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Exponential spreading and singular behavior of quantum dynamics near hyperbolic points

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Quantum dynamics of a particle in the vicinity of a hyperbolic point is considered. Expectation values of dynamical variables are calculated, and the singular behavior is analyzed. Exponentially fast extension of quantum dynamics is obtained, and conditions for this realization are analyzed.

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Hyperbolic (saddle) points are a source of instability in dynamical systems [1]. Therefore, the quantum dynamics of a particle in a saddle-point potential is an important problem in quantum chaos [2–4]. The hyperbolic point at the origin $(x,p)=(0,0)$ can be described locally by the Hamiltonian $H_{\rm loc}=xp$. The Lyapunov exponents that detect stable Λ_- and unstable Λ_+ manifolds are $\Lambda_\pm=\pm 1$. This leads to the exponential spreading in quantum dynamics and exponential growth of observable quantities [5,6]. The Hamiltonian $H_{\rm loc}$ has been studied in connection with the Riemann hypothesis [7,8], scattering of the inverted harmonic oscillator [9], and eigenstates near a hyperbolic point [10].

We consider a quantum system that is a second-order polynomial of $H_{\rm loc}$ with the Hamiltonian

$$\hat{H}_s = 2\omega\hat{x}\hat{p} - i\omega\hbar + \mu(2i\hat{x}\hat{p}+\hbar)^2 \equiv 2\omega\left[\hat{H}_{\rm loc}-\frac{i\hbar}{2}\right] - 4\mu\left[\hat{H}_{\rm loc}-\frac{i\hbar}{2}\right]^2, \qquad (1)$$

where ω and μ are the linearity and nonlinearity parameters, correspondingly, while the coordinate and momentum operators obey the standard commutation rule $[\hat{x},\hat{p}]=i\hbar$ with Planck's constant $\hbar$. This system was considered in [8] in connection with the Riemann hypothesis as well. Our primary interest in this Hamiltonian is related to a problem of quantum dynamics considered in the Heisenberg picture [11], where the expectation values of the operators $\hat{p}(t)$ and $\hat{x}(t)$ were calculated in the coherent states [12], especially prepared at the initial moment $t=0$. As shown in [11], the dynamics of these expectation values becomes singular at specific singularity times t_l. For example, for the observable value of $\hat{x}^2$ these singularities occur at times $t_l=\frac{\pi}{32\hbar\mu}+l\frac{\pi}{16\hbar\mu}, l=0,\pm 1,\pm 2,\ldots$. A remarkable property of these explosions is their pure quantum nature: in the classical counterpart, this corresponds to the separatrix motion without any singularities. As follows from the analysis of Ref. [11], this explosion behavior results from the interplay between the nonlinear term and the specific choice of zero boundary conditions on infinities. When $\mu=0$, the singularities are shifted to infinity and expectation values of operators are finite, and this result is independent of the boundary conditions.

We develop a different consideration of the problem to understand the nature of these singularities that, as will be shown, are related to the choice of the initial conditions. First we consider the quantum dynamics of a particle of a unit mass in the saddle potential described by the Hamiltonian (1). For calculation of the expectation values, following [11], the initial wave function is chosen in the form of the coherent state $\Psi_0(x)=\langle x|\alpha\rangle$. In the x representation, this is the Gaussian packet [11,12]:

$$\Psi_0(x)=\langle x|\alpha\rangle=(\hbar\pi)^{-1/4}e^{-|\alpha|^2/2\hbar-(x^2-2\sqrt{2}x\alpha+\alpha^2)/2\hbar}, \qquad (2)$$

where $\alpha\in\mathbb{C}$. It is worth mentioning that in this notation the dimension of x is $\sqrt{\hbar}$. For simplicity, we calculate the expectation value of the operator $\hat{x}^2(t)$. Thus we have

$$\langle\hat{x}^2(t)\rangle=\int_{-\infty}^{\infty}\Psi_0^*(x)\hat{U}^\dagger(t)x^2\hat{U}(t)\Psi_0(x)dx, \qquad (3)$$

where $\hat{U}(t)$ is the evolution operator.

The axis of integration is split into three intervals $(-\infty,-x_0],[-x_0,x_0],[x_0,\infty)$, and the expectation value is expressed by the following three integrals:

$$\langle\hat{x}^2(t)\rangle = I_s(t)+I_f^-(t)+I_f^+(t)=\int_{-x_0}^{x_0}\Psi_s^*(t)x^2\Psi_s(t)dx + \int_{-\infty}^{-x_0}\Psi_f^*(t)x^2\Psi_f(t)dx+\int_{x_0}^{\infty}\Psi_f^*(t)x^2\Psi_f(t)dx. \qquad (4)$$

The dynamics in the finite interval $[-x_0,x_0]$ is considered in the framework of the truncated interaction. Near the hyperbolic point, this dynamics is considered locally, such that $H=H_s$ for $x<x_0$ and the particle is, for example, free with $H=H_f$ outside the interaction region $|x|>x_0$, where $x_0>0$ determines arbitrarily the interaction range. Note that we do not consider a scattering task and just truncate the integration. Here $\hat{H}_s$ is determined by Eq. (1), while $\hat{H}_f=\hat{p}^2/2$ determines free motion. Therefore, the dynamics of an initial wave function Ψ_0 is determined in these two different regions,

$$\Psi_f(t)=\hat{U}_f(t)\Psi_0 \quad \text{and} \quad \Psi_s(t)=\hat{U}_s(t)\Psi_0, \qquad (5)$$

where the evolution operators $\hat{U}_f(t)=\exp[-\frac{i}{\hbar}\hat{H}_f t]$ and $\hat{U}_s(t)=\exp[-\frac{i}{\hbar}\hat{H}_s t]$ describe two independent processes, and a corresponding shift of the wave functions is supposed.

Integrals are calculated by substituting Eqs. (5) in Eq. (4) and taking into account the explicit form of the Hamiltonians. In the free motion window, the evolution of the square coordinate operator is

$$\hat{x}^2(t)=[\hat{U}_f^\dagger(t)\hat{x}\hat{U}_f(t)]^2=[\hat{x}+t\hat{p}]^2. \qquad (6)$$

Therefore, the free motion integrals $I_f^\pm(t)$ do not have any particular features. Their values can be expressed in the form of the error function $\Phi(z)=(2\pi)^{-1/2}\int_{-\infty}^{z}e^{\eta^2/2}d\eta$ [13]. Then, we obtain that $I_f^\pm(t)\sim t^2$, as expected.

The saddle-point integral possesses a more interesting behavior. Using calculations performed in [11] [Eq. (4.16)], we arrived at the expression

$$I_s(t) = (\hbar\pi)^{-1/2} e^{4\omega t + 24i\hbar\mu t} e^{-\frac{(\alpha+\alpha^*)^2}{2\hbar}} \int_{-x_0}^{x_0} dx\, x^2 \times \exp\left\{-\frac{x^2}{2\hbar}[(1+e^{32i\hbar\mu t}) - 2\sqrt{2}(\alpha^* + \alpha e^{16i\hbar\mu t})x]\right\}. \quad (7)$$

First, we admit the exponential quantum growth, obtained in Ref. [11]. This quantum behavior has the classical nature of the near separatrix motion, observed also for a kicked system [6]. At the singularity times $t_l = \frac{\pi}{32\hbar\mu} + \frac{l\pi}{16\hbar\mu}$, expression (7) is simplified, and the integral is calculated exactly. It reads

$$I_s(t_l) = \frac{(-i)^l}{(\hbar\pi)^{1/2}} e^{-\frac{(\alpha+\alpha^*)^2}{2\hbar}} e^{\omega\pi/8\hbar\mu + l\omega\pi/4\hbar\mu} e^{3i\pi/4} \times \left[\frac{\sqrt{2}\hbar^3}{\xi^3}\sinh\left(\frac{\sqrt{2}\xi x_0}{\hbar}\right) - \frac{2\hbar^2 x_0}{\xi^2}\cosh\left(\frac{\sqrt{2}\xi x_0}{\hbar}\right) - \frac{\sqrt{2}\hbar x_0^2}{\xi}\sinh\left(\frac{\sqrt{2}\xi x_0}{\hbar}\right)\right], \quad (8)$$

where $\xi = \alpha^* + i(-1)^l\alpha$. This expression is finite for finite x_0. When x_0 approaches infinity, the integral diverges and t_l are the singularity points. Note that x_0 is an arbitrarily defined scale [14].

To generalize the consideration of the explosion singularities, first we consider the eigenvalue problem for the Hamiltonian H_s. Since the operator $\hat{x}\hat{p} - i\hbar/2$ commutes with $\hat{H}_s$, this problem is reduced to the dimensionless equation for the eigenfunctions $\chi_\epsilon(x)$:

$$\frac{1}{i}\left(x\frac{d}{dx} + \frac{1}{2}\right)\chi_\epsilon(x) = \epsilon\chi_\epsilon(x) \quad (9)$$

with the solution

$$\chi_\epsilon(x) = \frac{1}{\sqrt{N_s|x|}}\exp[i\epsilon\ \ln|x|], \quad (10)$$

which satisfies the boundary conditions $\chi_\epsilon(x = \pm\infty) = 0$ and $N_s = 4\pi\hbar^{1/2}$ [15,16]. For the continuous spectrum, the normalization condition is

$$\int_{-\infty}^{\infty} \chi_{\epsilon'}^*(x)\chi_\epsilon(x)dx = \delta(\epsilon - \epsilon') \quad (11)$$

(see, e.g. [17]).

Now, expanding $\Psi_0(x)$ over the complete set of $\chi_\epsilon(x)$, we obtain

$$\Psi_0(x) = \int d\epsilon\, q(\epsilon)\chi_\epsilon(x). \quad (12)$$

Note that the explicit form of the expansion coefficients $q(\epsilon)$ is not important, since integration over energy ϵ will be performed with exactly the same form of $\chi_\epsilon(x)$. Substituting Eq. (12) in the integral I_s in Eq. (4) with $x_0 = \infty$, we obtain

$$I_s^\infty(t) = 2\int_0^\infty x^2 \int^{\prime} q^*(\epsilon')q(\epsilon)e{-}i(E - E')t\, d\epsilon\, d\epsilon' \times \chi_{\epsilon'}^*(x)\chi_\epsilon(x)dx, \quad (13)$$

where $E = 2\omega\epsilon - 4\hbar\mu\epsilon^2$ is the energy of $\hat{H}_s$, and we use that $\chi_\epsilon(-x) = \chi_\epsilon(x)$. The complex Gaussian exponents are presented in the form of the Fourier integrals:

$$e^{\pm i4\hbar\mu t\epsilon^2} = \int_{-\infty}^{\infty} \frac{e^{\mp i\tau\epsilon} d\tau}{\sqrt{\pm 16\pi i\hbar\mu t}} \exp\{\mp i\tau^2/16\hbar\mu t\}. \quad (14)$$

Substituting these expressions in Eq. (13) and taking into account the explicit form of $\chi_\epsilon(x)$, we obtain $\ln|x| - 2\omega t - \tau = \ln(|x|e^{-2\omega t}e^{-\tau})$. Then, one integrates over ϵ and ϵ' to obtain the following expression:

$$I_s(t) = 2\int_0^\infty x^2 \int \frac{d\tau\, d\tau'}{16\pi\hbar\mu t} e^{-i\frac{(\tau^2-\tau'^2)}{16\hbar\mu t}} e^{-2\omega t} e^{\frac{(\tau+\tau')}{2}} \times \Psi_0^*(xe^{-2\omega t}e^{-\tau'})\Psi_0(xe^{-2\omega t}e^{-\tau})dx. \quad (15)$$

The next step is integration over τ and τ'. To this end, we perform the following variables change $\tau = u + v$ and $\tau' = u - v$ with the Jacobian of the transformation equaling 2. Denoting $y = xe^{-2\omega t}e^{-(\tau+\tau')/2}$, integration over u is exact and gives the δ function $\delta(v - 8i\hbar\mu t)$. Therefore, integration over v is also exact. Finally, the expectation value reads

$$\langle\hat{x}^2(t)\rangle = 2e^{4\omega t}\int_0^\infty y^2\Psi_0^*(ye^{8i\hbar\mu t})\Psi_0(ye^{-8i\hbar\mu t})dy. \quad (16)$$

Here, we also use the symmetrical property of the wave function. Substituting Eq. (2) in Eq. (16), one obtains that at times $t = t_l$ this expression diverges. These are the same singularities obtained above in Eq. (7). Moreover, any "good" Gaussian and exponential functions lead to this kind of singular behavior for the expectation values of physical operators.

Obviously, these singularities result from a specific preparation of the initial wave packets $\Psi_0(x)$. Let us prepare the initial conditions "properly" to obtain the finite moments of the physical variables. Owing to Eq. (12), we present the initial condition as the spectral decomposition with the Gaussian weight $q(\epsilon) = [\frac{2a}{\pi}]^{\frac{1}{4}}\exp(-a\epsilon^2)$, where real $a > 0$. This yields the initial wave packet in the form of a log-normal distribution,

$$\Psi_0(x) = \frac{1}{\sqrt{N}}\exp\left(-\frac{1}{4a^2}\ln^2|x| - \frac{1}{2}\ln|x|\right), \quad (17)$$

with $N = 4a\sqrt{\pi}$. Using Eqs. (17) and (16), one obtains for the nth moment of $\hat{x}$

$$\langle\hat{x}^n(t)\rangle = \frac{1}{2a\sqrt{\pi}}e^{4\omega t}e^{\frac{16}{a^2}\hbar^2\mu^2t^2} \times \int_0^\infty \exp\left(-\frac{1}{2a^2}\ln^2 x + n\ \ln x\right)d(\ln x) = \frac{\sqrt{2}}{2}\exp\left(4\omega t + a^4n^2 + \frac{16}{a^2}\hbar^2\mu^2t^2\right). \quad (18)$$

This behavior of the expectation values is finite and spreads exponentially for the arbitrary long time scale. This exponential that increases with time consists of two values. The first is the above-mentioned classical term $e^{4\omega t}$, which is due to the classical motion near the hyperbolic point. The second, pure quantum, term $\frac{16}{a^2}\hbar^2\mu^2t^2$ is dominant and relates to the action of the evolution operator, which is a delation operator $e^{b\hat{x}\hat{p}}$, where $e^{b\hat{x}\hat{p}}f(x) = f(e^{-i\sqrt{\hbar}b})$ (see, e.g. [18]). Therefore, this term is due to the quantum dynamics near the hyperbolic point.

Both singular behavior and this exponential quantum growth are due to the nonlinear quantum parameter $\kappa = \hbar\mu T$, where T is a characteristic time scale. For example, for the explosion behavior of $\langle \hat{x}^2(t)\rangle$ in Eqs. (7) and (8) it is $T = \frac{\pi}{32\hbar\mu}$. For the quantum expansion, we can define this time scale parameter as $\hbar\mu/\omega$ to define the dimensionless growth of the expectation values $\exp[(\frac{\kappa}{a})^2\omega t]$.

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[14] The interaction range x_0 is always a finite value due to the local consideration near the hyperbolic point. For example, it can be evaluated as the difference between the normalization constants of the wave functions for the free motion N_f and for the saddle-point motion N_s (see Problem 133.1 in [17]). For the case of free motion, the normalization constant N_f is determined from the integral [16,17] $\int_{-\infty}^{\infty} e^{-ipx/\hbar}e^{ip'x/\hbar}dx = 2\pi\sqrt{\hbar}\delta(p-p') \equiv N_f\delta(p-p')$. Therefore, $N_f = 2\pi\hbar^{1/2}$. We also used here that x has the dimension $\hbar^{1/2}$. In order to find N_s, one uses the eigenfunction of the Hamiltonian $\hat{H}_s$ with an extended action on the whole x axis from minus to plus infinity. Eventually, we obtain that the interaction range is $x_0 = N_s - N_f = 2\pi\hbar^{1/2}$.

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