Exploring a noisy van der Pol type oscillator with a stochastic approach

Ruoshi Yuan,[1,2] Xinan Wang,[3] Yian Ma,[4] Bo Yuan,[3] and Ping Ao[1,5]

[1]*Key Laboratory of Systems Biomedicine, Ministry of Education, Shanghai Center for Systems Biomedicine, Shanghai Jiao Tong University, Shanghai, 200240, China*

[2]*School of Biomedical Engineering, Shanghai Jiao Tong University, Shanghai, 200240, China*

[3]*Department of Computer Science and Engineering, Shanghai Jiao Tong University, Shanghai, 200240, China*

[4]*Department of Applied Mathematics, University of Washington, Seattle, Washington 98195, USA*

[5]*Department of Physics, Shanghai Jiao Tong University, Shanghai, 200240, China*

(Received 6 March 2013; revised manuscript received 28 April 2013; published 7 June 2013)

Based on conventional Ito or Stratonovich interpretation, zero-mean multiplicative noise can induce shifts of attractors or even changes of topology to a deterministic dynamics. Such phenomena usually introduce additional complications in analysis of these systems. We employ in this paper a new stochastic interpretation leading to a straightforward consequence: The steady state distribution is Boltzmann-Gibbs type with a potential function severing as a Lyapunov function for the deterministic dynamics. It implies that an attractor corresponds to the local extremum of the distribution function and the probability is equally distributed right on an attractor. We consider a prototype of nonequilibrium processes, noisy limit cycle dynamics. Exact results are obtained for a class of limit cycles, including a van der Pol type oscillator. These results provide a new angle for understanding processes without detailed balance and can be verified by experiments.

DOI: 10.1103/PhysRevE.87.062109 PACS number(s): 05.40.−a, 05.45.−a, 05.70.Ln, 87.10.Mn

I. INTRODUCTION

The Langevin equation, or the stochastic differential equation in mathematics, is a more comprehensive description of natural phenomena than purely deterministic equations [1–3]. The Langevin equation alone, however, cannot determine a random process. Specifying a stochastic interpretation is needed and leads to different consequences [1–5]. The most widely applied stochastic interpretations are Ito's and Stratonovich's. Based on these two interpretations, when zero-mean multiplicative noise is introduced, the steady-state distribution of the process in general does not "correspond" to the deterministic counterpart: Shifts of attractors or even topology changes are observed [6,7]. As a result, additional difficulties are encountered in the analysis of a system; e.g., calculating transition probability between attractors, which is critical in applications [8–10], can be subtle [11].

Intuitively, the word "correspond" implies that a stable fixed point of the deterministic (part) dynamics is also a local maximum of the steady state distribution. More precisely, "correspond" means that the probability density function does not decrease along the trajectories of the deterministic dynamics and reaches maximum at the stable attractors. Based on Liouvilles's theorem, the probability density function for Hamiltonian dynamics keeps constant along trajectories; thus Hamiltonian dynamics has this correspondence property. A well-known concept in engineering, the Lyapunov function, does not increase along the trajectories of a deterministic dynamics. Therefore, we can define the word "correspond" as the steady state distribution multiplying negative one [12] being a Lyapunov function of the deterministic counterpart of the stochastic dynamics. Three questions arise here: First, does a stochastic interpretation with such a correspondence property exist? Second, if it exists, what are the new insights provided by this interpretation? Furthermore, is there a real process choosing this interpretation?

The answer to the first question is positive based on a recent framework [13–15]. A brief review of the framework is given in the next section. Afterwards, we will apply analytically this framework to a class of typical nonequilibrium processes: noisy limit cycle dynamics. Recently, they raised much research interest in physics [16–20] and other fields [21–23]. Rotationally symmetric and general planar limit cycles are handled separately in Secs. III A and III B. A van der Pol type oscillator with multiplicative noise is exactly solved. The analysis of these explicit results provides new insights of understanding processes without detailed balance.

II. A NEW STOCHASTIC INTEGRATION

The Langevin equation can be considered as a composition of a *deterministic dynamics* $\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})$ and a zero-mean multiplicative noise $N(\mathbf{q})\xi(t)$ [1]:

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}) + N(\mathbf{q})\xi(t), \quad (1)$$

where $\mathbf{q}$, $\mathbf{f}$ are n-dimensional vectors and $\mathbf{f}$ is a nonlinear function of the state variable $\mathbf{q}$. The noise $\xi(t)$ is k-dimensional Gaussian white with the zero mean, $\langle \xi(t) \rangle = 0$, and the covariance $\langle \xi(t)\xi^\tau(t') \rangle = \delta(t-t')I_k$. The notation $\delta(t-t')$ is the Dirac delta function, $\langle \cdots \rangle$ is an average over the noise distribution, and I_k is the k-dimensional identity matrix. The element of the $n \times k$ matrix $N(\mathbf{q})$ can be a nonlinear function of $\mathbf{q}$; then the noise considered in this framework can be a general multiplicative noise. This matrix is further described by $N(\mathbf{q})N^\tau(\mathbf{q}) = 2\epsilon D(\mathbf{q})$, the constant ϵ quantifying the noise strength and $D(\mathbf{q})$ being a $n \times n$ positive semidefinite diffusion matrix. Note that the noise may have less than n independent components $k < n$, leading to the zero eigenvalue of $D(\mathbf{q})$. During the study of a biological switch [13–15], a dynamics equivalent to Eq. (1) was discovered:

$$[S(\mathbf{q}) + A(\mathbf{q})]\dot{\mathbf{q}} = -\nabla\phi(\mathbf{q}) + \hat{N}(\mathbf{q})\xi(t). \quad (2)$$

The term $S(\mathbf{q})$ is a positive semidefinite matrix, where $-S(\mathbf{q})\dot{\mathbf{q}}$ denotes a frictional force; the term $A(\mathbf{q})$ is an antisymmetric matrix representing an embedded symplectic structure, and $-A(\mathbf{q})\dot{\mathbf{q}}$ is a rewritten form of the Lorentz force $e\dot{\mathbf{q}} \times \mathbf{B}$ in two- or three-dimensional cases and also a generalization to higher dimensions; the scalar function $\phi(\mathbf{q})$ is a potential function, e.g., the electrostatic potential, lying at the core of the discussion in this paper. The matrix $\hat{N}(\mathbf{q})$ is constrained by the fluctuation-dissipation theorem [24,25]: $\hat{N}(\mathbf{q})\hat{N}^\tau(\mathbf{q}) = 2\epsilon S(\mathbf{q})$. In the Appendix, we briefly discuss the transformation from Eqs. (1) to (2) and obtain a set of equations for a potential function.

A corresponding Fokker-Planck equation (FPE) for (2) [therefore for Eq. (1)] can be obtained with physical meaning (a zero mass limit) [5]:

$$\partial_t \rho(\mathbf{q},t) = \nabla \cdot [D(\mathbf{q}) + Q(\mathbf{q})] \cdot [\nabla\phi(\mathbf{q}) + \epsilon\nabla]\rho(\mathbf{q},t), \quad (3)$$

where ∇ in $\nabla\phi(\mathbf{q})$ does not operate on $\rho(\mathbf{q},t)$, $D(\mathbf{q})$ is the diffusion matrix, and the matrix $Q(\mathbf{q})$ is antisymmetric and can be calculated from the relation $[S(\mathbf{q}) + A(\mathbf{q})]\,[D(\mathbf{q}) + Q(\mathbf{q})] = I_n$. Equation (3) has the Boltzmann-Gibbs distribution with the potential $\phi(\mathbf{q})$ as a steady state solution:

$$\rho(\mathbf{q},t \to \infty) = \frac{1}{Z_\epsilon} \exp\left\{-\frac{\phi(\mathbf{q})}{\epsilon}\right\}, \quad (4)$$

where $Z_\epsilon = \int d^n\mathbf{q} \exp\{-\phi(\mathbf{q})/\epsilon\}$ is the partition function. The probability current density $\mathbf{j}(\mathbf{q},t) = [j_1(\mathbf{q},t),\ldots,j_n(\mathbf{q},t)]^\tau$ is commonly defined as

$$j_i(\mathbf{q},t) = \bar{f}_i(\mathbf{q})\rho(\mathbf{q},t) - \partial_j[\epsilon D_{ij}(\mathbf{q})\rho(\mathbf{q},t)], \quad (5)$$

where $\bar{f}_i(\mathbf{q}) = f_i(\mathbf{q}) + \epsilon[\partial_j D_{ij}(\mathbf{q}) + \partial_j Q_{ij}(\mathbf{q})]$, $f_i(\mathbf{q})$ is the ith component of the vector valued function $\mathbf{f}(\mathbf{q})$ in Eq. (1), and $D_{ij}(\mathbf{q})$ and $Q_{ij}(\mathbf{q})$ are the elements of the matrices $D(\mathbf{q})$ and $Q(\mathbf{q})$ in Eq. (3). In steady state, the probability distribution is given by Eq. (4). We have $\nabla \cdot \mathbf{j}(\mathbf{q},t \to \infty) = 0$, but $\mathbf{j}(\mathbf{q},t \to \infty)$ is usually not zero. One can check that $Q = 0$ is a sufficient condition for $\mathbf{j}(\mathbf{q},t \to \infty) = 0$; but when $Q(\mathbf{q}) \neq 0$, then generally $\mathbf{j}(\mathbf{q},t \to \infty) \neq 0$, since $\partial_j[Q_{ij}(\mathbf{q})\rho(\mathbf{q},t \to \infty)] \neq 0$. Therefore the framework encompasses the cases without detailed balance. The term "detailed balance" means the net current between any two states in the phase space is zero [26], identical to that for Markov process in mathematics. The dynamics studied in this paper corresponds to the nondetailed balance cases discussed in Ref. [27] as well.

Equation (3) defines a new stochastic interpretation for the Langevin equation (1), called A-type for short. The steady state distribution of the Langevin equation is a Boltzmann-Gibbs type with a potential function $\phi(\mathbf{q})$. For the deterministic dynamics, the time derivative of the potential function $\phi(\mathbf{q})$ along a trajectory is

$$\begin{aligned}\frac{d\phi(\mathbf{q})}{dt} &= \nabla\phi \cdot \mathbf{f}(\mathbf{q}) = -\nabla\phi(\mathbf{q}) \cdot [D(\mathbf{q}) + Q(\mathbf{q})] \cdot \nabla\phi(\mathbf{q}) \\ &= -\nabla\phi(\mathbf{q}) \cdot D(\mathbf{q}) \cdot \nabla\phi(\mathbf{q}) \leqslant 0,\end{aligned}$$

since the diffusion matrix $D(\mathbf{q})$ is nonnegative and symmetric. It means that the potential along the trajectory is nonincreasing and has the local extreme values at fixed points, limit cycles, or more complex attractors. Hence, the potential function $\phi(\mathbf{q})$ serves as a Lyapunov function [28] for the deterministic dynamics $\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})$. The probability density function (4) multiplying negative one is also a Lyapunov function,

$$-\frac{d}{dt}\frac{1}{Z_\epsilon}\exp\left\{-\frac{\phi(\mathbf{q})}{\epsilon}\right\} = \frac{1}{\epsilon Z_\epsilon}\exp\left\{-\frac{\phi(\mathbf{q})}{\epsilon}\right\}\frac{d\phi(\mathbf{q})}{dt} \leqslant 0,$$

which means that A-type interpretation has the correspondence property between deterministic and stochastic dynamics.

In view of the questions proposed in the introduction, five remarks are in order: First, A-type integration enables a straightforward use of the dynamical analysis for the deterministic dynamics in the presence of noise. Therefore, the calculation of the transition probability from one stable fixed point $\mathbf{q}_1$ to another one through a saddle point $\mathbf{q}_2$ is generally formulated as proportional to $\exp[-|\phi(\mathbf{q}_1) - \phi(\mathbf{q}_2)|/\epsilon]$. Second, the Boltzmann-Gibbs distribution is valid for arbitrary noise strength ϵ, not merely under the weak noise limit $\epsilon \to 0$, which is not held for conventional interpretations [29]. Third, the potential function obtained here serves also a Lyapunov function for the deterministic dynamics. The framework then contributes possible new approaches for the largely unsolved problem in engineering: constructing a Lyapunov function for general nonlinear dynamics. In addition, the A-type stochastic integration can be applied directly in the study of phase reduction. For a conventional phase reduction method [16], A-type integration does not lead to the noise-induced frequency shift. The last point, there exist processes in nature choosing A-type interpretation. A recent experiment [30] records the trajectories of the Brownian motion of a colloidal particle near a wall. It shows that A-type interpretation directly corresponds to the experimental data [5].

III. EXACT RESULTS FOR LIMIT CYCLE DYNAMICS

Noise disturbed limit cycle dynamics is now attracting considerable attention in the physics community [16–20]. A direct reason is that ubiquitous real systems can be modeled by them, e.g., from biological phenomena [21,22,31,32] such as cell cycle to chemical reaction [33] and oscillating electrical circuit [34]. The dynamics itself is a touchstone to study nonlinear dissipative process in the absence of detailed balance. Due to the difficulty arising out of nonlinearity and stochasticity, approximated methods based, for example, on phase reduction and weak noise perturbation are proposed from former studies [16–19,35,36], but exact results are rarely seen in the literature. Moreover, the existence of a potential function for processes without detailed balance is still suspected [20,23,37,38]; a specific argument is that, for a limit cycle system with nonconstant velocity along the cycle, the dual role potential (also the Lyapunov function) does not exist. In this paper, we examine such an example. The van der Pol oscillator [34] is a representative limit cycle dynamics; here we consider a stochastic version with a multiplicative noise $\zeta(\mathbf{q},t) = [\zeta_1(\mathbf{q},t),\zeta_2(\mathbf{q},t)]^\tau$ (the superscript τ denotes the transpose of a matrix) and a higher order term $h(q_1)$:

$$\begin{aligned}\dot{q}_1 &= q_2 + \zeta_1(\mathbf{q},t) \\ \dot{q}_2 &= -\mu\left(q_1^2 - 1\right)q_2 - q_1 + h(q_1) + \zeta_2(\mathbf{q},t).\end{aligned} \quad (6)$$

When $h(q_1) = 0$, the deterministic part of the dynamics reduces to the van der Pol oscillator. A specific system we

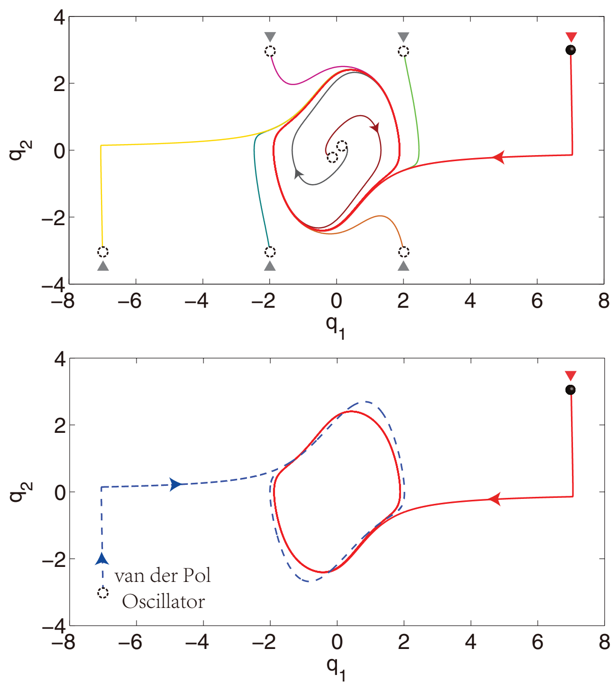


FIG. 1. (Color online) Upper panel: Trajectories (deterministic dynamics) for the system Eq. (6) with $h(q_1) = \mu^2 q_1^3/4 - \mu^2 q_1^5/16$ ($\mu = 1$). Lower panel: Comparison of two systems, the van der Pol oscillator $h(q_1) = 0$ is represented by the dashed blue line. The red line denotes the system in the upper panel.

would like to illustrate is $h(q_1) = \mu^2 q_1^3/4 - \mu^2 q_1^5/16$ [39]; we can observe from Fig. 1 the deterministic dynamical behavior of the system. It has a limit cycle without rotational symmetry and a position-dependent velocity along the cycle.

A. Rotationally symmetric limit cycles

The construction of a potential function relies on two relations: a potential condition [Eq. (A4)] and a generalized Einstein relation [Eq. (A5)]. From these two equations, we know the potential function is determined by the deterministic dynamics $\mathbf{f}(\mathbf{q})$ and the diffusion matrix $D(\mathbf{q})$. It can be proved that the potential function is invariant under a coordinate transformation ($\sigma : \mathbf{q} \to \mathbf{u}$) of the deterministic dynamics: $\phi(\mathbf{q}) = \phi[\sigma^{-1}(\mathbf{u})]$. The dynamical components, the matrices S, A, D, Q, vary in different coordinates, but a straightforward formulation can be achieved by multiplying the Jacobian matrix of the transformation.

For planar rotationally symmetric limit cycle dynamics with a constant diffusion matrix $D = D_0 I_2$, we can transform the deterministic part into polar coordinates ($q = \sqrt{q_1^2 + q_2^2}, \theta$):

$$\begin{aligned} \dot{q} &= R(q) \\ \dot{\theta} &= \psi(q) \end{aligned} \tag{7}$$

and provide an exact construction (some related results can be seen in Ref. [40]) of the potential function (note that the following result is represented in Cartesian coordinates):

$$\phi(\mathbf{q}) = -\frac{1}{D_0} \int R(q)\, dq \tag{8}$$

and corresponding dynamical components:

$$\begin{aligned} S(\mathbf{q}) &= \frac{R(q)^2}{D_0[R(q)^2 + q^2\psi(q)^2]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ A(\mathbf{q}) &= \frac{q\psi(q)R(q)}{D_0[R(q)^2 + q^2\psi(q)^2]} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ Q(\mathbf{q}) &= -\frac{q\psi(q)D_0}{R(q)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{9}$$

Note that for the weak noise limit, when $\epsilon \to 0$, this construction is still valid for the deterministic dynamics, and the potential function serves as a Lyapunov function of the system (see also Ref. [41]).

To illustrate, we examine a straightforward example:

$$\begin{aligned} \dot{q}_1 &= -q_2 + q_1\left(1 - q_1^2 - q_2^2\right) + \sqrt{2\epsilon}\xi_1(t) \\ \dot{q}_2 &= q_1 + q_2\left(1 - q_1^2 - q_2^2\right) + \sqrt{2\epsilon}\xi_2(t). \end{aligned} \tag{10}$$

By transforming the deterministic part into polar coordinates

$$\begin{aligned} \dot{q} &= R(q) = q(1 - q^2) \\ \dot{\theta} &= \psi(q) = 1 \end{aligned} \tag{11}$$

we can construct a potential function according to Eq. (8) with the diffusion matrix $D = I$. A Mexican hat shape potential function is then derived: $\phi(\mathbf{q}) = \frac{1}{4}(q_1^2 + q_2^2)(q_1^2 + q_2^2 - 2)$. Its corresponding Boltzmann-Gibbs steady state distribution is

$$\rho_{ss}(\mathbf{q}, t \to \infty) = \frac{1}{Z_\epsilon} \exp\left\{-\frac{\left(q_1^2 + q_2^2\right)\left(q_1^2 + q_2^2 - 2\right)}{4\epsilon}\right\}, \tag{12}$$

where $Z_\epsilon = e^{1/(4\epsilon)}\sqrt{\epsilon}\pi^{3/2}\{1 + \mathrm{erf}[1/(2\sqrt{\epsilon})]\}$. Meanwhile, we obtain

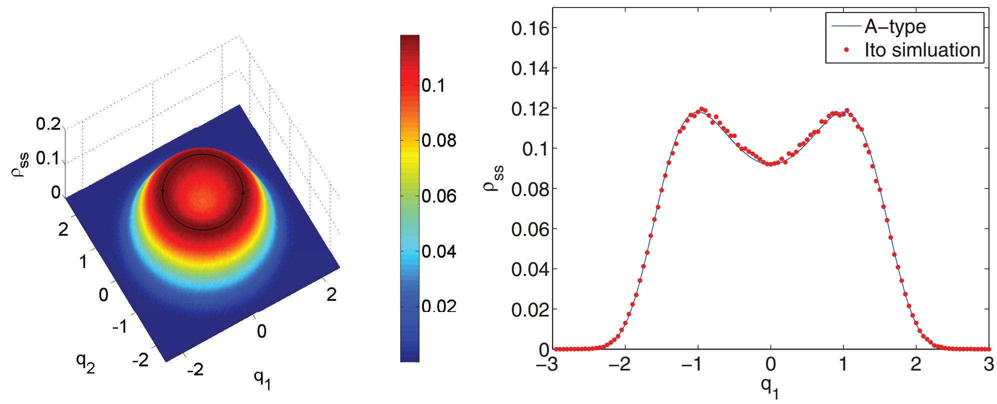
$$\begin{aligned} S(\mathbf{q}) &= \frac{\left(1 - q_1^2 - q_2^2\right)^2}{\left(1 - q_1^2 - q_2^2\right)^2 + 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ A(\mathbf{q}) &= \frac{\left(1 - q_1^2 - q_2^2\right)}{\left(1 - q_1^2 - q_2^2\right)^2 + 1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ Q(\mathbf{q}) &= \frac{1}{\left(1 - q_1^2 - q_2^2\right)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{13}$$

In this specific case, the steady state distribution of the Ito integration is identical to that of the A-type; see Fig. 2. However, for general situations the distributions are different even when the diffusion matrix is constant [42]. The reason here is that $\nabla \cdot Q(\mathbf{q}) \cdot \nabla\rho_{ss} = 0$, hence Eq. (12) is also a solution of the Ito FPE [by comparing with A-type FPE Eq. (3)].

B. General planar limit cycles

More generally, we can extend this method to systems without rotational symmetry through coordinate transformations (reversible smooth mappings, can be nonlinear) σ of the deterministic dynamics:

$$\begin{cases} \dot{q}_1 = f_1(\mathbf{q}) \\ \dot{q}_2 = f_2(\mathbf{q}) \end{cases} \overset{\sigma}{\underset{\sigma^{-1}}{\longleftrightarrow}} \begin{cases} \dot{u} = \bar{f}_1(u,v) \\ \dot{v} = \bar{f}_2(u,v) \end{cases} \overset{\text{polar}}{\underset{\text{polar}^{-1}}{\longleftrightarrow}} \begin{cases} \dot{q} = \rho(q)P(q,\theta) \\ \dot{\theta} = \varphi(q,\theta) \end{cases}.$$



(a) (b)

FIG. 2. (Color online) (a) Probability distribution function Eq. (12) with $\epsilon = 1$; (b) comparison with the distribution obtained from the Ito simulation at $q_2 = 0$ ($\epsilon = 1$).

If the property $P(q,\theta) \geqslant 0$ is satisfied, a potential function can be constructed by

$$\phi(q_1,q_2) \xleftarrow{\sigma^{-1}(u,v)} \bar{\phi}(u,v) \xleftarrow{\text{polar}^{-1}(q)} \bar{\phi}_p(q) = -\int \rho(q)\, dq,$$

since $d\phi(q_1,q_2)/dt = [d\bar{\phi}_p(q)/dq]\dot{q} = -\rho^2(q)P(q,\theta) \leqslant 0$. We list the protocol of this construction: First, for the deterministic dynamics (q_1,q_2), find a transformation $\sigma : (q_1,q_2) \to (u,v)$, calculate the dynamics under (u,v), that is, $\dot{u} = \bar{f}_1(u,v)$ and $\dot{v} = \bar{f}_2(u,v)$; Second, rewrite the obtained dynamics in polar coordinates $(u,v) \to (q,\theta)$, if the dynamics can be expressed as the requested form above, a potential function can be constructed as $\bar{\phi}_p(q)$; Third, transform $\bar{\phi}_p(q)$ back to $\bar{\phi}(u,v)$, and finally to $\phi(q_1,q_2)$.

Once the potential function (Lyapunov function) $\phi(\mathbf{q})$ for the deterministic dynamics $\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})$ is obtained, there are different ways to construct the dynamical components; one particular setting is provided in Ref. [41] [the binary operator of two n-dimensional vectors is defined as $\mathbf{x} \times \mathbf{y} = (x_i y_j - x_j y_i)_{n\times n}$, and the result is an $n \times n$ matrix]:

$$\begin{aligned} S(\mathbf{q}) &= -\frac{\nabla\phi \cdot \mathbf{f}}{\mathbf{f}\cdot\mathbf{f}} I, \quad A(\mathbf{q}) = -\frac{\nabla\phi \times \mathbf{f}}{\mathbf{f}\cdot\mathbf{f}}, \\ D(\mathbf{q}) &= -\left[\frac{\mathbf{f}\cdot\mathbf{f}}{\nabla\phi\cdot\mathbf{f}} I + \frac{(\nabla\phi\times\mathbf{f})^2}{(\nabla\phi\cdot\mathbf{f})(\nabla\phi\cdot\nabla\phi)}\right], \\ Q(\mathbf{q}) &= \frac{\nabla\phi\times\mathbf{f}}{\nabla\phi\cdot\nabla\phi}. \end{aligned} \tag{14}$$

Back to the example in Eq. (6) with $[\zeta_1(\mathbf{q},t),\zeta_2(\mathbf{q},t)]^\tau = N(\mathbf{q})\cdot[\xi_1(t),\xi_2(t)]^\tau$ and $h(q_1) = \mu^2 q_1^3/4 - \mu^2 q_1^5/16$. The deterministic part is a Liénard equation similar to the van der Pol oscillator ($0 < \mu < 2$; see Fig. 1) [39]. Through a nonlinear coordinate transformation σ^{-1}: $u = q_1$ and $v = q_2 - \mu q_1 + \mu q_1^3/4$, we obtain the dynamical system and its representation in polar coordinates:

$$\begin{cases} \dot{u} = \mu u - \frac{\mu}{4}u^3 + v \\ \dot{v} = -u - \frac{\mu}{4}u^2 v \end{cases} \leftrightarrow \begin{cases} \dot{q} = \frac{\mu}{4}(4-q^2)q\cos^2\theta \\ \dot{\theta} = -1 - \mu\cos\theta\sin\theta \end{cases},$$

where $\rho(q) = (4-q^2)q$ and $P(q,\theta) = \mu\cos^2\theta/4 \geqslant 0$ ($\mu > 0$). Therefore, we can provide an exact construction of potential function for Eq. (6) (see Fig. 3):

$$\begin{aligned} \phi(\mathbf{q}) = \frac{1}{4}&\left[q_1^2 + \left(q_2 - \mu q_1 + \frac{\mu}{4}q_1^3\right)^2\right] \\ &\times\left[q_1^2 + \left(q_2 - \mu q_1 + \frac{\mu}{4}q_1^3\right)^2 - 8\right]. \end{aligned} \tag{15}$$

We note that the potential function Eq. (15) has the minimal value at the stable limit cycle $\dot{q}_1 = \mu q_1 - \frac{\mu}{4}q_1^3 \pm \sqrt{4 - q_1^2}$ and a local maximum value at the unstable fixed point (0,0); see Fig. 3. Expressions for other dynamical components can be constructed through Eq. (14). We use the representation below with ($u = q_1, v = q_2 - \mu q_1 + \mu q_1^3/4$) and $J(\mathbf{q})$ the Jacobian matrix $J(\mathbf{q}) = \partial(u,v)/\partial(q_1,q_2) = \begin{pmatrix}1 & 0\\ -\mu+3\mu q_1^2/4 & 1\end{pmatrix}$:

$$\begin{aligned} S(\mathbf{q}) &= \frac{\mu(4-u^2-v^2)^2 u^2}{4(\dot{u}^2+\dot{v}^2)} J(\mathbf{q})^\tau J(\mathbf{q}), \\ A(\mathbf{q}) &= -\frac{(4-u^2-v^2)(u^2+v^2+\mu u v)}{\dot{u}^2+\dot{v}^2} J(\mathbf{q})^\tau \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} J(\mathbf{q}), \\ D(\mathbf{q}) &= \frac{\mu u^2}{4(u^2+v^2)} J(\mathbf{q})^{-1} J(\mathbf{q})^{-\tau}, \\ Q(\mathbf{q}) &= \frac{u^2+v^2+\mu u v}{(u^2+v^2)(4-u^2-v^2)} J(\mathbf{q})^{-1} \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} J(\mathbf{q})^{-\tau}. \end{aligned} \tag{16}$$

The result obtained can be understood as the following: The stochastic dynamics Eq. (6) with a position-dependent diffusion matrix $D(\mathbf{q})$ given in Eq. (16) has the explicitly constructed potential function $\phi(\mathbf{q})$ [Eq. (15)] and a corresponding Boltzmann-Gibbs steady state distribution [Eq. (4)]. For the matrix $Q(\mathbf{q})$, one can check $\partial_j[Q_{ij}(\mathbf{q})\rho_s(\mathbf{q})] \neq 0$, leading to the absence of detailed balance. The stochastic integration used is the A-type [see Eq. (3)] different from the traditional Ito's or Stratonovich's [5]. A clear difference can be viewed in Figs. 4(a) and 4(b): For Ito integration, the structure of the limit cycle disappears after the noise is introduced; but for the A-type, the limit cycle can be directly recognized.

The construction is valid for arbitrary noise strength. A criterion to roughly measure the stability of a deterministic dynamics under the perturbation of noise is $\Delta\phi/\epsilon$. In the case

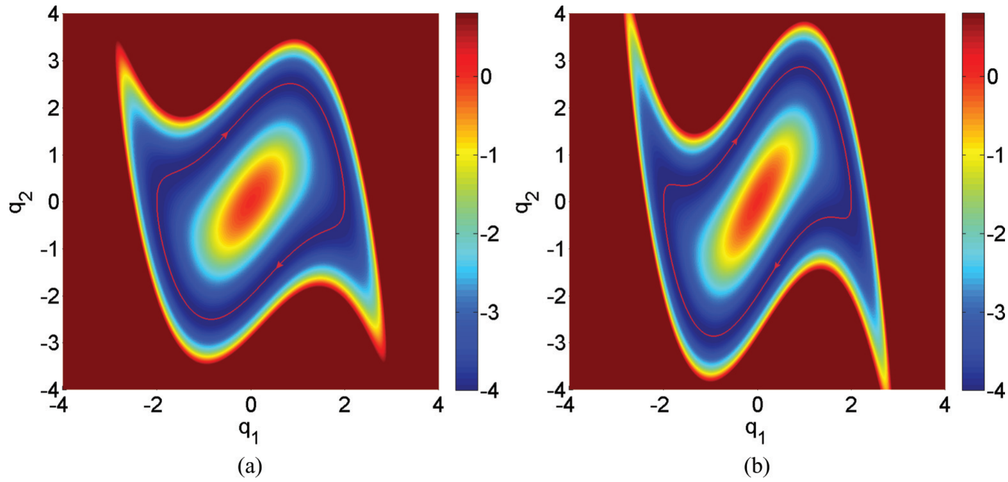


FIG. 3. (Color online) Potential function Eq. (15): The red lines denote the limit cycles. The graphs are drawn below a preset upper bound value 1, the phase variables are q_1 and q_2. (a) $\mu = 1$; (b) $\mu = 1.5$.

of the limit cycle in Fig. 3(a), the $\Delta\phi$ is the potential difference between the unstable point at (0,0) and the limit cycle, and the value is about 5. When noise is small, $\Delta\phi/\epsilon \approx 5 > 1$, the system behaves like a deterministic system; see Fig. 4(b). For $\Delta\phi/\epsilon \approx 1/2$, we can see from Fig. 4(c) the influence of the deterministic dynamics becomes weak. When noise is large $\Delta\phi/\epsilon \approx 1/100 \ll 1$, the distribution trends to be uniform and the dynamical behavior is nearly random; see Fig. 4(d).

Note that when approaching the limit cycle $(4 - u^2 - v^2) \to 0$, the force induced by the potential gradient goes

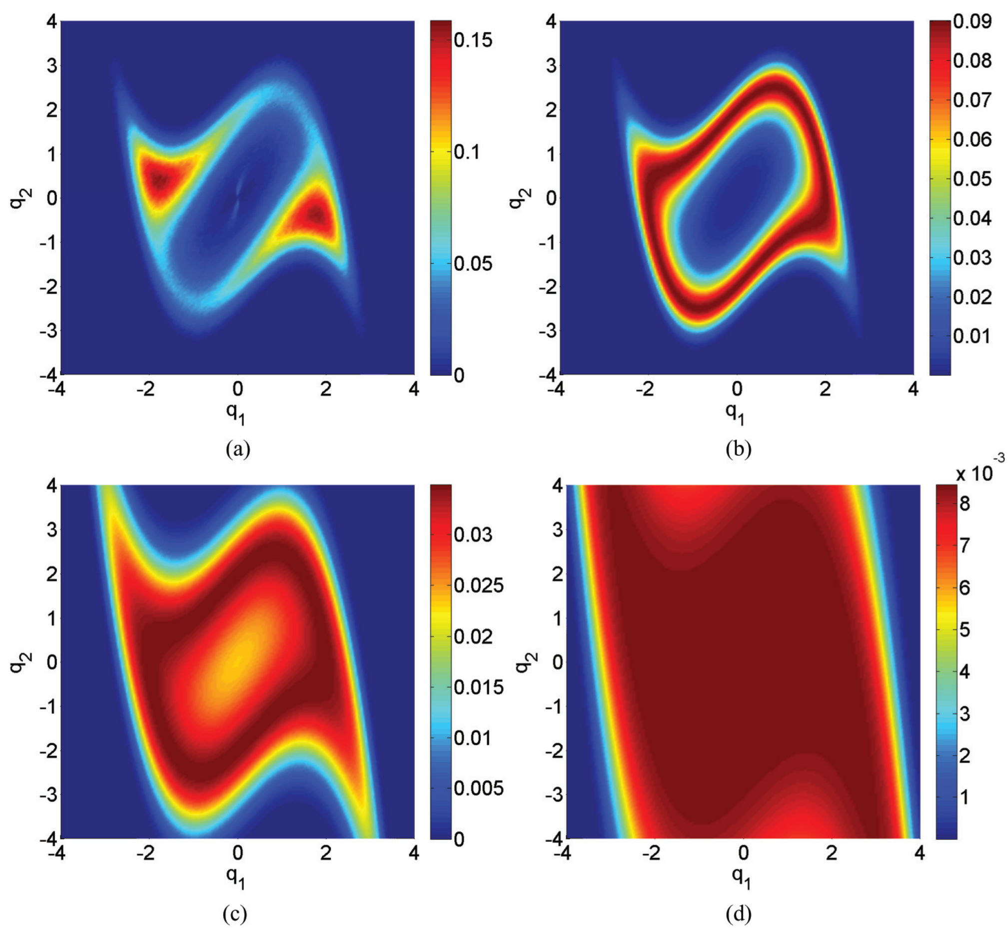


FIG. 4. (Color online) Comparison of probability distribution functions ($\mu = 1$): (a) Result of Ito simulation with $\epsilon = 1$, the structure of the limit cycle cannot be recognized from the distribution; (b) result of A-type integration with $\epsilon = 1$, $\Delta\phi/\epsilon \approx 5$, the limit cycle can be seen with uniformly distributed probability density; (c) A-type distribution with noise strength $\epsilon = 10$, $\Delta\phi/\epsilon \approx 1/2$; (d) A-type distribution with larger noise $\epsilon = 500$, $\Delta\phi/\epsilon \approx 1/100$, the region within the limit cycle is almost flat.

to zero; the Lorentz force matrix $A(\mathbf{q})$ goes zero in the same order and changes its sign at the limit cycle (since $0 < \mu < 2$); the friction matrix $S(\mathbf{q})$ goes to zero in a higher order. The dynamics at the limit cycle is no longer dissipative but conserved in this limit, reaching a stable cycle. Thus the potential should be equal on limit cycles where the system is conserved. We note that this is consistent with the definition of a Lyapunov function [28]. The particle is moving repeatedly along the cycle, the same as a conserved system moving along the Hamiltonian. The speed of the particle can be nonconstant. The singularity problem for this construction has been discussed in Ref. [41]. Previous works focus more on the diffusion matrix, ignoring the important role played by the friction matrix $S(\mathbf{q})$ and the Lorentz force matrix $A(\mathbf{q})$.

IV. CONCLUSION

Applying a new stochastic interpretation (A-type), we have exactly constructed a potential function (also Lyapunov function) for a class of limit cycles with noise, from rotationally symmetric to more general systems, where a specific example is a van der Pol type oscillator. These systems have been analyzed through the explicitly obtained dynamical components $S(\mathbf{q})$, $A(\mathbf{q})$, and $\phi(\mathbf{q})$: Near the limit cycle, the strength of the magnetic field $A(\mathbf{q})$ has the same order with that of the potential gradient $\nabla\phi(\mathbf{q})$, and the friction $S(\mathbf{q})$ goes to zero faster than that of the potential gradient. In the limit case, the dynamics is conserved at the limit cycle. Nevertheless, the diffusion matrix can be finite at the limit cycle. Using A-type integration, the steady state distribution of a system is the Boltzmann-Gibbs type. A correspondence between stochastic and deterministic dynamics is achieved. This property cannot be held by using the traditional Ito's or Stratonovich's integration. The framework is available for arbitrary noise strength. The stability of a limit cycle can be roughly measured by the ratio between potential depth and noise strength. Since new measuring techniques for Brownian motion are available, the theoretical results here may be experimentally verified.

ACKNOWLEDGMENTS

This work was supported in part by the National 973 Project No. 2010CB529200; by the Natural Science Foundation of China, Grants No. NSFC61073087 and No. NSFC91029738; and by the grants from the State Key Laboratory of Oncogenes and Related Genes (No. 90-10-11).

APPENDIX: EQUATIONS FOR POTENTIAL FUNCTION

Based on the dynamical equivalence between Eqs. (1) and (2), we can replace $\dot{\mathbf{q}}$ in Eq. (2) with the right-hand side of Eq. (1):

$$[S(\mathbf{q}) + A(\mathbf{q})]\,[\mathbf{f}(\mathbf{q}) + N(\mathbf{q})\xi(t)] = -\nabla\phi(\mathbf{q}) + \hat{N}(\mathbf{q})\xi(t). \tag{A1}$$

By an assumption that the deterministic and stochastic dynamics in Eq. (A1) are equal separately, we obtain

$$[S(\mathbf{q}) + A(\mathbf{q})]\,\mathbf{f}(\mathbf{q}) = -\nabla\phi(\mathbf{q}), \tag{A2}$$

$$[S(\mathbf{q}) + A(\mathbf{q})]\,N(\mathbf{q}) = \hat{N}(\mathbf{q}). \tag{A3}$$

Intuitively, this assumption on separation is plausible: The noise function is not differentiable but the deterministic forces are usually smooth, hence two very different mathematical objects; in addition, the stochastic and the deterministic forces have different physical origins. Replacing Eq. (A2) with an equivalent form, we obtain a potential condition (A4). The generalized curl operator is identical to the use in Eq. (14). Plugging Eq. (A3) into the fluctuation-dissipation theorem, $\hat{N}(\mathbf{q})\hat{N}^{\tau}(\mathbf{q}) = 2\epsilon S(\mathbf{q})$, we reach a generalized Einstein relation Eq. (A5):

$$\nabla \times \{[S(\mathbf{q}) + A(\mathbf{q})]\,\mathbf{f}(\mathbf{q})\} = 0, \tag{A4}$$

$$[S(\mathbf{q}) + A(\mathbf{q})]D(\mathbf{q})[S(\mathbf{q}) - A(\mathbf{q})] = S(\mathbf{q}). \tag{A5}$$

In principle, the potential function $\phi(\mathbf{q})$ can be derived by solving the $n(n-1)/2$ partial differential equations (under proper boundary conditions) [Eq. (A4)], together with the $n(n+1)/2$ equations given by Eq. (A5). Here we have n^2 unknowns in $[S(\mathbf{q}) + A(\mathbf{q})]$ and n^2 equations. It can also be calculated numerically through a gradient expansion [13].

In the one-dimensional case, $A = 0$, let $\epsilon = k_B T$, if the friction γ is a constant, then $S = \gamma/k_B T$, and Eq. (A5) reduces to $SD = \gamma D/k_B T = 1$, which is the Einstein relation [43]. Equation (A5) is a generalized form of the Einstein relation in two ways: The diffusion matrix can be nonlinear dependent of the state variable, and the detailed balance condition can be broken ($A(\mathbf{q}) \neq 0$).

[1] C. W. Gardiner, *Handbook of Stochastic Methods: For Physics, Chemistry and the Natural Cciences*, 3rd ed. (Springer-Verlag, Berlin, 2004).
[2] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. (Elsevier, Singapore, 2007).
[3] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer-Verlag, Berlin, 1996).
[4] R. Mannella and P. V. E. McClintock, Fluct. Noise Lett. **11**, 1240010 (2012).
[5] R.-S. Yuan and P. Ao, J. Stat. Mech. (2012) P07010.
[6] J. Wang, L. Xu, and E. Wang, Proc. Natl. Acad. Sci. USA **105**, 12271 (2008).
[7] J.-H. Shi, T.-Q. Chen, R.-S. Yuan, B. Yuan, and P. Ao, J. Stat. Phys. **148**, 579 (2012).
[8] X.-M. Zhu, L. Yin, L. Hood, and P. Ao, Funct. Integr. Genomics **4**, 188 (2004).
[9] J. X. Zhou, M. D. S. Aliyu, E. Aurell, and S. Huang, J. R. Soc. Interface **9**, 3539 (2012).
[10] R. Ma, J. Wang, Z. Hou, and H. Liu, Phys. Rev. Lett. **109**, 248107 (2012).
[11] We note that the attractors, e.g., stable fixed points, are obtained from the deterministic dynamics. Based on Ito's or Stratonovich's interpretation, when zero-mean noise is added, the position of the stable fixed points may shift or the point may

even disappear in the steady state distribution. As a result, the calculation of transition probability from one stable fixed point to another can be subtle.
[12] This negative sign can be eliminated by redefine ϕ as $-\phi$ in Eq. (2).
[13] P. Ao, J. Phys. A: Math. Gen. **37**, L25 (2004).
[14] C. Kwon, P. Ao, and D. J. Thouless, Proc. Natl. Acad. Sci. USA **102**, 13029 (2005).
[15] P. Ao, Commun. Theor. Phys. **49**, 1073 (2008).
[16] J.-N. Teramae and D. Tanaka, Phys. Rev. Lett. **93**, 204103 (2004).
[17] K. Yoshimura and K. Arai, Phys. Rev. Lett. **101**, 154101 (2008).
[18] J.-N. Teramae, H. Nakao, and G. B. Ermentrout, Phys. Rev. Lett. **102**, 194102 (2009).
[19] R. F. Galán, Phys. Rev. E **80**, 036113 (2009).
[20] H. Ge and H. Qian, Chaos **22**, 023140 (2012).
[21] J. C. Leloup, D. Gonze, and A. Goldbeter, J. Biol. Rhythms **14**, 433 (1999).
[22] J. J. Tyson, K. C. Chen, and B. Novak, Curr. Opin. Cell Biol. **15**, 221 (2003).
[23] M. Carneiro and D. L. Hartl, Proc. Natl. Acad. Sci. USA **107**, 1747 (2010).
[24] H. B. Callen and R. F. Greene, Phys. Rev. **86**, 702 (1952).
[25] R. Kubo, Rep. Prog. Phys. **29**, 255 (1966).
[26] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II: Nonequilibrium Statistical Physics*, 2nd ed. (Springer-Verlag, Heidelberg, 1995).
[27] M. San Miguel and S. Chaturvedi, Z. Physik B **40**, 167 (1980).
[28] J. P. La Salle, *The Stability of Dynamical Systems*, Vol. 25 (Society for Industrial and Applied Mathematics, Philadelphia, 1976).
[29] R. Graham and T. Tél, Phys. Rev. A **33**, 1322 (1986).
[30] G. Volpe, L. Helden, T. Brettschneider, J. Wehr, and C. Bechinger, Phys. Rev. Lett. **104**, 170602 (2010).
[31] T.-Z. Sun, R.-S. Yuan, W. Xu, F. Zhu, and P.-P. Shen, Phys. Biol. **7**, 036008 (2010).
[32] Y. Tang, R.-S. Yuan, and Y.-A. Ma, Phys. Rev. E **87**, 012708 (2013).
[33] H. Qian, S. Saffarian, and E. L. Elson, Proc. Natl. Acad. Sci. USA **99**, 10376 (2002).
[34] B. van der Pol, Phil. Mag. **2**, 978 (1926).
[35] K. Tomita and H. Tomita, Prog. Theor. Phys. **51**, 1731 (1974).
[36] M. K. Cameron, Physica D **241**, 1532 (2012).
[37] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (Perseus Books, Reading, 1994), p. 201.
[38] D. M. Weinreich, S. Sindi, and R. A. Watson, J. Stat. Mech. (2013) P01001.
[39] M. A. Abdelkader, J. Math. Anal. Appl. **218**, 308 (1998).
[40] X.-M. Zhu, L. Yin, and P. Ao, Int. J. Mod. Phys. B **20**, 817 (2006).
[41] R.-S. Yuan, Y.-A. Ma, B. Yuan, and P. Ao, arXiv:1012.2721v1 [nlin.CD] (2010).
[42] A sufficient condition for the equivalence of Ito and A-type is both D and Q are constant.
[43] A. Einstein, Ann. Phys. (Leipzig) **17**, 549 (1905).