ are known. For the evaluation of both h_d and $\tilde{h}_d$ a first orientation comes from truncating the expansion $\sum_{k=2}^{\infty} a_k(d) p^k$ at order $k = r$ and plotting the result vs some power of $1/r$. Let us first

consider the quantity $\tilde{h}_d$. Assuming that the series converges for $p = 1$ and that the coefficients $a_k(d)$ are all positive, then its successive truncations must provide an increasing sequence of lower bounds of the limit. First, we can check that the approximation of truncating the expansion at the highest known order is always consistent with the known upper bounds. However, in the cases of $d = 2$ and 3 the values thus obtained, i.e., $\tilde{h}_2 = 0.2865\ldots$ and $\tilde{h}_3 = 0.44916\ldots$, respectively, appear to be still too small. Therefore, one should properly extrapolate the sequences $S_r = \frac{1}{2}[\ln(2d) - 1] + \sum_2^r a_k(d)$ of the truncated expressions. Of course, the best way to do this depends on the behavior of the sequences. It is very encouraging to notice that for all values of d, the sequences are smooth and their behavior is well approximated by the simple ansatz $S_r = a + b/r^\alpha$ and so one has $a \approx \tilde{h}_d$. This procedure is very successful. We observe that α increases with d and ranges from $\alpha \approx 1$ for $d = 2$ to $\alpha \approx 2.6$ for $d = 6$. In dimensionality $d = 2$, this ansatz gives a good fit of the last 4–10 terms of the sequence and the extrapolated value $a = 0.2915(20)$ agrees with the exactly known value $\tilde{h}_2 = 0.291\,560\ldots$ in Eq. (7), within the estimated error. The uncertainty we have written is very conservative, although somewhat arbitrary. It is obtained both allowing for the spread of values resulting from small variations of the exponent α in the functional form used for fitting and from a comparison with other extrapolations obtained, for example, by evaluating the series $\sum_{k=2}^r a_k(d)p^k$ for $p = 1$, by Padé or differential approximants [32], and adding the result to the expression $\frac{1}{2}[\ln(2d) - 1]$. Analogously, for $d = 3$ the sequence S_r is well fitted by the ansatz $a + b/n^{1.3}$ and leads to the estimate $\tilde{h}_3 = 0.4499(2)$. This value is not far from the estimate $\tilde{h}_3 = 0.4479$ obtained by a Monte Carlo calculation [33] or from $\tilde{h}_3 = 0.453(1)$ obtained [34] by extrapolating a much shorter expansion and it is also completely consistent with the known bounds [9,14,35,36] $0.440\,075\,842 \leqslant \tilde{h}_3 \leqslant 0.457\,546\,930\,8$.

Proceeding along the same lines, we can determine the values of $\tilde{h}_d$ for any value of d. We notice that the apparent precision of the results improves rapidly as d grows, while the differences between the extrapolated values and the highest-order truncations of the series (as well as the estimated uncertainties) decrease rapidly. The final estimates are always completely consistent with the known bounds. Our estimates of $\lambda_d(1)$ for $d = 2,\ldots,8$ are reported in Table III. Note that evaluations of these quantities appear rarely [37] in the literature.

The computation of h_d requires only a quite short comment. Unsurprisingly, the sequences of truncated expansions $\sum_{k=2}^r a_k(d)p^k$ evaluated for $p < 1$ show a faster convergence than for $p = 1$. The estimates of h_d thus obtained agree well, within their uncertainties, with those already listed in Table VII of Ref. [19], which have been obtained by resumming via Padé approximants the expansion of $P(z)$ for $z = 1$. Therefore, the reader is referred to this source.

A. Series expansion for h_d for d large

As d goes to infinity, h_d tends to $\tilde{h}_d$. One can compute the rate with which the former approaches the latter by performing an expansion in $\frac{1}{\sqrt{d}}$.

To compute h_d one looks for a stationary point of Eq. (6). By setting $y = 1/\sqrt{2d}$, the stationarity equation can be written as

$$(1 - p_{\rm st})^2 - p_{\rm st} y^2 \exp\left(-2\sum_{k=2} k a_k p_{\rm st}^{k-1}\right) = 0. \qquad (32)$$

This equation can be solved for large d. Knowing a_k up to $k = 20$, one can solve iteratively the equation up to order y^{42}. Here we shall report only the first few terms

$$p_{st} = 1 - y + \tfrac{1}{2}y^2 + \tfrac{3}{8}y^3 - y^4 + \tfrac{201}{128}y^5 - \tfrac{5}{2}y^6 + \tfrac{7003}{1024}y^7 - 22y^8 + \cdots. \qquad (33)$$

At second order in y, it agrees with the value of p_{st} associated with the lower bound for h_d found in [5,14]

$$p_{\rm st} = \frac{4d + 1 - \sqrt{8d+1}}{4d}. \qquad (34)$$

Substituting Eq. (33) into Eq. (6) to get h_d and $p = 1$ into Eq. (6) to get $\tilde{h}_d$, one finds

$$h_d - \tilde{h}_d = y - \tfrac{1}{4}y^2 - \tfrac{11}{24}y^3 + \cdots, \qquad (35)$$

$$h_d = \frac{1}{2}(\ln 2d - 1) + \frac{1}{\sqrt{2d}} - \frac{11}{48\sqrt{2d^3}} + O(d^{-2}), \qquad (36)$$

in which we wrote only the first three terms out of the 40 terms we computed. Using 40 terms, this series expansion agrees with the difference $h_d - \tilde{h}_d$ computed numerically up to 2×10^{-6} for $d = 7$ and 10^{-15} for $d = 20$. From $d = 40$ up

TABLE III. Our estimates of $\tilde{h}_d = \lambda_d(1)$ for the hyper-simple-cubic lattices of dimensions $d = 2,3,\ldots,8$ with the known rigorous lower and upper bounds [15] defined by $(1/2)\ln(2d) - 1/2 \leqslant \lambda_d(1) \leqslant \ln[(2d)!]/4d$. While these rigorous bounds are valid for all d, for $d = 2$ we have simply reported the first eight digits of the exact value and for $d = 3$ we have reported the tighter bounds from Refs. [9,14,35,36]. The nonrigorous lower bounds (LB) are simply obtained by assuming the validity of the positivity conjecture for the coefficients $a_k(d)$ and truncating our expansions at the highest available order.

$\lambda_d(1)$	Lower bound	Nonrigorous LB	Our estimate	Upper bound
$\lambda_2(1)$	0.29156090	0.286521	0.2915(20)	0.29156090
$\lambda_3(1)$	0.44007584	0.449164	0.4499(2)	0.45754694
$\lambda_4(1)$	0.53972077	0.576517	0.57666(3)	0.66278769
$\lambda_5(1)$	0.65129254	0.679434	0.67949(2)	0.75522063
$\lambda_6(1)$	0.74245332	0.765301	0.765315(2)	0.83280061
$\lambda_7(1)$	0.81952866	0.838785	0.838789(1)	0.89968648
$\lambda_8(1)$	0.88629436	0.902947	0.902949(1)	0.95849563

to $d = 9000$ the precision is only 10^{-16}. The terms given in Eq. (36) give h_d with an error less than 3×10^{-3} for $7 \leqslant d < 100$ and 2×10^{-5} for $100 \leqslant d < 10000$. In particular, from Eqs. (33) and (35) one gets

$$\lim_{d\to\infty} \frac{h_d - \tilde{h}_d}{p_{\rm st}(d) - 1} = -1. \tag{37}$$

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APPENDIX

1. Conjecture that the coefficients a_k are positive in the case of the hsc lattices

We proved that the coefficients $a_k(d)$ are positive integers for $k \leqslant 20$ and $d \geqslant 1$ by locating in the complex d plane their real roots and counting the complex ones to make sure that none is missing. It is interesting to note that for $1 < d < 2$ or $2 < d < 3$ the $a_k(d)$ can be negative and that there are roots approaching 1 and 2 as k gets large. As we have already noticed, based on the conjecture that the $a_k(d)$ are positive, the computed values of $a_k(d)$ provide a lower bound of λ_d (see Table IV). For $d = 2$ in the case of h_d, this lower bound 0.662 798 966 is smaller than the estimate 0.662 798 972(1) obtained in Ref. [19] by Padé approximants. For $d \geqslant 3$ these lower bounds reproduce within the error the Padé estimates of Ref. [19].

2. Generalization of the positivity conjecture to other bipartite lattices

There is some evidence that the positivity conjecture can be extended to other *bipartite* lattices. Let us recall what is known about other lattices. The lower bound λ_d has been computed from the Mayer coefficients b_n on other lattices using the formula

$$\lambda_d = -\frac{1}{2} p \ln\left(\frac{p}{q}\right) - (1-p)\ln(1-p) - \frac{p}{2} + \frac{q}{2} \sum_{k=2} \frac{C_k \left(\frac{p}{q}\right)^k}{k(k-1)}, \tag{A1}$$

with q the lattice coordination number. The notation $a_k = \frac{q}{2}\frac{C_k/q^k}{k(k-1)}$ extends that used for the hypercubic case. From Eqs. (16) and (A1) it follows that

$$z = \frac{p}{q} \exp\left[-\sum_{k\geqslant 1} \frac{C_{k+1} - 2q^k}{k} \left(\frac{p}{q}\right)^k \right], \tag{A2}$$

corresponding to Eqs. (9) and (21) in Ref. [27], in which the first few coefficients for the square lattice and for some of the lattices discussed below were computed.

Let us now report the available data for other bipartite lattices. In the case of the tetrahedral lattice ($q = 4$), taking the Mayer coefficients b_n from Refs. [23,24], we obtain the following set of coefficients C_k: 1, 1, 1, 1, 31, 253, 1261, 4897, 16 201, 49 501, 161 239, 643 969, 3 006 823, 14 104 861, 60 942 421, 237 903 169, 854 124 745, and 2 955 594 097.

In the case of the hbcc lattices, the b_n for $n \leqslant 24$ have been computed in Ref. [19] for $d = 3, \ldots, 7$. The coefficients C_k computed from them are all positive. In Table V we list the coefficients $C_k(d)$ for hbcc lattices of dimensionalities $d =$ 3,4,5. The coordination numbers of these lattice are $q = 2^d$. For the hexagonal lattice [38] with $q = 3$, the coefficients C_k are 1, 1, 1, 1, 11, and 85.

Let us now turn to the case of nonbipartite lattices. For the triangular lattice ($q = 6$) the coefficients up to C_6 are listed in Ref. [38], while higher-order ones are obtained from Ref. [23]: 1, −3, −11, 1, 91, 141, −1651, −16 143, −87 329, −295 063, −72 533, 8 092 033, and 76 819 835. For the fcc lattice ($q = 12$), from Ref. [23] we obtain 1, −7, 19, 41, −779, 3557, 46 327, 118 529, and −557 909. These data imply that the positivity conjecture for the C_k has to be restricted to bipartite lattices.

On a Bethe lattice [39,40] the entropy is given by Eq. (A1) with $C_k = 1$ for all k. Notice that on any lattice $C_k = 1$ for $k < r$, where r is the length of the smallest nontrivial loop on the lattice, because the diagrams contributing to such C_k cannot distinguish between the given lattice and a Bethe lattice of the same coordination number. A stronger form of the positivity conjecture is that $C_k \geqslant 1$ for bipartite lattices.

3. Graphical expansion procedure for the Ising model

To make Sec. II more readable, we have included in this section some technical details on the graphical procedures used in the computation of the Ising model HT expansions.

TABLE IV. Real roots of $a_k(d)$ for $k \geqslant 10$.

$k = 10$	−0.65502486055142554	0.99997855862379883	
$k = 11$	1.0000010707811947	1.6603775954637132	
$k = 12$	−0.12473442164710268	1.0000008060184913	1.5835444714309055
$k = 13$	0.99999998817575145		
$k = 14$	1.0000000363954472	1.628126162558255	
$k = 15$	1.000000000099149	1.9594209128425236	
$k = 16$	0.99999999993591589		
$k = 17$	0.99999999999934955	2.0071302031011769	2.4952449117198663
$k = 18$	1.0000000000002549	1.9889667170409254	
$k = 19$	1.0000000000000033	1.9993853767904753	
$k = 20$	0.99999999999999901	2.0009898597900763	2.6230186617839066

TABLE V. Coefficients $C_k(d)$ in Eq. (A1) with $k = 2, \ldots, 24$ for the hyper-body-centered-cubic lattices of dimensionalities $d = 3,4,5$.

C_k	$d = 3$	$d = 4$	$d = 5$
C_2	1	1	1
C_3	1	1	1
C_4	37	151	541
C_5	241	1001	3601
C_6	1651	21241	290851
C_7	13861	276445	4136581
C_8	109873	4138275	185447641
C_9	850465	61222177	3766174561
C_{10}	6620401	903139171	134478272521
C_{11}	51657541	13527055301	3251891481301
C_{12}	403327651	201952069177	105463232417731
C_{13}	3151118881	3041256137921	2794164743354401
C_{14}	24647038963	45839858214697	86840903677417891
C_{15}	192950685061	69396577375846	2421252466929163141
C_{16}	1510882839217	105307033482448 51	73870429278903327001
C_{17}	11833222518145	160247978490447425	212302672147192177 1521
C_{18}	92728596423613	2444106838568935375	643066947198294147 61621
C_{19}	727194198560401	37359234126615235321	188389546112737380253 3921
C_{20}	5707071682914097	572176086489368008851	569612772101668885672268 41
C_{21}	44820667959807601	8779078842662089743601	169024263047852666983570 4401
C_{22}	352227866459521537	134925544759538198882283	511466246435457031932388 49401
C_{23}	2769671081569110445	20768686452939251241 33493	153152678051860809792754 5101821
C_{24}	21790699297032926587	32014374542692855556562921	464357676442230613585492 93433371

For simplicity, the whole graphical expansion procedure can be split into three steps. First, one has to list all graphs entering into the calculation up to the maximum order $L_{\max}$ of expansion. To begin with, one forms the simple, topologically distinct, one-vertex-irreducible graphs with $l \leqslant L_{\max}$ edges. One can further restrict these graphs to the subset of the bipartite graphs since only these can be embedded onto the bipartite hsc or hbcc lattices. This is the only memory intensive part of the procedure because there are many graphs [20,21,41] (approximately 3×10^5 graphs at order 20 and over 5×10^7 at order 24), but it took only a few hours. In a second step, the lattice embedding numbers and the symmetry numbers of these graphs are computed; one vertex of these graphs is marked in all possible ways and the graphs are decorated to have also multiple lines. This is the subset of the graphs from which the expansion of the magnetization can be reconstructed.

In the case of hsc lattices of high dimensionality, the most time-consuming part of this procedure is the computation of the embedding number for each graph. In the case of the hbcc lattices the timings are much smaller than for the hsc lattices and very slowly dependent on d, but unfortunately the expansion coefficients are not polynomials in d. One begins by appropriately ordering the graph vertices and then the first of them is placed at the lattice origin. The possible positions of the second vertex can be counted by exploiting the symmetries of the hypercube. After fixing the first two points of the embedding, the possible positions of the remaining vertices are restricted to relatively few configurations by the constraints given by the distances from the first two points and the count can go on in a relatively easy way. On the hsc lattices, the timings for computing the magnetization expansion of the d-dimensional Ising model at order $L_{\max}$ increase exponentially with the order of expansion and the lattice dimensionality d: roughly as $O(5.5^{L_{\max}} 2.5^d)$. In particular, the computation for the ten-dimensional Ising model at order 20 took 42 days of single-core time on a quad-core desktop computer with a CPU clock frequency of 2.8 GHz. Actually, less time was used since the calculation was appropriately distributed on the four cores of the computer. Using more extensive computer resources, it would be possible to compute only a few more orders, for not too high lattice dimensionalities.

The next step implements the algebraic vertex renormalization, namely, the procedure of reconstruction [29] of the magnetization from the one-vertex-irreducible graphs having a single marked vertex. By integrating the magnetization exactly with respect to the field one finally obtains the free energy in terms of the bare vertices (up to a standard constant of integration). This step of the calculation is based on codes written in the PYTHON language and is fast. The free energy thus computed is model independent: Eventually one has to specialize the precise form of the bare vertex functions to the particular model of interest.

[1] M. E. Fisher, Phys. Rev. **124**, 1664 (1961).
[2] P. W. Kasteleyn, Physica **27**, 1209 (1961).
[3] R. J. Baxter, J. Math. Phys. **9**, 650 (1968).
[4] P. Flajolet and R. Sedgewick, *Analytic Combinatorics* (Cambridge University Press, Cambridge, UK, 2009).
[5] S. Friedland, E. Kropp, P. H. Lundow, and K. Markström, J. Stat. Phys. **133**, 513 (2008).
[6] A. Hanany and K. Kennaway, arXiv:hep-th/0503149.
[7] R. Dijkgraaf, D. Orlando, and S. Reffert, Adv. Theor. Math. Phys. **13**, 1255 (2009).

[8] J. M. Hammersley, in *Research Papers in Statistics: Festschrift for J. Neyman*, edited by F. N. David (Wiley, London, 1966), p. 125.
[9] H. Minc, Linear Multilinear Algebra **8**, 235 (1980).
[10] P. Federbush, arXiv:0804.4220v1.
[11] P. Federbush, arXiv:0806.1941v1.
[12] P. Federbush, arXiv:0806.4158v1.
[13] P. Federbush, Phys. Lett. A **374**, 131 (2009).
[14] S. Friedland and U. N. Peled, Adv. Appl. Math. **34**, 486 (2005).
[15] P. Federbush and S. Friedland, J. Stat. Phys. **143**, 306 (2011).
[16] P. Federbush, arXiv:1101.4591.
[17] O. J. Heilmann and E. H. Lieb, Commun. Math. Phys. **25**, 190 (1972).
[18] D. Ruelle, *Statistical Mechanics* (Benjamin, Amsterdam, 1969).
[19] P. Butera and M. Pernici, Phys. Rev. E **86**, 011104 (2012).
[20] P. Butera and M. Pernici, Phys. Rev. E **85**, 021105 (2012).
[21] P. Butera and M. Pernici, Phys. Rev. E **86**, 011139 (2012).
[22] D. S. Gaunt, Phys. Rev. **179**, 174 (1969).
[23] D. A. Kurtze and M. E. Fisher, Phys. Rev. B **20**, 2785 (1979).
[24] S. McKenzie, Can. J. Phys. **57**, 1239 (1979).
[25] P. Federbush, arXiv:1207.1252.
[26] P. Federbush, J. Stat. Phys. **150**, 487 (2013).
[27] G. S. Rushbrooke, H. I. Scoins, and A. J. Wakefield, Discuss. Faraday Soc. **15**, 57 (1953).
[28] C. Domb, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green, Vol. 3 (Academic, New York, 1974), p. 357.
[29] M. Wortis, in *Phase Transitions and Critical Phenomena* (Ref. [28]), p. 113.
[30] M. E. Fisher and D. S. Gaunt, Phys. Rev. **133**, A224 (1964).
[31] P. Federbush, arXiv:1209.0987.
[32] A. J. Guttmann, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz, Vol. 13 (Academic, New York, 1989), p. 1.
[33] I. Beichl, D. P. O'Leary, and F. Sullivan, Phys. Rev. E **64**, 016701 (2001).
[34] P. Federbush, arXiv:0805.1195.
[35] A. Schrijver, in *Graph-Theoretic Concepts in Computer Science*, edited by H. L. Bodlaender, Lecture Notes in Computer Science, Vol. 2880 (Springer, Berlin, 2003).
[36] P. H. Lundow, Discrete Math. **231**, 321 (2001).
[37] S. R. Finch, *Mathematical Constants* (Cambridge University Press, Cambridge, 2003).
[38] P. Federbush, arXiv:1110.0684.
[39] J. F. Nagle, Phys. Rev. **152**, 190 (1966).
[40] J. F. Stilck and M. J. de Oliveira, Phys. Rev. A **42**, 5955 (1990).
[41] P. Butera and M. Pernici, Phys. Rev. B **83**, 054433 (2011).