Fidelity susceptibility of the quantum Ising model in a transverse field: The exact solution

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We derive an exact closed-form expression for fidelity susceptibility of the quantum Ising model in the transverse field. We also establish an exact one-to-one correspondence between fidelity susceptibility in the ferromagnetic and paramagnetic phases of this model. Elegant summation formulas are obtained as a by-product of these studies.

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I. INTRODUCTION

Quantum phase transitions happen when a small variation in an external field can fundamentally change ground-state properties of a quantum system [1,2]. They provide some of the most striking examples of the richness of many-body quantum physics. They can be studied in electronic [3], cold atom [4], and cold ion [5] systems (see Refs. [1,6] for an overview of the field). Traditional condensed matter approaches to quantum phase transitions rely on the identification of the order parameter and studies of correlation functions [1]. A different strategy has been recently proposed by the quantum information community and is known as the fidelity approach to quantum phase transitions [7,8].

Fidelity is a popular concept in quantum information science. It is defined here as the overlap between two ground states,

$$F(g,\delta) = |\langle\psi(g)|\psi(g+\delta)\rangle|,$$

where $|\psi(g)\rangle$ is a ground-state wave function of some Hamiltonian $\hat{H}(g)$, g is the external field whose variation induces a quantum phase transition, while δ is a small shift of this field. Since the ground states fundamentally change across the critical point, fidelity should have a minimum near the critical point [7]. This simple yet powerful observation is the basis of the fidelity approach to quantum phase transitions.

Recent studies suggest that fidelity is an efficient probe of quantum criticality [8]. In particular, the minimum of fidelity near a critical point has been established in several models. The scaling of fidelity with distance from the critical point $|g-g_c|$, the field shift δ, and the system size N has been shown to encode the critical exponent ν characterizing the power-law divergence of the correlation length near the critical point [7,9–11].

Fidelity has been also shown to play a crucial role in quantum phase transitions in quantum fields, where superpositions of ground states from different phases are created [12] (see also Refs. [13,14] for a different approach to quantum phase transitions in quantum fields). Moreover, fidelity has turned out to be useful in studies of the dynamics of quantum systems ranging from simple two-level [15] to many-body ones (see, e.g., Refs. [10,16] and Sec. V of Ref. [17]). Its relevance to the dynamics of decoherence of a central spin coupled to an environment undergoing a quantum phase transition has been described in Ref. [18]. There are two distinct regimes where fidelity can be studied.

The first one corresponds to the limit where the field shift δ is kept constant and the system size $N\to\infty$. In this limit one observes Anderson orthogonality catastrophe: Disappearance of the overlap of two ground states in the thermodynamic limit [19]. Quite interestingly, while the Anderson's seminal paper shows power-law decay of the overlap with the system size in a particular model that does not undergo a quantum phase transition, there is an exponential with the system size decay of the overlap, i.e., fidelity, near the generic quantum critical point [11,17,20,21] (see also Refs. [22,23]).

The second regime corresponds to the limit of the field shift $\delta\to 0$ taken at the constant system size N. In this limit fidelity is close to unity and it can be approximated by the lowest-order nontrivial Taylor expansion,

$$F(g,\delta) \simeq 1 - \chi(g)\frac{\delta^2}{2},$$

where the linear in δ term vanishes due to normalization of the ground states. The prefactor in the quadratic term, $\chi(g)$, is called fidelity susceptibility [24]. It has been recently intensively studied as a probe of quantum criticality [7–10].

II. MODEL

We will study fidelity susceptibility in the quantum Ising model in the transverse field. This is a paradigmatic model of quantum phase transitions: All the basic concepts about both equilibrium [1] and nonequilibrium [25] quantum phase transitions have been tested on this model. Its versatile experimental realization shall be possible in the nearest future in cold ion setups [26].

The Hamiltonian that we study is given by

$$\hat{H}(g) = -\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + g\sigma_i^z\right), \tag{1}$$

where g is the magnetic field and N is the number of spins. Phase diagram of this model reflects competition between the spin interactions and the magnetic field (Fig. 1). The spin interactions try to align spins in the $\pm x$ direction, while the magnetic field polarizes spins along the z direction. For large-enough magnetic fields the system is in the paramagnetic phase, while for small magnetic fields it is in the ferromagnetic phase. The two phases are separated by the critical point $g_c = 1$.

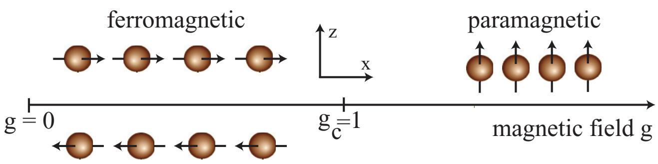

FIG. 1. (Color online) Schematic of the phase diagram of the Ising model in the transverse magnetic field.

To find the ground states of the Hamiltonian (1), we assume that the number of spins is *even* and proceed in the standard way following notation from Ref. [17]. One obtains after some calculations

$$F(g,\delta) = \prod_k \cos\frac{\theta_k(g+\delta) - \theta_k(g)}{2}, \quad \tan\theta_k(g) = \frac{\sin k}{g - \cos k},$$
$$k = \pi/N, 3\pi/N, \ldots, \pi - \pi/N. \tag{2}$$

Fidelity susceptibility then reads

$$\chi(g) = \frac{1}{4}\sum_k (\theta_k')^2 = \frac{1}{4}\sum_k \frac{\sin^2 k}{(g^2 - 2g\cos k + 1)^2}, \tag{3}$$

where the prime denotes the derivative with respect to g. Since fidelity susceptibility is symmetric around $g = 0$, it is sufficient to consider magnetic fields $g \geqslant 0$. Furthermore, it is useful to consider the vanishing magnetic field separately. We find with the help of Ref. [27] that

$$\chi(0) = \frac{N}{16}.$$

Hereafter, we assume that the magnetic field points in the $+z$ direction ($g > 0$). Moreover, we assume that $N \geqslant 4$ excluding the trivial $N = 2$ case.

There are at least four options for extracting information from Eq. (3); note that this sum appears not only in the studies of fidelity susceptibility but also in the similar studies of quantum geometric tensors [28], quantum Fischer information [29], and quantum adiabatic evolution [30]. The first option is to perform the summation numerically; see, e.g., Ref. [7]. This approach has obvious limitations. The second option is to expand the summand in the Taylor series; see, e.g., Refs. [7,10,29]. This technique is necessarily approximate, it produces results whose range of applicability is *a priori* unknown, and it cannot be deployed at arbitrary magnetic fields. The third option is to factor out small systems, replace $\sum_k$ by $\frac{N}{2\pi}\int dk$, and calculate the integral, see, e.g., Refs. [8,10,28,30,31]. The main drawback of this approach is that it produces results whose range of applicability can only be guessed. This range of applicability would be precisely known, had the "remainders" in the Euler-Maclaurin summation formula been studied [32], which is complicated and has *not* been yet done. In particular, the replacement of the sum by the integral produces meaningless results near the critical point (see the discussion below). The fourth, ultimate, option is to compute the sum exactly analytically, which we will do below.

III. EXACT SOLUTION

We start from an identity that can be found in Ref. [27],

$$\sum_k \left[\frac{\sin^2(k/2)}{\sinh(z)} + \frac{\tanh(z/2)}{2}\right]^{-1} = N\tanh\left(\frac{Nz}{2}\right), \tag{4}$$

valid for summations over the same k as in Eq. (2). Multiplying both sides of Eq. (4) by $\tanh(z/2)$ and taking the derivative of the resulting equation with respect to z, we obtain another identity,

$$\sum_k \frac{\sin^2(k/2)}{[\sinh^2(z/2) + \sin^2(k/2)]^2} = f(z),$$
$$f(z) = \frac{N}{\sinh(z)}\frac{d}{dz}\left[\tanh\left(\frac{Nz}{2}\right)\tanh\left(\frac{z}{2}\right)\right]. \tag{5}$$

Next, we multiply both sides of Eq. (5) by $\cosh^4(z/2)$ and again differentiate the resulting equation with respect to z. We obtain after some additional algebra

$$\frac{d}{dz}\sum_k \frac{\sin^2 k}{[\sinh^2(z/2) + \sin^2(k/2)]^2} = \frac{4}{\cosh^2(z/2)}\frac{d}{dz}\left[\cosh^4\left(\frac{z}{2}\right)f(z)\right]. \tag{6}$$

We then integrate Eq. (6) over z from 0 to x to obtain

$$\sum_k \frac{\sin^2 k}{[\sinh^2(x/2) + \sin^2(k/2)]^2} - 2N^2 + 2N = \frac{2N\tanh(Nx/2)}{\tanh(x)} - N^2\frac{\cosh(Nx)}{\cosh^2(Nx/2)}, \tag{7}$$

where the left-hand side is obtained with the help of

$$\sum_k \frac{1}{\sin^2(k/2)} = \lim_{z\to 0} f(z) = \frac{N^2}{2}, \tag{8}$$

following from Eq. (5). Equation (7) can now be cast into the following simple form:

$$\sum_k \frac{\sin^2 k}{[\sinh^2(x/2) + \sin^2(k/2)]^2} = \frac{N^2}{\cosh^2(Nx/2)} - 2N\left[1 - \frac{\tanh(Nx/2)}{\tanh(x)}\right]. \tag{9}$$

Substituting $x = \ln g$ into Eq. (9) and comparing the resulting expression to Eq. (3), we find

$$\chi(g) = \frac{N^2}{16g^2}\frac{g^N}{(g^N+1)^2} + \frac{N}{16g^2}\frac{g^N - g^2}{(g^N+1)(g^2-1)}. \tag{10}$$

This result is exact and remarkably simple. In particular, it works at and around the critical point, where the most interesting physics happens. For example, by rewriting

$$\frac{g^N - g^2}{g^2 - 1} = \frac{g^2}{g+1}(1 + g + \cdots + g^{N-3}), \tag{11}$$

we see that Eq. (10) is regular at $g_c = 1$.

We notice from Eq. (10) that

$$g^2\chi(g) = \left(\frac{1}{g}\right)^2\chi\left(\frac{1}{g}\right), \tag{12}$$

which can be also verified from Eq. (3). An analogical result was proposed for full fidelity in Ref. [22]. This symmetry reflects the Kramers-Wannier duality of the Ising model [33], which has not yet been discussed in the context of fidelity susceptibility.

Thus, $g^2\chi(g)$ is symmetric with respect to the

$$\text{ferromagnetic} \leftrightarrow \text{paramagnetic}, \quad g \leftrightarrow \frac{1}{g}, \tag{13}$$

mapping. Equation (12) establishes the ferromagnetic-paramagnetic duality of fidelity susceptibility: All information about fidelity susceptibility is contained in one of the phases and can be uniquely mapped to the other phase. We will now simplify Eq. (10).

We introduce another variable to properly organize the following discussion:

$$y = N \ln g, \quad g = \exp\left(\frac{y}{N}\right).$$

Its physical meaning is simple,

$$y \sim \mathrm{sgn}(g-1)\frac{N}{\xi(g)},$$

where $\xi(g)$ is the correlation length of the infinite Ising chain in the transverse field g. Note that we used the exact expression for the correlation length, $\xi(g) \sim 1/|\ln g|$, instead of the approximate one, $\xi(g) \sim 1/|g-1|$, valid near the critical point. The prefactors in both expressions are of the order of unity and depend on the type of spin correlations used to define the correlation length [34]. The ferromagnetic-paramagnetic mapping (13) is equivalent to the

$$y \leftrightarrow -y$$

mapping.

A. Away from the critical point

We define that the system is away from the critical point when

$$|y| > 1.$$

There are two advantages of the reparametrization of χ in terms of y. First, while $|g-1|$ is bounded from above in the ferromagnetic phase, $|y|$ can be arbitrarily large in both phases, making it a good parameter to consider the far-away limit. Second, it is also a convenient variable for the actual calculations (see the Appendix).

We start by rewriting fidelity susceptibility on the paramagnetic side, $g > 1$, as

$$\chi(g) = \frac{N}{16g^2(g^2-1)} + R(g), \tag{14}$$

$$R(g) = \frac{N}{16g^2(1+g^N)}\left(\frac{Ng^N}{1+g^N} - \frac{g^2+1}{g^2-1}\right), \tag{15}$$

$$\left|\frac{R}{\chi - R}\right| \leqslant 2e^2 y \, \exp\left[-y + \frac{2(y-N)}{N}\Theta(y-N)\right]. \tag{16}$$

Such splitting of fidelity susceptibility into the leading term and remainder $R(g)$ is exact. While the leading term was known before [8,28,31], remainder (15) was unknown. Knowledge of the remainder is crucial for determining where $\chi(g)$ can be approximated by the leading term. Bound (16), proven in the Appendix for $y \geqslant 1$, shows that for $y \gg 1$ the remainder is negligible (see also Fig. 2).

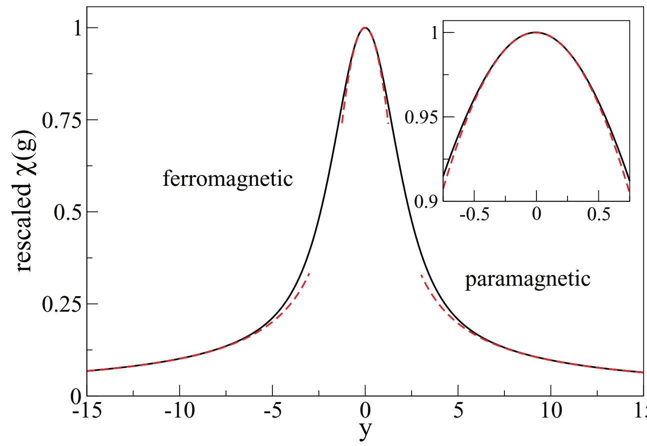

FIG. 2. (Color online) Rescaled fidelity susceptibility. Solid black line: Exact result (10). Red dashed lines: Approximations near and away from the critical point: Eqs. (19), (14), and (17) without remainders. The inset shows enlargement of the central part of the plot. The system size $N = 1000$. Fidelity susceptibility is rescaled by the $N(N-1)/32$ factor.

The leading term was known before because it follows from the above-mentioned replacement of the sum by the integral in Eq. (3). Note that without knowing the range of applicability of this approximation, one may draw a wrong conclusion about singularity of fidelity susceptibility near the critical point, i.e., in the limit of $g \to g_c = 1$ (see, e.g., Ref. [28] where one of the diagonal elements of the quantum geometric tensor is equal to fidelity susceptibility that we study). Our result shows why the singularity is absent: The singularity of the leading term in Eq. (14) is exactly canceled by the divergent part of remainder (15). This can be verified using Eq. (11). Thus, fidelity susceptibility near the critical point is regular rather than singular, which we will discuss below.

Using the ferromagnetic-paramagnetic duality (12), we readily obtain fidelity susceptibility on the ferromagnetic side, $0 < g < 1$,

$$\chi(g) = \frac{\chi(1/g)}{g^4} = \frac{N}{16(1-g^2)} + \tilde{R}, \quad \tilde{R} = \frac{R(1/g)}{g^4}, \tag{17}$$

and analogical remarks to the ones formulated to discuss Eqs. (14)–(16) apply here. In particular, bound on $|\tilde{R}/(\chi - \tilde{R})|$ is the same as in Eq. (16) after replacing y by $-y$. Despite differences in the functional form of the leading terms on both sides of the critical point—Eqs. (14) and (17)—the two expressions are in fact two sides of the same coin due to symmetry (12).

B. Near the critical point

We define that the system is near the critical point when

$$|y| < 1. \tag{18}$$

We start by noting that fidelity susceptibility does not have maximum at the critical point $g_c = 1$. This is a consequence of ferromagnetic-paramagnetic duality (12) allowing us to write

$$\chi(g) = \frac{h(g)}{g^2}, \quad h(g) = h\left(\frac{1}{g}\right).$$

From the symmetry of $h(g)$, we see that $h(g)$ has an extremum at $g_c = 1$ (as it is not the constant function here). This implies that $\chi(g)$ cannot have maximum at g_c due to the $1/g^2$ factor shifting the maximum to $g_m < g_c$.

To simplify the exact result for fidelity susceptibility near the critical point, we Taylor-expand $h(g(y))$ in y and keep the $1/g^2$ factor unchanged. Such an approximation is welcome for two reasons. First, it is efficient because $h(g(y))$ is an even function of y due to the $g \leftrightarrow 1/g$ symmetry of $h(g)$: odd expansion terms vanish. Second, it preserves symmetry (12). We get

$$\chi(g) = \frac{N(N-1)}{32g^2}\left[1 - \frac{N+1}{N}\frac{(N\ln g)^2}{6} + r(g)\right], \tag{19}$$

$$0 \leqslant r(g) \leqslant \frac{N}{N-1}\frac{(N\ln g)^4}{40}, \tag{20}$$

where $N\ln g = y$ is a small parameter near the critical point (18). We see from bound (20) that for $|y| \ll 1$ remainder $r(g)$ is negligible; $r(g)$ can be obtained by comparing Eqs. (10) and (19). The bound on $r(g)$ is derived in the Appendix. The lower bound is valid for any y, while the upper bound is valid for $|y| \leqslant \sqrt{10}$. The closest result to Eq. (19) was published in Ref. [29]. Apart from lacking the remainder, it suffers from computational errors that we list in Ref. [35].

Right at the critical point, Eq. (19) predicts

$$\chi(g_c = 1) = \frac{N^2}{32} - \frac{N}{32}. \tag{21}$$

This has to be compared to predictions of the scaling theory of fidelity susceptibility proposing that the leading (in the system size) contribution to $\chi(g_c)$ is proportional to N^2 [9,10]. Note that both the leading and the subleading term is exactly captured by Eq. (21). We mention also that the duality symmetry implies that

$$\chi'(g_c = 1) = -2\chi(g_c = 1) = -\frac{N^2}{16} + \frac{N}{16} \neq 0,$$

which explicitly shows that there is no extremum of fidelity susceptibility at the critical point.

Slightly away from the critical point, we enter the regime where the term $\sim(N\ln g)^2$ starts to play a role. This term is responsible for shifting the maximum of fidelity susceptibility away from the critical point. Indeed, forgetting about the remainder in Eq. (19) and solving equation $\chi'(g_m) = 0$, we obtain (see also Fig. 3)

$$g_m = 1 - \frac{6}{N^2} + \frac{6}{N^3} + O(N^{-4}). \tag{22}$$

The limit of $N \gg 1$ was assumed to simplify this result; analogical calculation done on the exact expression (10) predicts the same in the considered order. This shows that indeed the maximum of fidelity susceptibility is located near the critical point on the ferromagnetic side, which we proposed

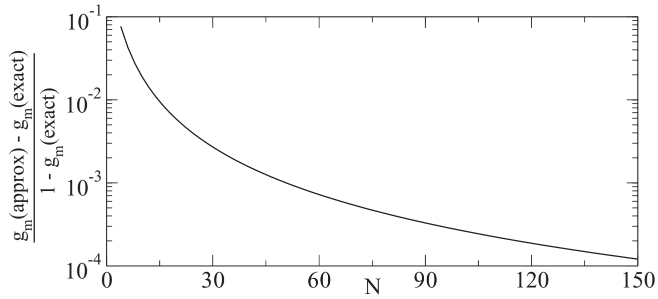

FIG. 3. Illustration of accuracy of the approximate expression for the position of maximum of fidelity susceptibility. g_m(approx) is given by Eq. (22), while g_m(exact) comes from numerical studies of the exact expression (10). For small system sizes N the shift of the maximum from the critical point predicted by Eq. (22) is off by a few percent and quickly decays with the system size.

from symmetry (12). The maximum moves towards the critical point when N increases.

Moving further from the critical point, we stay in the regime where the remainder is still negligible, while the term $\sim(N\ln g)^2$ describes decrease of fidelity susceptibility from the maximum (Fig. 2). Equation (19) shows then how the scaling prediction near the critical point, $\chi \sim N^2$, gradually breaks down. Finally, for $|N\ln g| = |y| \sim 1$ the remainder becomes non-negligible and the crossover to the far away limit begins.

IV. SUMMARY

Summarizing, we have derived an exact closed-form expression for fidelity susceptibility of the quantum Ising model in the transverse field—the quantity that has been intensively "approximately" studied in different contexts over the last couple of years [7,8,10,28–31]. Moreover, we have found an exact symmetry of fidelity susceptibility showing that all information about it is contained in one of the phases and can be easily mapped to the other phase. This symmetry follows from the Kramers-Wannier duality of the Ising model [33]. These two results allow for an elegant illustration of the fidelity approach to quantum phase transitions in arguably the most important model of a quantum phase transition. They also pave the way for analytical improvements of several earlier studies involving similar summations; see, e.g., Refs. [29,30]. Finally, they should stimulate analytical investigations of other exactly solvable models.

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APPENDIX

We bound below the remainders in Eqs. (14) and (19).

1. Away from the critical point

We will bound here $|R/(\chi - R)|$. This is conveniently done after first replacing g by $\exp(y/N)$,

$$\frac{R}{\chi - R} = N\frac{\exp(2y/N)-1}{[\exp(y/2)+\exp(-y/2)]^2} - \frac{1+\exp(2y/N)}{1+\exp(y)}.$$

Assuming that $y \geqslant 1$ and $N > 2$,

$$\left|\frac{R}{\chi - R}\right| \leqslant N\frac{\exp(2y/N)-1}{[\exp(y/2)+\exp(-y/2)]^2} + \frac{1+\exp(2y/N)}{1+\exp(y)},$$

after applying the standard inequality, $|a-b| \leqslant |a| + |b|$. Note that it is sufficient to study only positive y, i.e., to focus on the paramagnetic side, thanks to the duality symmetry.

The subsequent "bounding" proceeds as follows:

$$\begin{aligned} &N\frac{\exp(2y/N)-1}{[\exp(y/2)+\exp(-y/2)]^2} \\ &\quad\leqslant N[\exp(2y/N)-1]\exp(-y) \\ &\quad\leqslant (e^2-1)y\exp\left[-y+\frac{2(y-N)}{N}\Theta(y-N)\right]. \end{aligned}$$

The first step above is obvious. To perform the second step, we use the following inequalities from Ref. [36]:

$$\begin{aligned} a^r - b^r &< r(a-b)b^{r-1} \quad \text{for} \quad 0 < r < 1, \\ a^r - b^r &< r(a-b)a^{r-1} \quad \text{for} \quad r > 1, \end{aligned}$$

which are valid for positive and unequal a and b (equality happens for $r = 0$, $r = 1$, or $a = b$). We substituted $a = e^2$, $b = 1$, and $r = y/N$ to bound $\exp(2y/N) - 1$.

We then bound

$$\frac{1+\exp(2y/N)}{1+\exp(y)} \leqslant [\exp(2y/N)+1]\exp(-y).$$

Next we combine the above results and proceed as follows.

For $1 \leqslant y \leqslant N$ we get

$$\begin{aligned} \left|\frac{R}{\chi - R}\right| &\leqslant y\ \exp(-y)\left[e^2 - 1 + \frac{1+\exp(2y/N)}{y}\right] \\ &\leqslant 2e^2 y\exp(-y). \end{aligned}$$

For $y \geqslant N$ we get

$$\begin{aligned} \left|\frac{R}{\chi - R}\right| &\leqslant y\ \exp(-y+2y/N)\left[\frac{e^2-1}{e^2} + \frac{1+\exp(-2y/N)}{y}\right] \\ &\leqslant 2y\exp(-y+2y/N). \end{aligned}$$

Combining these two bounds we get

$$\left|\frac{R}{\chi - R}\right| \leqslant 2e^2 y\exp\left[-y+\frac{2\,(y-N)}{N}\Theta(y-N)\right].$$

Equality in this bound is reached only at $y = \infty$.

2. Near the critical point

We will bound remainder r using the following inequalities:

$$\begin{aligned} \text{I:}\ \tanh(x) &> \frac{x}{1+x^2/3}, \\ \text{II:}\ \tanh(x) &< x, \\ \text{III:}\ \tanh^2(x) &> x^2 - \frac{2x^4}{3}, \\ \text{IV:}\ \tanh(x) &< x - \frac{x^3}{3} + \frac{2x^5}{15}, \\ \text{V:}\ \frac{1}{\tanh(x)} &> \frac{1}{x} + \frac{x}{3} - \frac{2x^3}{15}. \end{aligned}$$

All of them are valid for $x > 0$ of interest in our calculations (equalities hold at $x = 0$).

Inequality I follows from the discussion presented in Sec. 3.6.13 of Ref. [37]. Inequality II is proven by considering

$$q(x) = x - \tanh(x) \quad \Rightarrow \quad q'(x) = \tanh^2(x).$$

Since $q(0) = 0$ and $q'(x > 0) > 0$, we have $q(x > 0) > 0$, which establishes inequality II. Inequality III follows from

$$\tanh^2(x) - x^2 + \frac{2x^4}{3} > \frac{x^6(9+2x^2)}{3(3+x^2)^2} > 0,$$

where inequality I was employed to bound $\tanh^2(x)$. Inequality IV is proven by considering

$$\begin{aligned} q(x) &= x - \frac{x^3}{3} + \frac{2x^5}{15} - \tanh(x) \\ &\Rightarrow \quad q'(x) = \tanh^2(x) - x^2 + \frac{2x^4}{3} > 0, \end{aligned}$$

where the bound for $q'(x)$ follows from the proof of inequality III. Since $q(0) = 0$ and $q'(x > 0) > 0$, we have $q(x > 0) > 0$, which establishes inequality IV. Inequality V straightforwardly follows from inequality IV.

To bound remainder

$$\begin{aligned} r = {}& \frac{2N}{N-1}\frac{g^N}{(1+g^N)^2} + \frac{2}{N-1}\frac{g^N-g^2}{(1+g^N)(g^2-1)} \\ &+ \frac{N(N+1)}{6}(\ln g)^2 - 1, \end{aligned}$$

we replace g by $\exp(y/N)$ and rewrite the resulting expression to the form

$$\begin{aligned} r = {}& -\frac{N\tanh^2(y/2)}{2(N-1)} + \frac{\tanh(y/2)}{(N-1)\tanh(y/N)} + \frac{N+1}{N}\frac{y^2}{6} \\ &- \frac{N}{2(N-1)}. \end{aligned}$$

Note that the remainder is an even function of y thanks to the duality symmetry.

We bound the remainder from above by bounding $\tanh^2(y/2)$, $\tanh(y/2)$, and $\tanh(y/N)$ with inequalities III, IV, and I, respectively. It leads to

$$r \leqslant \frac{y^4}{720}\frac{18N^2 - 10 + y^2}{N(N-1)} \leqslant \frac{N}{N-1}\frac{y^4}{40},$$

where the last step is valid for $|y| \leqslant \sqrt{10}$.

Next, we show that the remainder is bounded from below by zero by bounding $\tanh^2(y/2)$, $\tanh(y/2)$, and $1/\tanh(y/N)$

with inequalities II, I, and V, respectively. This gives us

$$r \geqslant \frac{y^4}{120} \frac{5N^4 - 20N^2 - 96}{N^3(N-1)(12+y^2)} \geqslant 0$$

for

$$N \geqslant \sqrt{2 + \frac{2}{5}\sqrt{145}} = 2.6108\ldots.$$

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[35] Section III B of Ref. [29] focuses on studies of quantum Fisher information at zero temperature, G_J. After setting $J = 1, h = g$, and $L = N$ in Eq. (14) of Ref. [29], we find that $\chi(g) = G_J(g)/4g^2$. G_J is expanded in Ref. [29] in the Taylor series in $N(g-1)$ being similar to our $N \ln g$. To get the final result similar to our Eq. (19), one needs

$$\sum_k \cot^2(k/2) = \frac{N^2}{2} - \frac{N}{2}, \quad \sum_k \frac{\cot^2(k/2)}{\sin^2(k/2)} = \frac{N^4}{6} - \frac{N^2}{6},$$

valid for summations over the same k as in Eq. (2). For the first sum authors of Ref. [29] claim that there is an additional $O(N^0)$ term with respect to the above result, while the expression that they list for the second sum is incorrect. We got the first sum above from Eq. (8). The second sum is obtained after noting that it equals $\sum_k \sin^{-4}(k/2) - \sum_k \sin^{-2}(k/2)$, where $\sum_k \sin^{-2}(k/2)$ is given by Eq. (8), while

$$\sum_k \sin^{-4}(k/2) = \frac{N^4}{6} + \frac{N^2}{3}. \quad (2)$$

This comes from multiplying Eq. (4) by $1/\sinh(z)$, then calculating derivative of the resulting expression with respect to z and, finally, after rearrangement of terms, from taking the limit of $z \to 0$.
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