Left passage probability of Schramm-Loewner Evolution

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SLE($\kappa,\vec{\rho}$) is a variant of Schramm-Loewner Evolution (SLE) which describes the curves which are not conformal invariant, but are self-similar due to the presence of some other preferred points on the boundary. In this paper we study the left passage probability (LPP) of SLE($\kappa,\vec{\rho}$) through field theoretical framework and find the differential equation governing this probability. This equation is numerically solved for the special case $\kappa = 2$ and $h_\rho = 0$ in which h_ρ is the conformal weight of the boundary changing (bcc) operator. It may be referred to loop erased random walk (LERW) and Abelian sandpile model (ASM) with a sink on its boundary. For the curve which starts from ξ_0 and conditioned by a change of boundary conditions at x_0, we find that this probability depends significantly on the factor $x_0 - \xi_0$. We also present the perturbative general solution for large x_0. As a prototype, we apply this formalism to SLE($\kappa,\kappa - 6$) which governs the curves that start from and end on the real axis.

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I. INTRODUCTION

The recent breakthrough in two-dimensional (2D) critical phenomena, referred to Schramm-Loewner Evolution (SLE), has provided us with a new interpretation of the traditional conformal field theory (CFT) and Coulomb gas approaches. According to Schramm's idea [1] one can describe the interfaces of 2D critical statistical models via a stochastic growth process in which statistical models fall into one-parameter classes labeled by a diffusivity parameter, namely κ. The examples of statistical models which are described by SLE are the Ising model [2], the Potts model [3], the O(n) model [4], the Abelian sandpile model (ASM) [5], etc., and some geometrical models such as self-avoiding walks [6], percolation [7], loop erased random walk (LERW) [8], watershed lines in random landscapes [9], etc. This description focuses on some nonlocal objects in the statistical models, in contrast to CFT in which one deals with local fields. These nonlocal objects can be interfaces of statistical models, be it the boundary of clusters in the Ising model, or the boundary of avalanches in the ASM, or loops in the O(n) model.

A very crucial step towards understanding SLE was taken by Bauer *et al.* [10] to connect this theory to CFT. They found a simple relation between the diffusivity parameter κ in SLE and the central charge c in CFT. This connection helps to conjecture CFT universality classes for less-known statistical models and obtain the operator content of the CFT corresponding to SLE observables. One way to extract κ from the statistical models lies within using these observables such as crossing probability and left passage probability (LPP) [11,12]. This makes it crucial to investigate these mathematical quantities exactly. The LPP of chordal SLE can be expressed in terms of κ within Schramm's formula [11]. This probability is the solution of a differential equation obtained by conformal invariance of the probability measure of the growing SLE curve.

SLE($\kappa,\vec{\rho}$) is a variant of SLE(κ) in which there are some more preferred points on the boundary, affecting the growth process of the SLE curve. The relation of this generalization of SLE to CFT and its operator content and also its correspondence to the Coulomb gas is widely studied [13–15]. Since in some models, one deals with the interfaces which have some preferred points on the boundary of their domain, analyzing the statistical observables for SLE($\kappa,\vec{\rho}$) seems to be crucial. The example which we have investigated in this paper is the LPP of loop erased random walk (LERW) (with $\kappa = 2$ corresponding to $c = -2$ CFT) with a preferred point on the boundary with conformal weight $h_\rho = 0$. This case fits the problem of avalanche frontiers in the Abelian sandpile model (ASM) in the presence of a sink point on the boundary in which the grains (defined in the model) dissipate [5,16].

In the next section we briefly introduce SLE and its variant SLE($\kappa,\vec{\rho}$). Sections III and IV are devoted to the LPP of the SLE($\kappa,\kappa - 6$) and the more general case SLE($\kappa,\vec{\rho}$). In Sec. V we present the numerical solution for the case $\kappa = 2$ and $h_\rho = 0$. Section V is devoted to the perturbative analytic solution for large x_0 at which the boundary condition changes.

II. SLE

SLE theory describes the critical behavior of 2D statistical models by focusing on their geometrical features such as their interfaces and classifying them to the one-parameter classes SLE$_\kappa$. These domain walls are some nonintersecting curves which directly reflect the status of the system in question and are supposed to have two properties: conformal invariance and the domain Markov property. For good introductory review see Refs. [14,17]. There are three kinds of SLE: chordal SLE in which the random curve starts from zero and ends at infinity, dipolar SLE in which the curve starts from and ends at the boundary, and radial SLE in which the curve starts from the boundary and ends in the bulk. In this paper we deal with chordal and dipolar SLEs.

A. Chordal SLE

Let us denote the upper half plane by H and γ_t as the SLE trace grown up to time t. SLE$_\kappa$ is a growth process defined via conformal maps which are solutions of stochastic Loewner's

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equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \tag{1}$$

in which the initial condition is $g_0(z) = z$ and the driving function ξ_t is proportional to a one-dimensional Brownian motion B_t, i.e., $\xi_t = \sqrt{\kappa} B_t$ in which κ is the diffusivity parameter defined above. Define τ_z as the time for which for fixed z, $g_t(z) = \xi_t$ and the hull as $K_t = \overline{\{z \in H : \tau_z \leqslant t\}}$. It is notable that the complement $H_t := H \backslash K_t$ is simply connected so that one can conclude that every point which is separated from the infinity by the SLE trace will be involved in K_t. The map $g_t(z)$ is well defined up to time τ_z. This map is the unique conformal mapping $H_t \to H$ with $g_t(z) = z + \frac{2t}{z} + O(\frac{1}{z^2})$ as $z \to \infty$ known as hydrodynamical normalization.

There are three phases for SLE traces. For $0 < \kappa \leqslant 4$ the trace is non-self-intersecting and it does not hit the real axis; in this case the hull and the trace are identical: $K_t = \gamma_t$. This is called the "dilute phase." For $4 < \kappa < 8$, the trace touches itself and the real axis so that a typical point is surely swallowed as $t \to \infty$ and $K_t \neq \gamma_t$. This phase is called the "dense phase." Finally for $\kappa \geqslant 8$ the trace is space filling. There is a connection between the first two phases: for $4 \leqslant \kappa \leqslant 8$ the frontier of K_t, i.e., the boundary of H_t minus any portions of the real axis, is a simple curve which is locally a SLE$_{\tilde{\kappa}}$ curve with $\tilde{\kappa} = \frac{16}{\kappa}$, i.e., it is in the dilute phase [18]. The crucial question about the connection between SLE and CFT has been addressed by M. Bauer *et al.* [10] in which it was shown that the bcc operator in CFT corresponding to the change of the boundary condition at the point from which the SLE trace starts or ends is the operator having a null vector at the second level with conformal weight $h_1(\kappa) = \frac{6-\kappa}{2\kappa}$ and the central charge $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$. This observation helps us to construct the CFT correspondence of the observables in SLE as we will see in the following sections.

B. SLE$(\kappa,\vec{\rho})$

SLE$(\kappa,\vec{\rho})$ is a generalization of SLE. As above, we consider the upper half plane. The parameter κ, as was defined above, identifies the local properties of the model in hand, and the parameters $\vec{\rho} \equiv (\rho_1,\rho_2,\ldots,\rho_n)$ have to do with the boundary condition changes (bc) imposed on the points on the real axis $x_1,x_2,\ldots,x_n$ (except the origin from which the curve starts). For example, for the dipolar setup of SLE on the upper half plane in which the planar curves start from the origin and end on a point on the real axis (we name it x_∞), we have $n = 1$ and $\rho = \kappa - 6$ [19]. The stochastic equation governing SLE$(\kappa,\vec{\rho})$ is the same as formula (1) but the driving function has a different form:

$$d\xi_t = \sqrt{\kappa} dB_t + \frac{\rho_1}{\xi_t - g_t(x_1)} dt + \frac{\rho_2}{\xi_t - g_t(x_2)} dt + \cdots + \frac{\rho_n}{\xi_t - g_t(x_n)} dt. \tag{2}$$

For review see [14,17]. In determining the operator content of CFT corresponding to SLE$(\kappa,\vec{\rho})$, the important feature is that the conformal weight of the bcc operator corresponding to boundary changing in x_i, denoted by h_{ρ_i}, is related to ρ_i via the simple relation $h_{\rho_i}(\kappa) = \frac{\rho_i(\rho_i+4-\kappa)}{4\kappa}$, $(i = 1,2,\ldots,n)$ [20]. We will use this in the following sections.

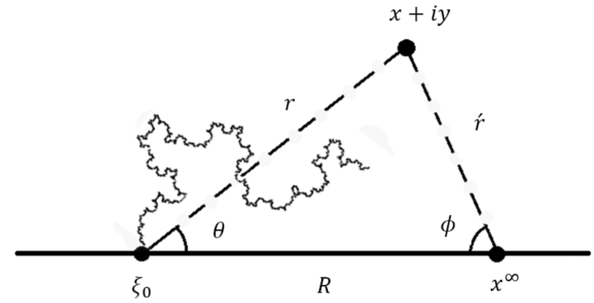

FIG. 1. The schematic picture of a triangle involving the points ξ_0, x^∞ and $x + iy$.

III. LPP OF SLE $(\kappa, \rho_c = \kappa - 6)$

In this section we consider the probability of the event that a point (x,y) lies within the hull of a dipolar SLE at $t = \infty$, i.e., is separated from the infinity. We name this probability $P(w,\bar{w},\xi_0,x^\infty)$ in which $w = x + iy$ is the detection point, $\bar{w}$ is its complex conjugate, ξ_0 is the point from which the SLE trace starts, and x^∞ is the point on the real axis at which the curve ends. This probability has been calculated for the chordal case [11]. Figure 1 schematically shows the situation. To proceed, we introduce the coordinates θ and ϕ, indicated in the figure as follows:

$$\begin{aligned} z - \xi_0 &= re^{i\theta} = R\frac{\sin\phi}{\sin(\phi+\theta)}e^{i\theta}, \\ z - x^\infty &= re^{i(\pi-\phi)} = R\frac{\sin\theta}{\sin(\phi+\theta)}e^{-i\phi}, \\ x^\infty - \xi_0 &= R, \quad z - \bar{z} = R\frac{\sin\phi}{\sin(\phi+\theta)}(r^{i\theta} - r^{-i\theta}). \end{aligned} \tag{3}$$

According to Eq. (2), the driving function obeys the following equation for this case [19]:

$$d\xi_t = \sqrt{\kappa} dB_t + \frac{\kappa - 6}{\xi_t - g_t(x_\infty)} dt, \tag{4}$$

which shows that the driving function acquires a drift term. Suppose that the curve is grown up to time δt. Let us uniformize the domain $H\backslash\gamma_{\delta t}$ to H by a hydrodynamically normalized Loewner map $g_{\delta t}$. Using conformal invariance and martingale property [15], one can conclude that the ensemble average [21] of the LPP for the mapped points [denoted by $P_{\delta t}(w_{\delta t},\bar{w}_{\delta t},\xi_{\delta t},x^\infty_{\delta t})$ in which the points are evolved according to Eqs. (1) and (4)] is the same as the original LPP [at $t = 0$ in the domain H denoted by $P(w,\bar{w},\xi_0,x^\infty)$], i.e., $\delta P \equiv \mathbf{E}[P_{\delta t} - P] = 0$ in which $\mathbf{E}[\ ]$ denotes the ensemble average. Then one can write ($\delta t \to dt$)

$$P(w,\bar{w},\xi_0,x^\infty) = \mathbf{E}\left[P\left(w + \frac{2dt}{w-\xi_0},\bar{w} + \frac{2dt}{\bar{w}-\xi_0},\xi_0 + d\xi_0, x^\infty + \frac{2dt}{x^\infty - \xi_0}\right)\right]. \tag{5}$$

After some Ito calculations one obtains

$$\left[\frac{2}{w-\xi_0}\partial_w + \frac{2}{\bar{w}-\xi_0}\partial_{\bar{w}} + \rho_c\frac{1}{\xi_0 - x^\infty}\partial_{\xi_0} - \frac{2}{\xi_0 - x^\infty}\partial_{x^\infty} + \frac{\kappa}{2}\partial^2_{\xi_0}\right]P(w,\bar{w},\xi_0,x^\infty) = 0. \quad (6)$$

It is obvious that this equation has the symmetry under transformation $\xi_0 \to \xi_0 + a$, Re$[w]\to$ Re$[w]$ $+a$, $x^\infty \to x^\infty + a$. So we can replace $\partial_{x^\infty} = -\partial_{\xi_0} - \partial_x$, $(x = \text{Re}[w])$. To solve the above equation, we need to predict the more exact form of P from its CFT counterpart. This reduces Eq. (6) to a single variable differential equation.

A. CFT Background

In this subsection we study the CFT interpretation of LPP of SLE$(\kappa,\kappa-6)$. Suppose that, in CFT corresponding to the underling model, $\hat{O}$ is the operator which detects the left passage, i.e., the left passage probability is the expectation value of this operator. As we have boundary conformal field theory (real axis) with two boundary changing operators (one in ξ_0 and another in x^∞), the LPP can be written as

$$P(w,\bar{w},\xi_0,x^\infty) = \frac{\langle \hat{O}(x,y)\hat{O}(x,-y)\psi(\xi_0)\psi(x^\infty)\rangle}{\langle\psi(\xi_0)\psi(x^\infty)\rangle}. \quad (7)$$

In this equation, the operator $\hat{O}(x,-y)$ is the image of $\hat{O}(x,y)$ with respect to the real axis. ψ is the boundary changing operator for the CFT corresponding to underling SLE whose conformal weight is $h_1(\kappa) = \frac{6-\kappa}{2\kappa}$ with second level null vector:

$$\left(\frac{\kappa}{2}L^2_{-1} - 2L_{-2}\right)\psi = 0 \quad (8)$$

in which L_n's are the generators of the Virasoro algebra satisfying

$$[L_n,L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \quad (9)$$

In this equation c is the central charge of the CFT model in hand. Equation (8) leads to the following equation for the mentioned correlation function [22]:

$$\left(\frac{\kappa}{2}\mathcal{L}^2_{-1} - 2\mathcal{L}_{-2}\right)f(w,\bar{w},\xi_0,x^\infty) = 0 \quad (10)$$

with

$$\begin{aligned} f(w,\bar{w},\xi_0,x^\infty) &\equiv \langle\hat{O}(x,y)\hat{O}(x,-y)\psi(\xi_0)\psi(x^\infty)\rangle \\ &= \frac{1}{(\xi_0 - x^\infty)^{2h_1(\kappa)}}P(w,\bar{w},\xi_0,x^\infty), \\ \mathcal{L}_{-1} &= \frac{\partial}{\partial\xi_0}, \\ \mathcal{L}_{-2} &= \sum_i\left(\frac{h_i}{(z_i-\xi_0)^2} - \frac{1}{z_i-\xi_0}\frac{\partial}{\partial z_i}\right). \end{aligned} \quad (11)$$

The above sum is over each field in the correlation function Eq. (7) except ξ_0. So f satisfies the equation

$$\left[\frac{\kappa}{4}\partial^2_{\xi_0} - h_O\text{Re}\left(\frac{1}{(z-\xi_0)^2}\right) + \frac{1}{z-\xi_0}\frac{\partial}{\partial z} + \frac{1}{\bar{z}-\xi_0}\frac{\partial}{\partial\bar{z}} + \frac{1}{x^\infty - \xi_0}\frac{\partial}{\partial x^\infty} - \frac{h_1(\kappa)}{(x^\infty-\xi_0)^2}\right]f = 0. \quad (12)$$

In this equation h_O is conformal weight of $\hat{O}$. It is notable that this equation can be written in terms of x^∞ in which one exchanges the rule of ξ_0 and x^∞. The chordal case can be obtained in the limit $x^\infty \to \infty$. In this limit we have $f = P(w,\bar{w},\xi_0,x^\infty)$ in Eq. (10) and

$$\left[\frac{\kappa}{4}\partial^2_{\xi_0} - h_O\text{Re}\left(\frac{1}{(z-\xi_0)^2}\right) + \frac{1}{z-\xi_0}\frac{\partial}{\partial z} + \frac{1}{\bar{z}-\xi_0}\frac{\partial}{\partial\bar{z}}\right]f = 0. \quad (13)$$

Comparing Eq. (12) with the equation of LPP of the chordal case [14], one obtains $h_O = 0$. Substituting f from Eq. (10) into Eq. (12), it easy to check that the equation governing P is the same as Eq. (6). It is known that the global conformal symmetry can fix four point functions up to a function of the crossing ratios [22]. In this case, letting $h_O = 0$ we have

$$\begin{aligned} P &= y^{(2/3)h_1(\kappa)}(x^\infty - \xi_0)^{(2/3)h_1(\kappa)}[(x-x^\infty)^2 + y^2]^{(-1/3)h_1(\kappa)} \\ &\quad\times[(x-\xi_0)^2 + y^2]^{(-1/3)h_1(\kappa)}g(\kappa,\eta,\bar{\eta}) \\ &= (\eta\bar{\eta})^{(-1/3)h_1(\kappa)}\, g(\kappa,\eta,\bar{\eta}) \equiv \tfrac{1}{2}(\eta+\bar{\eta})\, h(\kappa,\eta,\bar{\eta}) \end{aligned} \quad (14)$$

in which η is the crossing ratio, i.e., $\frac{(w-\xi_0)(\bar{w}-x^\infty)}{y(x^\infty-\xi_0)}$ and $\bar{\eta}$ is its complex conjugate and $g(\kappa,\eta,\bar{\eta})$ and $h(\kappa,\eta,\bar{\eta})$ are functions of crossing ratios which should be determined. So the finding of P reduces to finding h. Let $u \equiv \text{Re}[\eta] = \frac{x(x-x^\infty)+y^2}{yx^\infty}$ (we set $\xi_0 = 0$). After some calculations one obtains

$$4u\partial_u P + \frac{\kappa}{2}(u^2+1)\partial^2_u P = 0. \quad (15)$$

In terms of θ and ϕ, u is equal to $\cot(\theta+\phi)$. The solution of this equation, with the boundary conditions $P = 1$ for $\theta = \pi$ and $P = 0$ for $\theta = 0$, is

$$P = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)}\,{}_2F_1\left(\frac{1}{2},\frac{4}{\kappa},\frac{3}{2},-\cot^2(\theta+\phi)\right) \times \cot(\theta+\phi). \quad (16)$$

This result can be derived directly from the chordal case (the $x^\infty \to \infty$ limit, or equivalently $\phi \to 0$). In the chordal case [19]

$$P_{\text{chordal}} = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)}\,{}_2F_1 \times\left[\frac{1}{2},\frac{4}{\kappa},\frac{3}{2},-\left(\frac{x-\xi_0}{y}\right)^2\right]\frac{x-\xi_0}{y}. \quad (17)$$

The corresponding probability for the dipolar SLE can be obtained using the map $\varphi = \frac{x^\infty w}{x^\infty - w}$. Under this map, $x + iy \to \frac{x^\infty(xx^\infty - x^2 - y^2)}{(x-x^\infty)^2+y^2} + i\frac{x^2_\infty y}{(x-x^\infty)^2+y^2}$. The probability that the point $x+iy$ is swallowed by a SLE curve in the dipolar case is equal to the probability of the left passage of the same point in the chordal setup, i.e., the left passage probability of the mapped point $\varphi(x+iy)$. Using this point, one can write

$$P_{\text{dipolar}}(x+iy)=P_{\text{chordal}}[\varphi(x+iy)]=\frac{1}{2}+\frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\Gamma\left(\frac{8-\kappa}{2\kappa}\right)}{}_2F_1\left[\frac{1}{2},\frac{4}{\kappa},\frac{3}{2},-\left(\frac{(x-\xi_0)(x^{\infty}-x)-y^2}{y(x^{\infty}-\xi_0)}\right)^2\right]\times\frac{(x-\xi_0)(x^{\infty}-x)-y^2}{y(x^{\infty}-\xi_0)}, \tag{18}$$

which is exactly the same as Eq. (16) (setting $\xi_0=0$).

IV. LPP OF SLE(κ,ρ)

In this section we apply the CFT formalism developed in the previous section to the SLE(κ,ρ) [$n=1$ in Eq. (2)] curves. Let us consider a curve growing from origin to infinity, conditioned by a change in the value of fields on the boundary which correspond to a scaling operator on this point with the weight $h_\rho=\frac{\rho(\rho+4-\kappa)}{4\kappa}$ [20]. The left passage probability equals a five point function in the corresponding conformal field theory:

$$P(x,y,\xi_0,x_0)=\frac{\langle\hat{O}(x,y)\hat{O}(x,-y)\psi(x_0)\psi(\xi_0)\psi(\infty)\rangle}{\langle\psi(x_0)\psi(\xi_0)\psi(\infty)\rangle}. \tag{19}$$

So the problem reduces to the calculation of three-point and five-point functions which satisfy the boundary conditions. As above, we define $f(x,y,\xi_0,x_0)$ the numerator of the right-hand side of Eq. (19). We have

$$P(x,y,\xi_0,x_0)=(x_0-\xi_0)^{h_\rho}f(x,y,\xi_0,x_0). \tag{20}$$

Using the null vector equation for ψ and after some calculations we obtain the following equation for P:

$$\left\{\frac{\kappa}{2}\partial_{\xi_0}^2+\frac{2}{z-\xi_0}\partial+\frac{2}{\bar{z}-\xi_0}\bar{\partial}+\frac{2}{x_0-\xi_0}\partial_{x_0}+\frac{\kappa h_\rho}{x_0-\xi_0}\partial_{\xi_0}+\frac{\frac{1}{2}h_\rho[\kappa(h_\rho+1)-8]}{(x_0-\xi_0)^2}\right\}P=0. \tag{21}$$

Using global conformal invariance, one can fix f up to a function of crossing ratios and prove that

$$P=[(x-\xi_0)^2+y^2]^{-1/3h_1+1/6h_\rho}[(x-x_0)^2+y^2]^{1/3h_1-1/2h_\rho}\times(x_0-\xi_0)^{-1/3h_1+1/2h_\rho}y^{1/3h_1+1/6h_\rho}g(\eta_1,\eta_2) \tag{22}$$

in which we have considered two independent crossing ratios $\eta_1\equiv\frac{(z-\xi_0)(\bar{z}-x_0)}{y(x_0-\xi_0)}$ and $\eta_2\equiv\frac{(z-\xi_0)}{y}=\lim_{x^{\infty}\to\infty}\frac{(z-\xi_0)(\bar{z}-x^{\infty})}{y(x^{\infty}-\xi_0)}$ and as before, $g(\eta_1,\eta_2)$ is a function of crossing ratios to be determined. It would be more convenient to work with the dimensionless variables $a\equiv\frac{x-\xi_0}{y}$ and $b\equiv\frac{x-x_0}{x_0-\xi_0}$. It is not difficult to check that P can be written in the following form:

$$P(x,y,\xi_0,x_0)=\left\{\frac{1+b}{a(1+a^2)}\left[1+\left(\frac{ab}{1+b}\right)^2\right]\right\}^{1/3h_1}\times\left(\frac{a^3(1+a^2)}{(1+b)^2[1+(\frac{ab}{1+b})^2]^3}\right)^{1/6h_\rho}g(a,b). \tag{23}$$

From the above formula, one realizes that all coefficients can be absorbed in g and so P would be a function of a and b, i.e., $P(x,y,\xi_0,x_0)=P(\frac{x-\xi_0}{y},\frac{x-x_0}{x_0-\xi_0})$. Combining Eqs. (21) and (23) and writing the derivatives in terms of a and b, the following differential equation for P is obtained:

$$\left[\lambda_{a^2}\partial_a^2+\lambda_{b^2}\partial_b^2+\lambda_{ab}\partial_a\partial_b+\lambda_a\partial_a+\lambda_b\partial_b+\lambda\right]P=0, \tag{24}$$

where

$$\begin{aligned}\lambda_{a^2}&=\frac{\kappa}{2},\quad \lambda_{b^2}=\frac{\kappa}{2}\left(\frac{b(1+b)}{a}\right)^2,\\ \lambda_{ab}&=-\kappa\frac{b(1+b)}{a},\quad \lambda_a=-\kappa h_\rho\frac{1+b}{a}+4\frac{a}{1+a^2},\\ \lambda_b&=2(1+b)\left[\frac{1}{1+a^2}-\left(\frac{1+b}{a}\right)^2\right]+\kappa(h_\rho+1)b\left(\frac{1+b}{a}\right)^2,\\ \lambda&=\frac{h_\rho}{2}[\kappa(h_\rho+1)-8]\left(\frac{1+b}{a}\right)^2\end{aligned} \tag{25}$$

with the boundary conditions

$$\begin{aligned}&\text{region: } x<0,\quad y\to 0^+(a\to-\infty)\Rightarrow P\to 0,\\ &\text{region: } x>0,\quad y\to 0^+(a\to+\infty)\Rightarrow P\to 1,\\ &\text{region: } x_0\to\infty(b\to-1)\Rightarrow P\to P_{\text{chordal}}.\end{aligned} \tag{26}$$

It is notable that in the limit $b\to-1$, Eq. (24) becomes

$$4a\partial_aP+\frac{\kappa}{2}(a^2+1)\partial_a^2P=0, \tag{27}$$

which is exactly Eq. (15) in which $a=u|_{x_\infty\to\infty}$, so the requirement of the last line of Eq. (26) is confirmed.

V. NUMERICAL RESULTS FOR LOOP ERASED RANDOM WALK

In this section we present the result for the loop erased random walk (LERW) model in the presence of a boundary condition changing at x_0 whose conformal weight in its corresponding CFT, h_ρ, is zero. One of the most important examples of this case is the Abelian sandpile model (ASM) with a sink on the boundary [16,23]. This model corresponds to $c=-2$ CFT and the frontier of the avalanches of sinkless ASM has been numerically proved to be LERW with $\kappa=2$ [5]. When the model contains a sink at which the grains dissipate, the statistical properties of the avalanche frontiers may change. In this model the boundary condition changing (bcc) operator corresponding to the change from open to close boundary condition is the twisting operator μ with the conformal weight $\frac{-1}{8}$. It has been proved that in the scaling limit, the operator corresponding to a sink on the boundary results from operator product expansion (OPE) of two twist operators, which is $\tilde{I}=-:\bar{\theta}\theta:(z)$ with the conformal weight 0 in which θ and $\bar{\theta}$ are Grassman variables, living in the ghost action in $c=-2$ CFT. This operator is the logarithmic partner of the identity operator I [16].

For this case we set $\kappa=2$ and $h_\rho=0$ in Eq. (24) and obtain

$$\left[a^2\partial_a^2+b^2(1+b)^2\partial_b^2-2ab(1+b)\partial_a\partial_b+\frac{4a^3}{1+a^2}\partial_a-2(1+b)\frac{1+b(1+a^2)}{(1+a^2)}\partial_b\right]P=0 \tag{28}$$

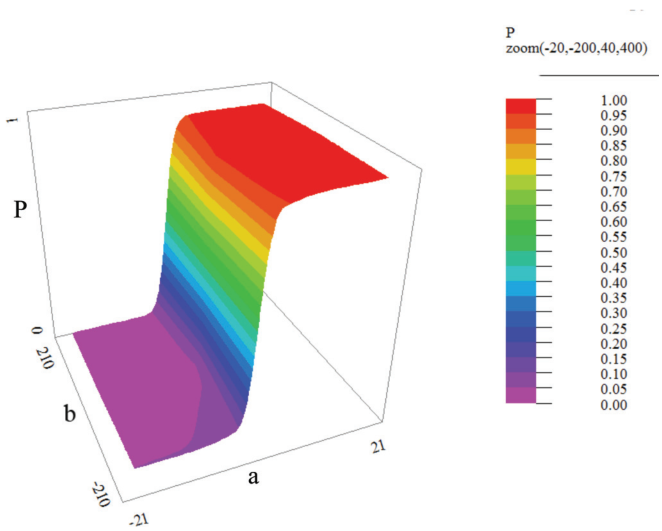

FIG. 2. (Color online) The 3D result of PDE Eq. (28).

with the same boundary conditions like Eq. (26). To solve this equation we have used the finite difference method. We have considered a $10^4 \times 10^4$ lattice and analyzed the solutions up to the distance 500 from the origin. Due to the singular behavior of Eq. (28) near $a = 0$ and $b = -1$, one should mesh the space near these lines more exact (smaller) than the other points. The numerical solution has been indicated in Fig. 2 in which the overall shape of LPP has been sketched in terms of both a and b variables. Figure 3 shows the contour plot of $P(a,b)$ in terms of a and b in which the amount of each contour is indicated in the graph. It is obvious in both graphs that the solution tends to unity in the rightmost part of the graph and to zero in the leftmost region. To be more exact, we have shown the LPP in terms of a for fixed values of b in Figs. 4 and 5. In Fig. 4 the result for $b = -1$ and the exact result of the chordal case have been sketched and compared. The agreement of results shows the reliability of our numerical solution. The solution for the other values of coordinate b have been shown in Fig. 5. As can be seen in this figure, by enlarging b, the graphs widen. This widening follows from a simple relation, i.e., $P(a,b) = P(\lambda a,-1)$ in which λ is the rescaling parameter and depends on the coordinate b. This parameter has been reported in Table I.

The other important case is the variation of LPP in terms of θ for the fixed x_0 and radius, i.e., $r = \sqrt{x^2 + y^2} = \text{const}$. To investigate this dependence, we note that $b = -1 + \alpha \frac{a}{\sqrt{1+a^2}}$ in which $\alpha = \frac{r}{x_0}$. Figure 6 shows LPP for various amounts of α. The interesting feature of this figure is that the LPP does significantly change for large values of α ($x_0 \ll r$) in which $x_0 \to \xi_0$ and for other values of α it does not change significantly. The direct consequence of this result is that the LPP depends mainly on $x_0 - \xi_0$ and the dependence on $x_0 - x$ is small.

Now we can fix a and observe the dependence of LPP on the b coordinate. This dependence has been shown in Fig. 7. The

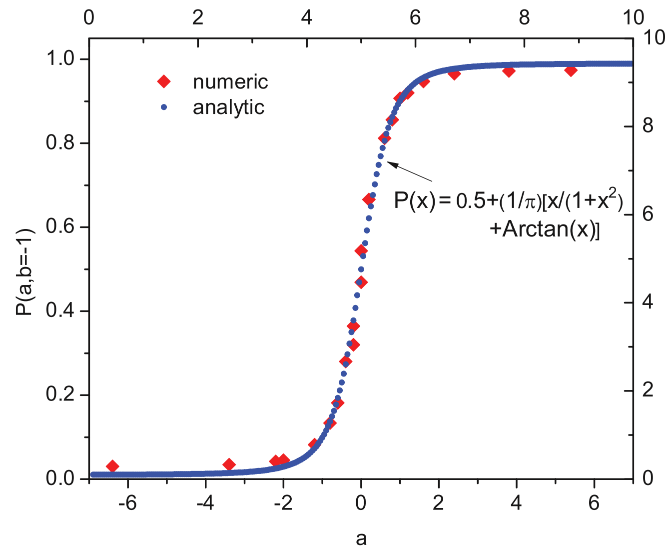

FIG. 4. (Color online) The result of PDE Eq. (28) along a axis in $b = -1$.

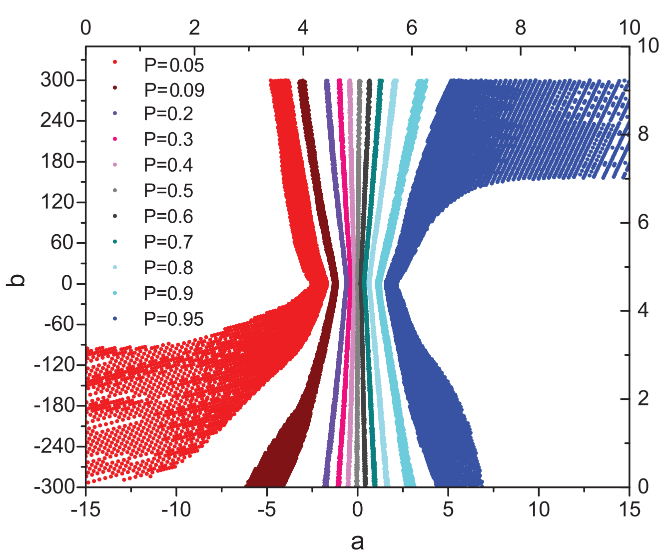

FIG. 3. (Color online) The contour plot of the numeric solution of PDE Eq. (28).

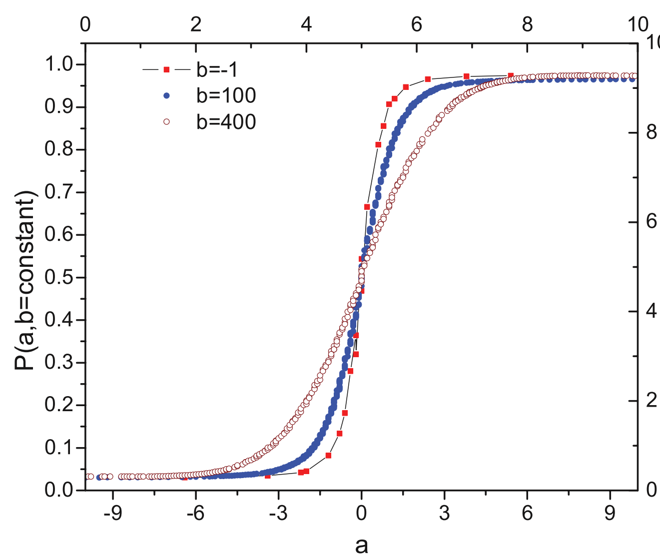

FIG. 5. (Color online) The LPP for various b.

TABLE I. The amounts of λ in terms of b.

b	$\lambda(b) \mp 0.02$
10	0.98
20	0.93
50	0.83
100	0.62
150	0.53
200	0.44
250	0.39

LPP decreases linearly with b and with the slopes depending on the lattice size. The important feature of this graph is the singular behavior in $b=-1$ for all amounts of a. For large values of a, it manifests itself in an immediate change of slope and for small values, in a sharp pick around $b=-1$. This signals a change of behavior, when one of the parameters $x-\xi_0$ or $x_0-\xi_0$ changes sign. In other words when $x_0 \to \xi_0$ or $x \to \xi_0$ we see a singular behavior as expected in Eq. (23).

A. Perturbative equation for large x_0

In this subsection we perturbatively analyze Eq. (28) in the large x_0 limit (equivalent to $b \to -1$). Let us define $\chi \equiv \frac{y(x_0-\xi_0)}{(x-x_0)^2+y^2} = \frac{a(1+b)}{(1+b)^2+a^2b^2}$. In the limit $x_0 \to \infty$, χ becomes $\epsilon = \frac{1+b}{a}$ which is a small quantity and we take it as the perturbation parameter. To the first order of ϵ, Eq. (28) can be written as

$$\left[\partial_{a^2} + \epsilon\partial_a\partial_b + \frac{2a\epsilon}{1+a^2}\partial_b + \frac{4a}{1+a^2}\partial_a\right]P = 0. \qquad (29)$$

We expand P in terms of ϵ. To the first order of ϵ, P is

$$P = P_0(a) + \frac{y(x_0-\xi_0)}{(x-x_0)^2+y^2}P_1(a) + O(\chi^2)|_{x_0\to\infty}$$
$$= P_0(a) + \epsilon P_1(a) + O(\epsilon^2). \qquad (30)$$

The above ansatz is the only answer satisfying the following conditions: in the limit $x_0 \to \infty$ or $y \to \infty$ it retrieves the

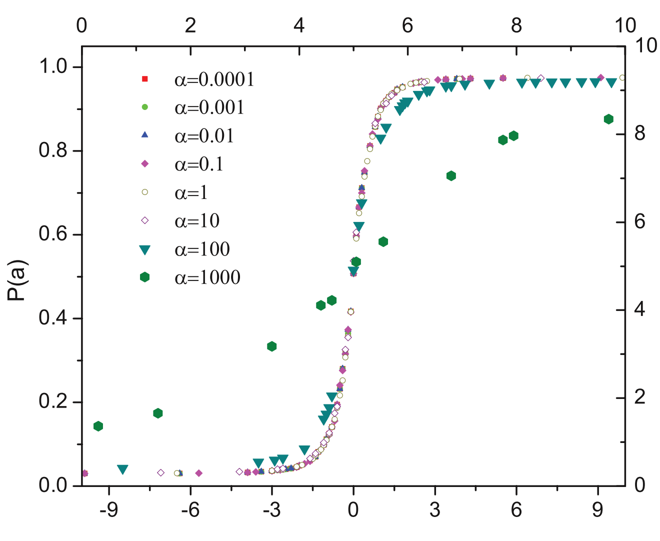

FIG. 6. (Color online) The LPP for various rates of x_0.

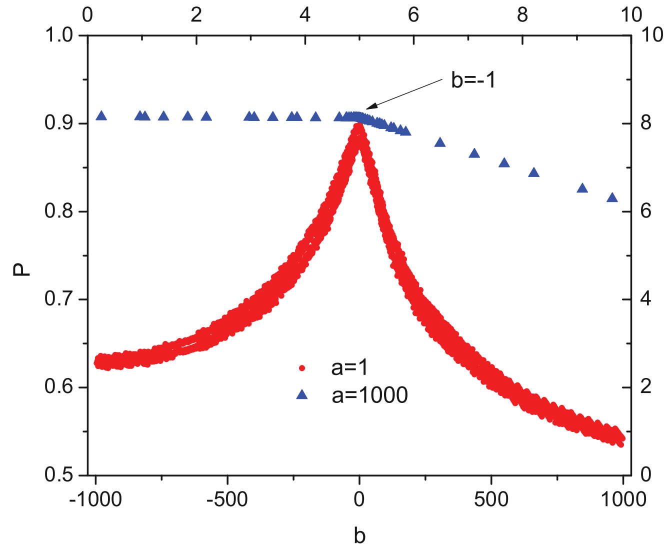

FIG. 7. (Color online) The LPP in terms of b for various rates of a. A singular behavior is seen in $b=-1$.

ρ-free solution [LPP of SLE($\kappa,\rho=0$)] as expected. Substituting this into Eq. (29), to the leading order we obtain ($\partial_b P_0 = \partial_b P_0 = 0$)

$$\partial_a^2 P_0 + \frac{4a}{1+a^2}\partial_a P_0 + \epsilon\left(\partial_a^2 P_1 + \frac{4a}{1+a^2}\partial_a P_1 - \frac{2}{1+a^2}P_1\right)$$
$$= 0. \qquad (31)$$

From the above we can conclude that P_0 is exactly the solution of Eq. (15), i.e., Eq. (16). So P_1 should be the solution of the following equation:

$$\partial_a^2 P_1 + \frac{4a}{1+a^2}\partial_a P_1 - \frac{2}{1+a^2}P_1 = 0. \qquad (32)$$

The general solution of Eq. (32) is

$$P_1 = A\,{}_2F_1\left(\frac{3-\sqrt{17}}{4},\frac{3+\sqrt{17}}{4},\frac{1}{2},-a^2\right) + B\,{}_2F_1$$
$$\times\left(\frac{5-\sqrt{17}}{4},\frac{5+\sqrt{17}}{4},\frac{3}{2},-a^2\right)a. \qquad (33)$$

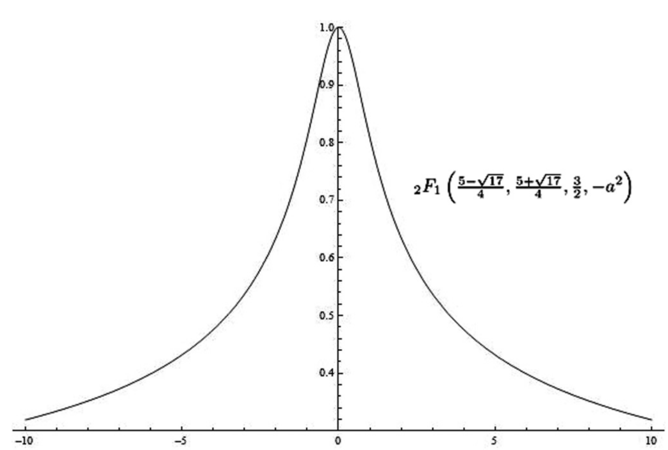

FIG. 8. The plot of ${}_2F_1(\frac{5-\sqrt{17}}{4},\frac{5+\sqrt{17}}{4},\frac{3}{2},-a^2)$ in terms of a.

The boundary conditions of P imply that $\lim_{a\to\infty}\frac{1}{a}P_1(a)=\lim_{a\to-\infty}\frac{1}{a}P_1(a)=0$ and $\lim_{a\to 0}\frac{1}{a}P_1(a)=\text{finite}$, from which we conclude that $A=0$. The plot of ${}_2F_1(\frac{5-\sqrt{17}}{4},\frac{5+\sqrt{17}}{4},\frac{3}{2},-a^2)$ has been presented in Fig. 8. Due to the insufficient precision, the determination of B is beyond our analysis. Therefore to first order of ϵ we suffice to present the general solution of Eq. (29):

$$P=\frac{1}{2}+\frac{1}{\pi}\left(\frac{a}{1+a^2}+\arctan(a)\right)+B(1+b){}_2F_1\times\left(\frac{5-\sqrt{17}}{4},\frac{5+\sqrt{17}}{4},\frac{3}{2},-a^2\right). \quad (34)$$

VI. CONCLUSION

In this paper we have calculated the left passage probability for SLE(κ,ρ). As an example we have analyzed the exact solution for $\rho=\kappa-6$. For general ρ we have obtained the differential equation and numerically solved it for the case $\kappa=2$ and $h_\rho=0$. We also obtained perturbative result for large x_0 (x_0 is the point on the real axis at which the boundary conditions change) up to one undetermined parameter.

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