Dynamics of a nonlocal discrete Gross-Pitaevskii equation with defects

Yue Jian, Ai-Xia Zhang, Cai-Xia He, Xiu-Ying Qi, and Ju-Kui Xue[*]
College of Physics and Electronic Engineering, Northwest Normal University, Lanzhou 730070, China
(Received 12 January 2013; revised manuscript received 27 March 2013; published 28 May 2013)

We study the dynamics of dipolar gas in deep lattices described by a nonlocal nonlinear discrete Gross-Pitaevskii equation. The stabilities and the propagation properties of traveling plane waves in the system with defects are discussed in detail. For a clean lattice, both energetic and dynamical stabilities of the traveling plane waves are studied. It is shown that the system with attractive local interaction can preserve the stabilities, i.e., the dipoles can stabilize the gas because of repulsive nonlocal dipole-dipole interactions. For a lattice with defects, within a two-mode approximation, the propagation properties of traveling plane waves in the system map onto a nonrigid pendulum Hamiltonian with quasimomentum-dependent nonlinearity (induced by the nonlocal interactions). Competition between defects, quasimomentum of the gas, and nonlocal interactions determines the propagation properties of the traveling plane waves. Critical conditions for crossing from a superfluid regime with propagation preserved to a normal regime with defect-induced damping are obtained analytically and confirmed numerically. In particular, the critical conditions for supporting the superfluidity strongly depend on the defect type and the quasimomentum of the plane waves. The nonlocal interaction can significantly enhance the superfluidity of the system.

DOI: 10.1103/PhysRevE.87.053201 PACS number(s): 05.45.Yv, 42.65.Wi, 03.75.Lm

I. INTRODUCTION

In recent years, the interplay between nonlinearity, discreteness, and disorder (i.e., small random impurities or defects) has been the subject of intensive theoretical and experimental investigations [1–11]. The competition between nonlinearity, discreteness, and disorder can induce rich phenomena and plays a crucial role in nonlinear discreteness systems such as Anderson localization [12] and disorder-induced inhibition of transportation [13–16]. In particular, the transportation properties of the disordered nonlinear discrete system have become a challenging issue. A key property is that in such system, the propagation of traveling plane waves experiences a crossover from a superfluid regime with propagation preserved to a normal regime with disorder-induced damping [3], in which nonlinearity plays a crucial role. Because of the controllability of both disorder (which can be introduced in the system in a controlled way by using optical means [17], atomic mixtures [18], or inhomogeneous magnetic fields [19]) and nonlinearity (which, as a consequence of interactions between particles, can be controlled by the Feshbach technique [20]), ultracold bosons in deep lattices with defects provide an ideal physical system to study this issue.

At low temperature, bosons in deep lattices are well described by the nonlinear discrete Gross-Pitaevskii (GP) equation [3,21], which has played a central role in our understanding of the system. In the discrete GP equation the cubic nonlinearity arising in the case of local interaction is characterized by a two-body nonlinear term through a contact interaction that is parametrized by the s-wave scattering length a, whose sign determines the type of interaction, i.e., $a < 0$ indicates that the interaction among the particles in the system is attractive, while $a > 0$ indicates that the interaction is repulsive. Importantly, systems with dominant attractive local interactions are fundamentally unstable against collapse [22–25]. The transportation properties of bosons in lattices with defects were originally predicted and explored in the context of this local discrete GP equation. The transportation properties of bosons in the disordered nonlocal discrete GP equation are still not clear; however, the long-range nonlocal character of the dipolar interaction results in the dipolar condensate trapped in deep optical lattices [26,27] promoting discussion of this issue. A dipolar condensate loaded into the deep lattices can be described by a nonlinear discrete GP equation with nonlocal interaction [28–30], i.e., a nonlocal nonlinear discrete GP equation. Stable solitons [31–37] and the condensate [38–42] should be observable.

In this paper we investigate the stability and superfluidity of a dipolar condensate in lattices within a nonlocal nonlinear discrete GP equation with and without defects. The stability and the propagation properties of traveling plane waves in the system are discussed in detail. For a clean lattice, both energetic and dynamical stabilities of the traveling plane waves are studied. It is shown that there is a critical scattering length a_c and when $a > a_c$, the system is stable. Interestingly, we find that, in a system with nonlocal interaction, a_c is always negative. This is different from the case with only a local interaction, when $a_c > 0$. That is, the dipoles can stabilize the condensate because of the repulsive nonlocal dipole-dipole interaction. For a lattice with defects, we discuss the superfluidity of the condensate in a deep annular lattice with defects, i.e., the propagation properties of the traveling plane waves in the system with competition between defects and the nonlocal interaction. Within a two-mode approximation, the dynamics of the system described by the nonlocal nonlinear discrete GP equation maps onto a nonrigid pendulum Hamiltonian. We find that there can also exist a critical scattering length a_c that divides the system into two regime: $a > a_c$, in which a plane wave coherently passes through the defects and the system is in a superfluid state, and $a < a_c$, in which the system is in a normal regime with defect-induced damping. Importantly, a_c and the superfluidity of the system strongly depend on the quasimomentum of the plane waves. In particular, the

[*]xuejk@nwnu.edu.cn

nonlocal interaction can enhance the superfluidity of the system.

The paper is organized as follows. In Sec. II we present the physical model for the dipolar condensate in a deep one-dimensional lattice. In Sec. III, by using the perturbative approximation, we analyze the stabilities of the clean system. In Sec. IV, within a two-mode approximation, the dipolar condensate is mapped onto a nonrigid pendulum Hamiltonian. We study the dynamical properties of the system with a single defect and Gaussian defects. Finally, we summarize in Sec. V.

II. MODEL

We consider a dipolar condensate trapped in a deep one-dimensional lattice, with magnetic dipolar moment $\vec{\mu}$ oriented perpendicular to the lattice by an external magnetic $\vec{B}$. By using the tight-binding approximation, the system can be described by the dimensionless nonlocal discrete nonlinear GP equation [28–30]

$$i\frac{\partial\psi_n}{\partial\tau} = -\frac{1}{2}(\psi_{n-1}+\psi_{n+1}) + \epsilon_n\psi_n + [(a\chi + C_{DD0})|\psi_n|^2 + C_{DD1}(|\psi_{n+1}|^2+|\psi_{n-1}|^2) + C_{DD2}(|\psi_{n+2}|^2 + |\psi_{n-2}|^2)]\psi_n, \tag{1}$$

where ψ_n is the wave function of the condensate in the nth site of the array, $n = 1,\ldots,\mathcal{N}$ ($\mathcal{N}$ the number of sites). The first term on the right-hand side of Eq. (1) is the tunneling term, which denotes the tunneling between the adjacent sites; ϵ_n, which is proportional to any external field superimposed on the lattice (i.e., $\epsilon_n \propto \int d\vec{r}\,[(\hbar^2/2mJ)|\nabla\phi_n|^2 + V_{\rm ext}|\phi_n|^2]$, where ϕ_n are wave functions localized in each site of the periodic potential), is the on-site energy. For a clean lattice, ϵ_n is a constant; for a defected lattice, ϵ_n in each lattice is different and expresses the defect distribution. The defects ϵ_n can be created by additional lasers and/or magnetic fields. In the physical systems we have discussed, the defects ϵ_n can be spatially localized or extended. In Eq. (1), the nonlinearity is induced by the atomic contact interaction a, the on-site dipolar interaction C_{DD0}, the nearest-neighbor dipolar interaction C_{DD1}, and the next-nearest-neighbor interaction C_{DD2}. The s-wave scattering length a is in units of the Bohr radius a_0. The local on-site dipolar interaction C_{DD0} and the nonlocal intersite dipolar interaction C_{DDj} $(j = 1,2)$ are given in Ref. [28], i.e.,

$$C_{DD0} = \frac{\mu_0\mu^2}{4\pi J}\frac{1}{l_\perp^3 c^3}\sqrt{\frac{2}{\pi}}\left(\frac{c(3-c^2)}{3\sqrt{1-c^2}} - \arcsin(c)\right),$$
$$C_{DDj} = \frac{\mu_0\mu^2}{4\pi J}\frac{1}{3l_\perp^3}\sqrt{\frac{2}{\pi}}F\left(c,\frac{jb}{l_\perp}\right)$$

$(j = 1,2)$, where $\chi = \frac{4\pi\hbar^2}{mJ}\frac{a_0}{(2\pi)^{3/2}l_\perp^2 l}$, $l_\perp = \sqrt{\hbar/mw_\perp}$, and $l = bs^{-1/4}/\pi$, with $b = \pi/k_L$ the lattice step, $w_\perp = 290$ Hz the vertical trapping frequency, and $k_L = 2\pi/\lambda$ the laser wave vector ($\lambda = 1064$ nm). Further, $J = \frac{4}{\sqrt{\pi}}s^{3/4}e^{-2\sqrt{s}}E_R$, $E_R = \hbar^2\pi^2/2Md^2$ is the recoil energy of the optical lattices, d is the lattice period ($d = \lambda/2$), s is the strength of the optical lattice, $c = \sqrt{1 - l^2/l_\perp^2}$, μ_0 is the vacuum permeability, and μ is the magnetic dipole moment ($\mu = 6\mu_B$ for $^{52}C_r$ with μ_B the Bohr magneton). Here

$$F(u,v) = \int_0^1 ds\frac{3s^2-1}{(1-u^2s^2)^{3/2}}\left(1 - \frac{v^2s^2}{1-u^2s^2}\right) \times e^{-[v^2s^2/2(1-u^2s^2)]}.$$

In this article we study the propagation of a plane wave $\psi_n(\tau = 0) = e^{ikn}$ in system (1); here k is the quasimomentum of the plane wave. We will use periodic boundary conditions (due to the annular geometry); thus we have $k = 2\pi l/\mathcal{N}$, where l is an integer ($l = 0,\ldots,\mathcal{N}-1$). The Hamiltonian of the Eq. (1) is

$$H = \sum\Bigg[-\frac{1}{2}(\psi_{n+1}\psi_n^* + \psi_n\psi_{n+1}^*) + \epsilon_n|\psi_n|^2 + \Bigg(\frac{(a\chi + C_{DD0})}{2}|\psi_n|^2 + C_{DD1}(|\psi_{n+1}|^2 + |\psi_{n-1}|^2) + C_{DD2}(|\psi_{n+2}|^2 + |\psi_{n-2}|^2)\Bigg)|\psi_n|^2\Bigg]. \tag{2}$$

III. STABILITIES OF THE SYSTEM WITHOUT DEFECTS

Let us consider the stabilities of the system with $\epsilon_n = 0$. We employ the plane wave $\psi = \psi_0 e^{i(kn-u_0t)}$, which is the stationary solution of Eq. (1), where k is the quasimomentum of the condensate. The stability analysis of such a state can be carried out by perturbing the carrier wave with small-amplitude phonons: $\psi = [\psi_0 + u(t)e^{iqn} + v^*(t)e^{-iqn}]e^{i(kn-u_0t)}$, where q is the quasimomentum of the excitation. The perturbation functions $u(t)$ and $v(t)$ have the same periodicity as the lattices; then Eq. (1) becomes

$$i\frac{\partial}{\partial t}\begin{pmatrix}u\\ v\end{pmatrix} = \hat{\sigma}\hat{A}\begin{pmatrix}u\\ v\end{pmatrix}, \tag{3}$$

where $\hat{A} = \begin{pmatrix} L_+ & C\psi_0^2 \\ C(\psi_0^*)^2 & L_-\end{pmatrix}$ with $L\pm = \cos(k) - \cos(q\pm k) + C|\psi_0|^2$ and the effective interaction parameter $C = a\chi + C_{DD0} + 2C_{DD1}\cos(q) + 2C_{DD2}\cos(2q)$. It is important to note that this effective atom interaction depends on the quasimomentum of the excitation. This momentum-dependent atom interaction is induced by the nonlocal dipolar interaction. For a nondipolar gas (i.e., $C_{DD0} = C_{DD1} = C_{DD2} = 0$), C does not depend on q. Here $\hat{\sigma}$ is the Pauli matrix. By straightforward calculation, the eigenvalues of $\hat{A}$ are easily found as

$$\lambda_\pm = 2\cos(k)\sin^2\left(\frac{q}{2}\right) + C|\psi_0|^2 \pm \sqrt{P^2 + C^2|\psi_0|^4} \tag{4}$$

and the discrete nonlinear GP equation excitation spectrum (eigenvalues of $\hat{\sigma}\hat{A}$) is given by

$$\eta_\pm = P \pm \sqrt{Q^2 - 2CQ|\psi_0|^2}, \tag{5}$$

where $P = \sin(q)\sin(k)$ and $Q = \cos(k)[\cos(q) - 1]$.

Base on Eq. (4), we can easily find that the boundary of energetic stability of Bloch waves is described by (λ should be real positive) $\cos^2(\frac{q}{2}) \leqslant \cos(k)[\cos(k) + C|\psi_0|^2]$. Clearly, the energetic instability can be completely excited when $\cos(k) < 0$. For long-wavelength perturbation ($q \to 0$), this condition can be reduced to a critical contact scattering

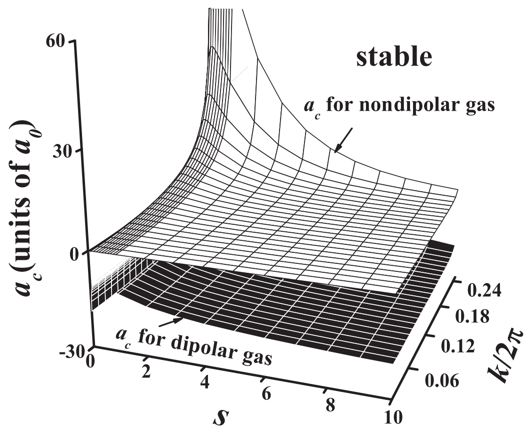

FIG. 1. Energetic stability diagram of the system. Here a_0 is the Bohr radius.

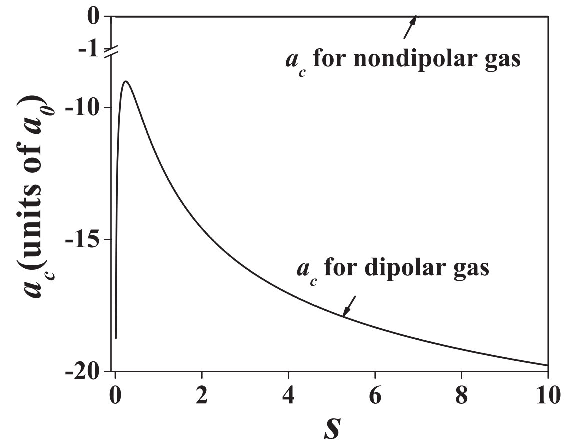

FIG. 2. Modulational stability diagram of the system. The critical scattering length a_c is shown as a function of the lattice depth s. Here a_0 is the Bohr radius.

length a_c for maintaining the stability in a dipolar condensate:

$$a \geqslant a_c = \frac{1}{\chi}\left(\frac{\sin^2(k)}{\cos(k)|\psi_0|^2} - C_{DD0} - 2C_{DD1} - 2C_{DD2}\right). \tag{6}$$

In Fig. 1 we plot the energetic stability diagram of the system; the area above the critical scattering length a_c corresponds to the energetically stable region. The critical parameter a_c for both the dipolar condensate and the nondipolar condensate is shown in Fig. 1. We find that the critical scattering length a_c decreases with increasing strength of optical lattices s and the quasimomentum k of the plane wave fixed; for a fixed s, a_c increases with increasing k. Interestingly, we find the dipolar condensate is more stable than the nondipolar system, i.e., the dipolar gas can preserve the stability with an attractive local interaction (contact interaction). With increasing s, the critical scattering length a_c of the nondipolar gas tends to 0, while the critical scattering length a_c of the dipolar condensate tends to $-20a_0$.

For a nondipolar condensate with purely contact interaction, the system with attractive contact interaction (attractive local interaction) is fundamentally unstable against collapse, while the system with repulsive contact interaction prevents the collapse and is stable. For a dipolar gas, the nonlocal repulsive dipolar interaction can compensate for the local attractive contact interaction and the effective interaction of the system can be repulsive. Thus the system with attractive contact interaction could be stable as long as the effective interaction of the system is repulsive (i.e., $C > 0$). That is, dipoles can stabilize the condensate due to the effectively nonlocal repulsive dipolar interaction.

Furthermore, the modulational instability (dynamical instability) can be induced when the eigenfrequency η in Eq. (5) becomes imaginary, i.e., $C|\psi_0|^2 \geqslant -\cos(k)\sin^2(\frac{q}{2})$. Therefore, when the effective atomic interaction is repulsive ($C > 0$), the system suffers an exponential growth of perturbations with $\cos(k) < 0$. For long-wavelength perturbation ($q \to 0$), this condition reduces to a critical scattering length a_c for preserving the modulational stability of the system:

$$a \geqslant a_c = -\frac{1}{\chi}(C_{DD0} + 2C_{DD1} + 2C_{DD2}). \tag{7}$$

The critical scattering length a_c given by Eq. (7) is plotted in Fig. 2. The area above the line corresponds to the stable region. When s is small, we can find that the critical scattering length a_c rapidly grows to a maximum, which is due to the influence of the intersite dipolar interaction. Then a_c decreases gradually with increasing lattice depth s until close to $-20a_0$. Also, we can observe that a_c for different s is less than zero. However, it is well known that the dynamical stability in the nondipolar gas can be induced when the scattering length a is positive. Thus the dipolar condensate is more stable in dynamics than the system without dipolar interaction. In contrast, dipoles can suppress both energetic and dynamical instabilities because of the effective nonlocal repulsive dipolar interaction. Our results are in good agreement with recent experiments [38,39].

IV. SUPERFLUIDITY WITH DEFECTS

We now consider the dynamical properties of Eq. (2) with defects. As discussed above, when $\cos(k) < 0$, the system becomes unstable, so we consider the case in which $\cos(k) > 0$. The angular momentum of this system is defined as

$$L(\tau) = i\sum(\psi_n\psi_{n+1}^* - \psi_n^*\psi_{n+1}). \tag{8}$$

The angular momentum $L(\tau)$ oscillates between the initial value L_0 to $-L_0$, corresponding, respectively, to plane waves with wave vectors k and $-k$ in a liner system. Moreover, rotational states with opposite wave vectors k and $-k$ are degenerate in clear optical lattices. However, the defects split the degeneracy by coupling the two k and $-k$ waves, very much as the tunneling barrier in a double-well potential between the left and right localized states. For this reason, the relative population of the two waves oscillates according to an effective Josephson Hamiltonian [3,43]. In this limit, one can employ a two-mode ansatz for the dynamical evolution of the wave function:

$$\psi_n(\tau) = A(\tau)e^{ikn} + B(\tau)e^{-ikn}. \tag{9}$$

We set $A,B = \sqrt{n_{A,B}(\tau)}e^{i\phi_{A,B(\tau)}}$, $z = n_A - n_B$, and $\phi = \phi_A - \phi_B$.

To understand the dynamics of the system, we discuss the variation of the angular momentum with some related parameters. Thus, using ansatz (9) in Eq. (8), we get

$$L = 2\mathcal{N} z \sin(k). \tag{10}$$

Therefore, we observe that the angular momentum is proportional to z. Note that $\langle L\rangle = 0$ implies that the wave is completely reflected, which means that the system is in a normal state with defect-induced damping, and $\langle L\rangle \neq 0$ implies that the wave is only partially reflected by the defects, that is, the system is in a superfluid state. Here the angular brackets stand for a time average. The latter case corresponds to a self-trapping of the angular momentum. The incident wave cannot be reflected completely and coherence is preserved. The observation of a persistent current is associated with a superfluid regime of the system (1).

A. Single defect

Let us consider the case of a single defect

$$\epsilon_n = \epsilon \delta_{n,\bar{n}}. \tag{11}$$

Defining the effective Lagrangian as $\pounds = \sum \frac{i}{2}(\dot{\psi}_n \psi_n^* - \psi_n \dot{\psi}_n^*) - H$, both H and the norm $\sum |\psi_n|^2 = N$ are conserved and using ansatz (9) in this effective Lagrangian, we have

$$\frac{\pounds}{\mathcal{N}} = -n_A\dot{\phi}_A - n_B\dot{\phi}_B - \frac{2\epsilon}{\mathcal{N}}\sqrt{n_A n_B}\cos(\phi_A - \phi_B + 2k\bar{n}) - C n_A n_B, \tag{12}$$

with the relation $\sum e^{2kn} = 0$. Using the Euler-Lagrange equations $\frac{d}{dt}\frac{\partial \pounds}{\partial \dot{q}_i} = \frac{\partial \pounds}{\partial q_i}$ for the variational parameters $q_i(\tau) = n_{A,B}, \phi_{A,B}$ in Eq. (12) and with the replacement $\phi + 2k\bar{n} \to \phi$, we obtain

$$\dot{z} = -\frac{2\epsilon}{\mathcal{N}}\sqrt{1-z^2}\sin(\phi), \tag{13}$$

$$\dot{\phi} = \frac{2\epsilon}{\mathcal{N}}\frac{z}{\sqrt{1-z^2}}\cos(\phi) + Cz, \tag{14}$$

where $C = a\chi + C_{DD0} + 4C_{DD1}\cos(2k) + 4C_{DD2}\cos(4k)$ is the effective atom interaction, which, interestingly, depends on the quasimomentum k (induced by the nonlocal dipolar interaction). That is, within a two-mode ansatz in Fourier space, the dynamics of the system map onto a nonrigid pendulum with quasimomentum-dependent nonlinearity. The effective Hamiltonian (i.e., the total conserved energy) becomes

$$H = -\frac{2\epsilon}{\mathcal{N}}\sqrt{1-z^2}\cos(\phi) + \frac{Cz^2}{2}. \tag{15}$$

Let us derive the critical condition for supporting a superfluid flow, that is, the occurrence of transition between the regimes with $\langle L\rangle = 0$ and the regime with $\langle L\rangle \neq 0$. Equation (10) indicates that the angular momentum L is proportional to z. Therefore, $\langle z\rangle = 0$ (i.e., z oscillates around 0 and $\langle L\rangle = 0$) implies that the wave is completely reflected by the defects; $\langle z\rangle \neq 0$ (i.e., z oscillates around a nonzero value and $\langle L\rangle \neq 0$) implies that the wave is only partially reflected by the defects and the system is in a superfluid regime. Hence, if z cannot reach the value 0, then we can have $\langle z\rangle \neq 0$ and the system is in a superfluid regime. To prevent the system from reaching the state $z = 0$, the initial energy of the system H_0 should be larger than the energy of this state, i.e., $H_0 > H(z=0)$. Initially, we set $z(0) = 1$ and $\phi(0) = 0$, so the conserved initial energy is $H_0 = C/2$. Because $H(z=0) = -\frac{2\epsilon}{\mathcal{N}}\cos(\phi)$, we clearly see that the maximum value of $H(z=0)$ is $2\epsilon/\mathcal{N}$, i.e., $H(z=0) = -\frac{2\epsilon}{\mathcal{N}}\cos(\phi) \leqslant \frac{2\epsilon}{\mathcal{N}}$. Hence, if $H_0 = C/2 \geqslant 2\epsilon/\mathcal{N}$, i.e., $C \geqslant 4\epsilon/\mathcal{N}$, then $H_0 > H(z=0)$ should be satisfied for all values of ϕ. That is, when $C \geqslant 4\epsilon/\mathcal{N}$, z cannot reach the value 0 and the system will be in a superfluid regime. Thus we find a critical condition for supporting the superfluid flow $a\chi + C_{DD0} + 4C_{DD1}\cos(2k) + 4C_{DD2}\cos(4k) = 4\epsilon/\mathcal{N}$. From this condition we can obtain a critical atomic scattering length a_c for maintaining the superfluidity

$$a_c = \frac{1}{\chi}\left(\frac{4\epsilon}{\mathcal{N}} - C_{DD0} - 4C_{DD1}\cos(2k) - 4C_{DD2}\cos(4k)\right). \tag{16}$$

The system can be divided into two regimes by the critical condition: a normal regime when $a < a_c$ and a superfluid regime when $a > a_c$. When $a < a_c$, L oscillates around 0; when $a = a_c$, L asymptotically approaches 0, with $a > a_c$, $\langle L\rangle \neq 0$. In a normal state a plane wave is reflected by the defect, while a plane wave travels coherently through the defects in a superfluid state. Importantly, Eq. (16) indicates that the superfluidity of the system strongly depends on the defect ϵ, the dipolar interaction, and the quasimomentum k of the gas. The competition between the defect, dipolar interaction, and quasimomentum of the gas provides a critical scattering length a_c for maintaining the superfluidity. For fixed defect, the presence of nonlocal dipolar interaction can reduce a_c, even results in a negative a_c, and enhance the superfluidity of the system.

Figure 3 shows the critical a_c plotted against lattice depth s and the quasimomentum k with a single defect given by Eq. (16). Clearly, we find the critical scattering length a_c for dipolar gas decreases from a positive value to $a_c = -30a_0$ with increasing lattice depth and fixed quasimomentum k. Thus, in the deep lattice regime, the superfluid can be more easily preserved due to the relatively small critical scattering length a_c. Furthermore, the critical scattering length a_c for nondipolar gas is positive and tends gradually to zero in the deep lattice regime. That is, the dipolar condensate with a single defect can more easily support the superfluid state than the system without

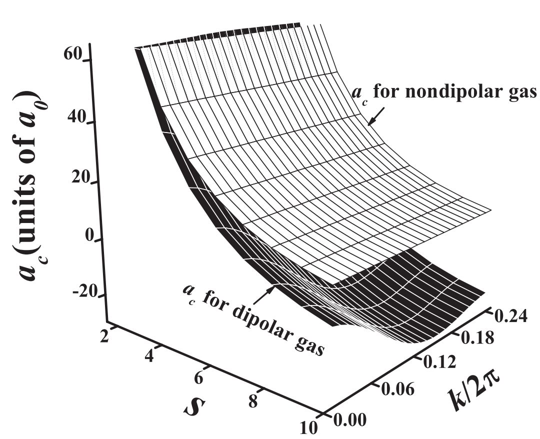

FIG. 3. Critical scattering length a_c plotted against lattice depth s and the quasimomentum k of the gas associated with a single defect. Here a_0 is the Bohr radius, $\epsilon = 0.05$, and $\mathcal{N} = 100$.

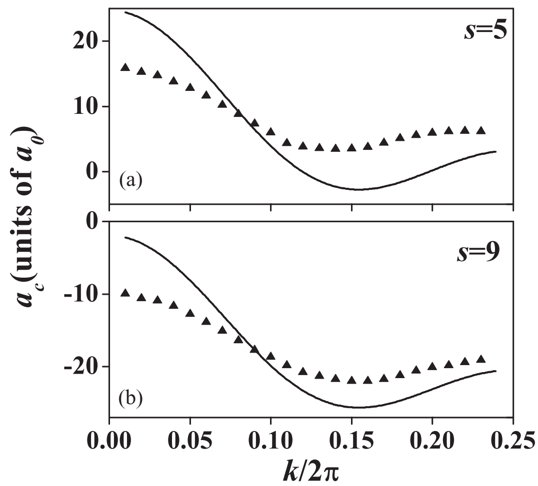

FIG. 4. Critical scattering length a_c as a function of the quasimomentum k for different lattice depths: (a) $s = 5$ and (b) $s = 9$. Here a_0 is the Bohr radius, $\epsilon = 0.05$, and $\mathcal{N} = 100$. The points are numerical simulations of Eq. (1) and the solid lines are the analytical results of Eq. (16).

dipolar interaction. The dipoles can enhance the superfluidity of the condensate with a single defect. In order to clearly show the relationship between a_c and the quasimomentum k, the critical scattering length a_c of dipolar condensate vs the quasimomentum k for different lattice depths of $s = 5$ and 9 is plotted in Fig. 4. It is clear that the value of a_c for $s = 9$ is smaller than that for $s = 5$. Interestingly, there is a critical k_c, when $k < k_c$, for which a_c decreases with increasing k, while when $k > k_c$, a_c increases with k. This nonmonotonic behavior of a_c with respect to k is induced by nonlocal dipolar interaction (note that C depends on k). To confirm the analytical results, numerical results obtained by direct numerical integration of Eq. (1) with a fourth-order Runge-Kutta method are also shown in Fig. 4. We find that the analytical results qualitatively agree with the numerical results.

B. Gaussian defect

Furthermore, we now consider a Gaussian defect with width σ centered on the site $\bar{n}$:

$$\epsilon_n = \frac{\epsilon}{\sqrt{\pi}\sigma} e^{-(n-\bar{n})^2/\sigma^2}. \tag{17}$$

For sufficiently large $\mathcal{N}$ and $\sigma \gtrsim 1$, we can set $\sum \epsilon_n \approx \int dn \epsilon_n = \epsilon$. In the same way as in the case of a single defect and setting $\phi + 2k\bar{n} \to \phi$, the effective Hamiltonian reduces to

$$H \approx -\frac{2\varepsilon e^{-k^2\sigma^2}}{\mathcal{N}} \sqrt{1 - z^2}\cos(\phi) + \frac{Cz^2}{2}. \tag{18}$$

We can clearly see that the system is equal to that of a single defect with an effective defect $\epsilon_{\text{eff}} = \epsilon e^{-k^2\sigma^2}$. Thus the critical a_c for supporting the superfluid is

$$a_c = \frac{1}{\chi}\left(\frac{4\epsilon_{\text{eff}}}{\mathcal{N}} - C_{DD0} - 4C_{DD1}\cos(2k) - 4C_{DD2}\cos(4k)\right). \tag{19}$$

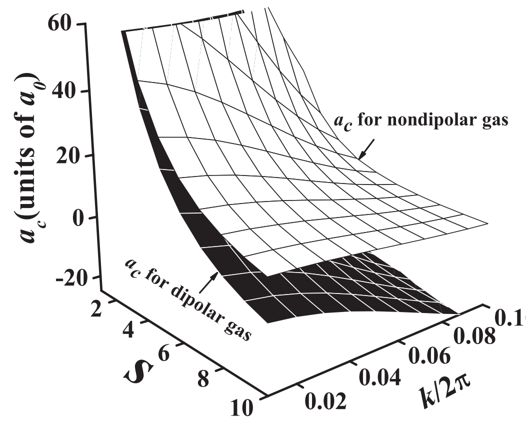

FIG. 5. Critical scattering length a_c plotted against the lattice depth s and the quasimomentum k with the Gaussian defect. Here a_0 is the Bohr radius, $\epsilon = 0.05$, $\mathcal{N} = 100$, and $\sigma = 2$.

For fixed defect, the critical scattering length a_c plotted against the lattice depth s and the quasimomentum k is shown in Fig. 5. Just like the case of a single defect, we can see that the critical scattering length a_c decreases with increasing lattice depth s and fixed quasimomentum k. The dipolar gas with the Gaussian defect also requires a smaller critical scattering length a_c to preserve the superfluidity than that in the system of the nondipolar gas. In Fig. 6 we plot the critical scattering length a_c as a function of the quasimomentum k with respect to the lattice depths $s = 5$ and 9. We can see that the critical scattering length a_c decreases with quasimomentum k. We note that when $k\sigma \gg 1$, $\epsilon_{\text{eff}} \to 0$. This means that the dipolar system with a Gaussian defect and large quasimomentum will always pass through the defect. This is different from the case of a single defect, where a_c varies nonmonotonically with k (see Fig. 4). We also find that for the Gaussian defect,

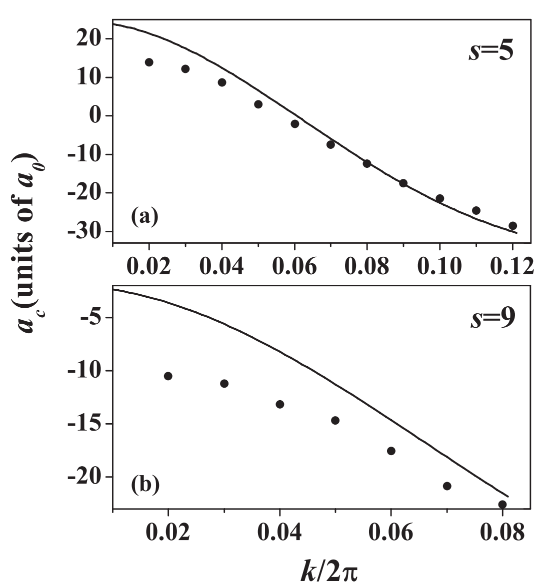

FIG. 6. Critical scattering length a_c vs the quasimomentum k with respect to (a) the shallow lattice depth $s = 5$ and (b) the deep lattice depth $s = 9$. The points indicate the numerical solutions of Eq. (1) and lines the analytical results of Eq. (19). Here a_0 is the Bohr radius, $\epsilon = 0.05$, $\mathcal{N} = 100$, and $\sigma = 2$.

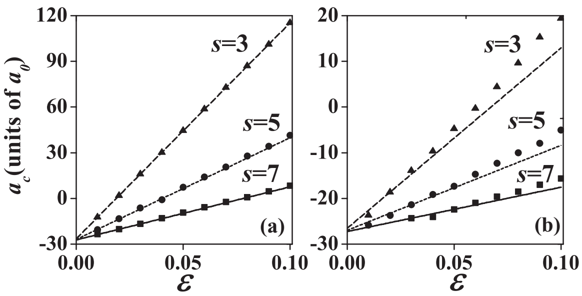

FIG. 7. Critical scattering length a_c vs defects ϵ for different lattice depths s with (a) a single defect and (b) a Gaussian defect. The points indicate the numerical simulations of Eq. (1) and the solid lines the analytical results of Eqs. (16) and (19). Here a_0 is the Bohr radius, $k/2\pi = 0.09$, $\mathcal{N} = 100$, and $\sigma = 2$.

the analytical result is in good agreement with the numerical result.

The critical scattering length a_c plotted against the defect ϵ for different lattice strengths s is shown in Fig. 7. One can find that a_c increases with increasing ϵ and decreases with increasing s. In particular, a_c for the system with a Gaussian defect is much lower than that for the system with a single defect. In contrast, the system with a Gaussian defect can more easily support the superfluid state than the system with a single defect.

V. CONCLUSION

In this work we have investigated the stability and the superfluidity of a dipolar $^{52}C_r$ condensate in a deep one-dimensional lattice. By using the perturbative and tight-binding approximation, we analyzed energetic stability and modulational stability (dynamical stability) of a dipolar condensate in a clean lattice. There is a critical scattering length and the system is stable when $a > a_c$. We showed that the system is more stable in the deep lattice regime due to the decrease of the critical scattering length with the lattice depth. Through a comparison with the nondipolar gas, it was found that it is easier for a dipolar gas to maintain stability because of the nonlocal dipolar interactions. Furthermore, the superfluidity of the dipolar condensate in a deep annular lattice with defect was discussed both analytically and numerically. Within a two-mode approximation, the dynamics of the system can be considered as a single nonrigid Hamiltonian with quasimomentum-dependent nonlinearity (induced by the nonlocal interaction). We found that the superfluid state can exist beyond a critical scattering length. The analytical expression of the critical scattering length for supporting a superfluid flow was obtained and we found that it is determined by the competition between the defect, the quasimomentum of the gas, and the nonlocal dipolar interaction. The system can easily support a superfluid state in deep lattices. In particular, the dipolar system can easily support a superfluid state with low scattering length relative to the nondipolar gas. The present results give deep insight into the dynamics of a dipolar condensate in a disordered optical lattice.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grants No. 11274255 and No. 10975114, the Natural Science Foundation of Gansu province under Grant No. 2011GS04358, and Creation of Science and Technology of Northwest Normal University, China, under Grants No. NWNU-KJCXGC-03-48 and No. NWNU-LKQN-12-12.

[1] A. C. Scott, *Nonlinear Science: Emergence and Dynamics of Coherent Structures* (Oxford University Press, Oxford, 1999).
[2] S. Flach and C. R. Willis, Phys. Rep. **295**, 182 (1998).
[3] A. Trombettoni, A. Smerzi, and A. R. Bishop, Phys. Rev. E **67**, 016607 (2003); Phys. Rev. Lett. **88**, 173902 (2002).
[4] B. Deissler *et al.*, Nat. Phys. **6**, 354 (2010).
[5] V. A. Brazhnyi and B. A. Malomed, Phys. Rev. E **83**, 016604 (2011).
[6] K. Ø. Rasmussen, T. Cretegny, P. G. Kevrekidis, and N. Grønbech-Jensen, Phys. Rev. Lett. **84**, 3740 (2000).
[7] P. L. Christiansen, Y. B. Gaididei, K. Ø. Rasmussen, V. K. Mezentsev, and J. J. Rasmussen, Phys. Rev. B **54**, 900 (1996).
[8] Y. B. Gaididei, D. Hendriksen, P. L. Christiansen, and K. Ø. Rasmussen, Phys. Rev. B **58**, 3075 (1998).
[9] P. L. Christiansen, Y. B. Gaididei, M. Johansson, and K. Ø. Rasmussen, Phys. Rev. B **55**, 5759 (1997).
[10] G. Kalosakas, K. Ø. Rasmussen, and A. R. Bishop, Phys. Rev. Lett. **89**, 030402 (2002).
[11] X.-D. Bai and J.-K. Xue, Phys. Rev. E **86**, 066605 (2012).
[12] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[13] Y. P. Chen, J. Hitchcock, D. Dries, M. Junker, C. Welford, and R. G. Hulet, Phys. Rev. A **77**, 033632 (2008).
[14] Ch. Skokos and S. Flach, Phys. Rev. E **82**, 016208 (2010).
[15] S. Flach, D. O. Krimer, and Ch. Skokos, Phys. Rev. Lett. **102**, 024101 (2009).
[16] R. A. Vicencio and S. Flach, Phys. Rev. E **79**, 016217 (2009).
[17] M. White, M. Pasienski, D. McKay, S. Q. Zhou, D. Ceperley, and B. DeMarco, Phys. Rev. Lett. **102**, 055301 (2009).
[18] U. Gavish and Y. Castin, Phys. Rev. Lett. **95**, 020401 (2005).
[19] H. Gimperlein, S. Wessel, J. Schmiedmayer, and L. Santos, Phys. Rev. Lett. **95**, 170401 (2005).
[20] S. Inouye, M. R. Andrews, J. Stenger, H.-J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, Nature (London) **392**, 151 (1998).
[21] A. Trombettoni and A. Smerzi, Phys. Rev. Lett. **86**, 2353 (2001).
[22] G. Baym and C. J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).
[23] P. M. Lushnikov, Phys. Rev. A **66**, 051601(R) (2002).
[24] L. Bergé, T. J. Alexander, and Y. S. Kivshar, Phys. Rev. A **62**, 023607 (2000).
[25] T. J. Alexander and L. Bergé, Phys. Rev. E **65**, 026611 (2002).
[26] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, Rep. Prog. Phys. **72**, 126401 (2009).
[27] M. A. Baranov, Phys. Rep. **464**, 71 (2008).
[28] M. Fattori, G. Roati, B. Deissler, C. D'Errico, M. Zaccanti, M. Jona-Lasinio, L. Santos, M. Inguscio, and G. Modugno, Phys. Rev. Lett. **101**, 190405 (2008).

[29] A.-X. Zhang and J.-K. Xue, Phys. Rev. A **82**, 013606 (2010).
[30] Z.-W. Xie, Z.-X. Cao, E. I. Kats, and W. M. Liu, Phys. Rev. A **71**, 025601 (2005).
[31] I. Tikhonenkov, B. A. Malomed, and A. Vardi, Phys. Rev. A **78**, 043614 (2008); Q. Wang, W. Chen, G. Xavier, M. Swillo, T. Zhang, S. Sauge, M. Tengner, Z.-F. Han, G.-C. Guo, and A. Karlsson, Phys. Rev. Lett. **100**, 090501 (2008).
[32] D. J. Frantzeskakis, J. Phys. A **43**, 213001 (2010).
[33] R. Nath, P. Pedri, and L. Santos, Phys. Rev. Lett. **101**, 210402 (2008).
[34] P. L. Christiansen, Y. B. Gaididei, M. Johansson, K. Ø. Rasmussen, V. K. Mezentsev, and J. J. Rasmussen, Phys. Rev. B **57**, 11303 (1998).
[35] M. Johansson, Y. B. Gaididei, P. L. Christiansen, and K. Ø. Rasmussen, Phys. Rev. E **57**, 4739 (1998).
[36] Y. B. Gaididei, S. F. Mingaleev, P. L. Christiansen, and K. Ø. Rasmussen, Phys. Rev. E **55**, 6141 (1997).
[37] V. V. Konotop and V. E. Vekslerchik, J. Phys. A **25**, 4037 (1992).
[38] S. Müller, J. Billy, E. A. L. Henn, H. Kadau, A. Griesmaier, M. Jona-Lasinio, L. Santos, and T. Pfau, Phys. Rev. A **84**, 053601 (2011).
[39] T. Koch *et al.*, Nat. Phys. **4**, 218 (2008).
[40] H. Sakaguchi and B. A. Malomed, Phys. Rev. A **84**, 033616 (2011).
[41] L. E. Young, S. P. Muruganandam, and S. K. Adhikari, J. Phys. B **44**, 101001 (2011).
[42] S. Rojas-Rojas, R. A. Vicencio, M. I. Molina, and F. Kh. Abdullaev, Phys. Rev. A **84**, 033621 (2011).
[43] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997).