Subdiffusive rocking ratchets in viscoelastic media: Transport optimization and thermodynamic efficiency in overdamped regime

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(Received 11 March 2013; published 15 May 2013)

We study subdiffusive overdamped Brownian ratchets periodically rocked by an external zero-mean force in viscoelastic media within the framework of a non-Markovian generalized Langevin equation approach and associated multidimensional Markovian embedding dynamics. Viscoelastic deformations of the medium caused by the transport particle are modeled by a set of auxiliary Brownian quasiparticles elastically coupled to the transport particle and characterized by a hierarchy of relaxation times which obey a fractal scaling. The most slowly relaxing deformations which cannot immediately follow to the moving particle imprint long-range memory about its previous positions and cause subdiffusion and anomalous transport on a sufficiently long time scale. This anomalous behavior is combined with normal diffusion and transport on an initial time scale of overdamped motion. Anomalously slow directed transport in a periodic ratchet potential with broken space inversion symmetry emerges due to a violation of the thermal detailed balance by a zero-mean periodic driving and is optimized with frequency of driving, its amplitude, and temperature. Such optimized anomalous transport can be low dispersive and characterized by a large generalized Peclet number. Moreover, we show that overdamped subdiffusive ratchets can sustain a substantial load and do useful work. The corresponding thermodynamic efficiency decays algebraically in time since the useful work done against a load scales sublinearly with time following to the transport particle position, but the energy pumped by an external force scales with time linearly. Nevertheless, it can be transiently appreciably high and compare well with the thermodynamical efficiency of the normal diffusion overdamped ratchets on sufficiently long temporal and spatial scales.

DOI: 10.1103/PhysRevE.87.052119 PACS number(s): 05.40.−a, 05.10.Gg, 87.16.Uv

I. INTRODUCTION

Brownian motors can be described as prototype models for the biomolecular motors in biological cells that work out of thermal equilibrium [1–3]. Most research in this area is devoted to normal classical ratchet transport, where both the mean position and the position variance grow linearly in time, i.e., $\langle \delta x(t)\rangle \propto t$ and $\langle \delta x^2(t)\rangle \propto t$, respectively. At the same time, anomalous diffusion and transport [4–6] become increasingly popular as indicated by the huge literature produced during the past 20 years (see, e.g., recent overview papers in Ref. [7] and Fig. 1 in Ref. [8] therein). Experimental works have discovered that both viscoelasticity of dense polymer solutions [9–11] and colloids [12,13] such as, e.g., cytosol of biological cells [14–19], and spatial inhomogeneity and disorder [20,21] in such media can entail subdiffusion, $\langle \delta x^2(t)\rangle \propto t^\alpha$, with a power-law exponent $0 < \alpha < 1$. This naturally inspires the question on how natural molecular motors can operate in such crowded environments featured by a large macroscopic (zero-frequency) viscosity which causes subdiffusion of macromolecules on a transient mesoscopic spatial (several micrometers) and time (up to several minutes) scales [15,22,23]. In fact, such transient transport phenomena are faster on mesoscopic scales than one expects [22,23] for such highly viscous media from their effective macroscopic viscosity coefficient, which depends on the Brownian particle size [24,25]. Clarifying the problem requires, first, generalizing well-known toy Brownian ratchets models such as a rocking ratchet or flashing ratchet towards anomalous viscoelastic dynamics with memory. The first related steps were done recently in Refs. [26,27] for rocking ratchets and in Ref. [28] for flashing ratchets within the framework of a nonlinear generalized Langevin equation (GLE) approach [29,30] applied to viscoelastic stochastic dynamics with memory within a generalized Maxwell-Langevin Markovian multidimensional embedding dynamics [31,32]. In particular, in Ref. [27], it has been shown that such viscoelastic rocking ratchets are genuine ratchets capable to sustain a sufficient load in the direction opposite to rectified motion and perform thus a useful work. Moreover, an optimal subdiffusive ratchet transport reflects synchronization between the potential periodic tilts and advancing the Brownian particle over one or two spatial substrate periods in the transport direction. Such a synchronization can be interpreted as stochastic resonance occurring in a highly non-Markovian dynamics on a thermal noise intensity variation [27]. Furthermore, a synchronization between the potential periodic flashes and nonlinear oscillations within the potential wells is responsible for an optimization of the ratchet transport in the case of flashing ratchets [28]. Clearly, similar features are simply impossible within an alternative subdiffusive transport mechanism based on continuous time random walks (CTRWs) with divergent mean residence times in traps or associated

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fractional Fokker-Planck dynamics (FFPD) [6]. Moreover, such anomalous transport can be low dispersive, with a large or even diverging (for vanishingly small temperature, cf. in Ref. [28]) generalized Peclet number, which presents the ratio of the scaled subvelocity and subdiffusion coefficient. This is in sharp contrast with highly dispersive CTRW and FFPD transport featured by a vanishing generalized Peclet number [32]. In spite of these recent advances, many important fundamental questions remain open. In particular, in our previous works [26–28] the inertial effects in the anomalous ratchet transport were very essential. Will the anomalous ratchet transport persist also in the overdamped limit, where the inertial effects are entirely neglected? This is the first important question which is answered in the affirmative in this work. The second important question concerns thermodynamic efficiency of such anomalous isothermal anomalous Brownian motors, namely which portion of the energy provided by an external field for transport can be transformed into useful work against a load? Do the conventional notions of power, i.e., the work done per unit of time, remain meaningful for anomalous transport or does it requires a generalization? This is one of the fundamental questions about anomalous ratchet transport which we address in this work.

In this article, we study subdiffusive *overdamped* Brownian motors operating in viscoelastic media and periodically rocked by an external force. It will be shown that a subdiffusive current can be optimized with the driving frequency and temperature also in the overdamped limit. Such anomalous ratchets can be characterized by a very good transport quality (coherence) at sufficiently low temperatures in spite of the anomalous character of transport. Studying dependence of subvelocity on load we will show that the considered Brownian ratchet is a genuine one. It does a useful work against a load and we find the corresponding thermodynamic efficiency which turns out to be a slowly decaying function of time. This is because the energy pumped into rectified motion scales linearly with time, but the subdiffusive transport is sublinear. Nevertheless, this efficiency is not vanishingly small and it compares well with the efficiency of the corresponding normal diffusion ratchets on the time scale of simulations, which is a surprise.

II. MODEL

Let us consider the following model of anomalous Brownian motion. A Brownian particle moving with velocity $dx(t)/dt \equiv \dot{x}(t)$ in a dense water solution of polymers (e.g., cytosol of biological cells) experiences Stokes memoryless friction with friction coefficient η_0 (corresponding to water) and, in addition, a frequency-dependent friction with memory which is characterized by a fractional friction coefficient η_α. It corresponds to a polymeric fluid. Considering a general case of linear friction with memory, $f_{\rm mem}(t) = -\int_0^t \eta(t-t')\dot{x}(t')dt'$, for a particle starting its motion at $t_0 = 0$, the normal contribution corresponds to the memory kernel $\eta(t) = 2\eta_0\delta(t)$ and frictional force, $f_{\rm Stokes}(t) = -\eta_0 dx(t)/dt$, while an anomalous one emerges for $\eta(t) = \eta_\alpha t^{-\alpha}/\Gamma(1-\alpha)$, where $0 < \alpha < 1$ and $\Gamma(\cdot)$ is a Γ function. The corresponding term with memory, which captures, e.g., viscoelastic effects, can be abbreviated as $f_\alpha(t) = -\eta_\alpha d^\alpha x(t)/dt^\alpha$ using the notion of fractional Caputo derivative, just per its definition [33]. In accordance with the second fluctuation-dissipation theorem (FDT) [29], these dissipative forces are complemented by the corresponding mutually independent thermal fluctuation forces $\xi_0(t)$ and $\xi_\alpha(t)$, which are Gaussian, zero-mean, and completely characterized by the autocorrelation functions $\langle \xi_0(t)\xi_0(t')\rangle = 2k_BT\eta_0\delta(t-t')$ and

$$\langle \xi_\alpha(t)\xi_\alpha(t')\rangle = k_BT\eta_\alpha|t-t'|^{-\alpha}/\Gamma(1-\alpha), \qquad (1)$$

at the environmental temperature T. In the presence of external force field $f(x,t) = -\partial V(x,t)/\partial x$ the motion of an overdamped Brownian particle is described by an overdamped generalized Langevin equation (GLE) [29,30],

$$\eta_0\frac{dx}{dt} + \eta_\alpha\frac{d^\alpha x}{dt^\alpha} = f(x,t) + \xi_0(t) + \xi_\alpha(t). \qquad (2)$$

Furthermore, we shall consider a spatially asymmetric periodic ratchet potential [34],

$$U(x) = -U_0\left[\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{4}\sin\left(\frac{4\pi x}{L}\right)\right], \qquad (3)$$

with amplitude U_0 and spatial period L. The motion is driven also by an external periodic force $f_{\rm ext}(t) = A\cos(\Omega t)$ with amplitude A and driving frequency Ω. The FDT (1) ensures that the energy dissipated is always balanced by the energy gained from the environment at the thermal equilibrium ($f_{\rm ext} \to 0$) so there is no net heat exchange and the kinetic degree of freedom has energy $k_BT/2$ on average. The external force $f_{\rm ext}$ is expected to violate thermal detailed balance and cause a net directed motion of the Brownian particles beyond thermal equilibrium. This net motion can be directed against a loading in the opposite direction constant force f_0 and do a useful work against such a load. Altogether, $V(x,t) = U(x) - f_{\rm ext}(t)x + f_0x$.

Let us scale further the coordinate x in the units of L and time t in the units of $\tau_r = (4\pi^2 U_0/L^2\eta_\alpha)^{-1/\alpha}$. In these units, the GLE (2) reads

$$\eta_0\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \frac{1}{2\pi}[f(x,t) + \sqrt{2T\eta_0}\zeta_0(t) + \sqrt{T/\Gamma(1-\alpha)}\zeta_\alpha(t)], \qquad (4)$$

where the friction coefficient η_0 is scaled in the units of $\eta_\alpha\tau_r^{1-\alpha}$, temperature T in the units of U_0/k_B and

$$f(x,t) = \cos(2\pi x) + (1/2)\cos(4\pi x) + A\cos(\Omega t) - f_0, \qquad (5)$$

where A and f_0 are scaled in the units of $2\pi U_0/L$. Moreover, $\langle\zeta_0(t)\zeta_0(t')\rangle = \delta(t-t')$, and $\langle\zeta_\alpha(t)\zeta_\alpha(t')\rangle = 1/|t-t'|^\alpha$ in these units. Next, we follow to the road of Markovian embedding in Refs. [31,32] and approximate the memory kernel by a sum of exponentials, $\eta(t) = \sum_{i=1}^N k_i\exp(-\nu_i t)$ and the corresponding noise $\xi_\alpha(t)$ by a sum of Ornstein-Uhlenbeck processes. By choosing the spectrum of relaxation rates scaled as $\nu_i = \nu_0/b^{i-1}$ via a maximal relaxation rate ν_0 and a scaling parameter $b > 1$ the corresponding power dependence $t^{-\alpha}$ can be nicely approximated over about $r = N\log_{10}b - 2$ time decades between two time cutoffs $\tau_l = \nu_0^{-1}$ and $\tau_h = b^{N-1}\tau_l$. To ensure a power-law scaling one chooses $k_i \propto \nu_i^\alpha$, or $k_i = C_\alpha(b)\nu_i^\alpha/\Gamma(1-\alpha)$, where $C_\alpha(b)$ is a numerical fitting constant which mostly depends on α and b for a sufficiently large r and N. Weak dependencies of r on N and b ensure a very powerful numerical approach to integrate fractional stochastic non-Markovian dynamics with a well-controlled numerical accuracy (of several percentages in this work). Alternatively, it can be considered an independent approach to anomalous transport which is even not bounded by a strict requirement on the power scaling. Approximation of the memory kernel by a sum of exponentials can be derived from an experiment; see a practical example in Fig. 3 in Ref. [32], where the sum of just four exponentials suffices to fit a power-law memory kernel extending over four time decades. This approach allows also for a vivid physical interpretation in terms of viscoelastic forces $u_i = -k_i(x - x_i)$ caused by overdamped Brownian particles modeling viscoelastic degrees of freedom of the environment and corresponding to viscoelastic deformations of the medium (principal modes). These auxiliary quasiparticles are elastically attached to the transport particle with spring constants k_i and subjected to Stokes frictional forces with frictional constants $\eta_i = k_i/\nu_i$. This leads to the following Markovian embedding dynamics with uncorrelated white noise sources, $\langle \zeta_i(t)\zeta_j(t')\rangle = \delta_{ij}\delta(t - t')$:

$$\begin{aligned}\eta_0 \dot{x} &= \frac{1}{2\pi} f(x,t) - \sum_{i=1}^{N} k_i(x - x_i) + \frac{\sqrt{2T\eta_0}}{2\pi}\zeta_0(t),\\ \eta_i \dot{x}_i &= k_i(x - x_i) + \frac{1}{2\pi}\sqrt{2T\eta_i}\zeta_i(t).\end{aligned} \tag{6}$$

Initial $x_i(0)$ must be thermally (Gaussian) distributed around $x(0)$, with $\langle [x_i(0) - x(0)]^2\rangle = T/k_i$, in order to have complete equivalence with above GLE description for a memory kernel being a sum of exponentials, which is, of course, an approximation to the considered power-law memory kernel [32]. The accuracy of this approximation is, however, well controlled. We do the corresponding sampling of $x_i(0)$ below. Otherwise, there would be a transient force present in GLE reflecting thermal equilibration of the medium disturbed initially by the Brownian particle (e.g., on its insertion) or corresponding aging effects [32]. The general presence of such transients adds a flexibility to this modeling approach. However, we shall not consider such transient aging effects because we are interested in an asymptotic transport regime, which is not influenced by initial transients. Thereafter we fix $\alpha = 1/2$ and choose $b = 10$, $C_{1/2}(10) = 1.3$, and $\nu_0 = 100$. Most results presented below were obtained using an ensemble averaging over 10^4 trajectories. Simulations are done with the help of the stochastic Heun method [35] on the graphical processor units (GPUs) with double precision [36]. This technique provided an effective acceleration of numerics by a factor of about 100 for the studied system over the standard CPUs computing on modern commodity processors. The total integration time of GLE (6) was varied in the interval $t_{\text{total}} \in [3 \times 10^5 \cdots 10^6]$ and the time step was $\Delta t = 2 \times 10^{-3}$. The number of auxiliary particles was fixed to $N = 12$.

As discussed previously [28,31,32], N auxiliary particles can be roughly divided into the groups of fast, N_f, and slow $N_s = N - N_f$ particles. This division can be made on comparison of the mean time of transitions made by central Brownian particle to the neighboring potential well with the relaxation times ν_i^{-1} of the corresponding viscoelastic force components u_i. Fast medium's deformations move together with the Brownian particle, forming together a quasiparticle. It reminds polaron in condensed matter physics, i.e., a naked particle plus deformation of its nearest environment which are considered together as a compound particle. The mean viscoelastic force created by such a nearest-neighbor environment equals to zero, on average, on the time scale of slow motion. However, the most sluggish deformations temporally imprint the medium's memory about the previous particle's positions and create a quasielastic slow varying retarding force acting in the direction opposite to the transport direction. These viscoelastic deformations introduce long-lived negative correlations in the particle displacements. Such a mechanism leads to anomalously slow diffusion and transport with $\langle \delta x^2(t)\rangle \propto t^\alpha$ and $\langle x(t)\rangle \propto t^\alpha$, respectively, where $\langle \delta x^2(t)\rangle = \langle x^2(t)\rangle - \langle x(t)\rangle^2$. The corresponding subdiffusion coefficient D_α and the subvelocity v_α are defined as follows:

$$\begin{aligned}D_\alpha &= \frac{1}{2}\Gamma(1+\alpha)\lim_{t\to\infty}\frac{\langle \delta x^2(t)\rangle}{t^\alpha},\\ v_\alpha &= \Gamma(1+\alpha)\lim_{t\to\infty}\frac{\langle x(t)\rangle}{t^\alpha}.\end{aligned} \tag{7}$$

The limit should be understood as a physical limit in the following sense: t is large but yet much smaller than the time cutoff $\tau_0 b^{N-1}$. The latter time scale is made not attainable (and, thus, irrelevant) in our simulations. To characterize the coherence and the quality of the transport we shall use a generalized Peclet number $\mathrm{Pe}_\alpha = v_\alpha L/D_\alpha$ [26]. Such a Peclet number is a natural quantifier for the coherence quality of stochastic transport in periodic potentials. It measures the ratio of the mean traveling distance $\langle x(t)\rangle$ (in units of L) to the diffusional spread $\langle \delta x^2(t)\rangle$ (in units of L^2) [37].

III. RESULTS AND DISCUSSION

First, we tested numerics and compared the results for the ensemble-averaged position variance $\langle \delta x^2(t)\rangle$ with the exact result in the absence of potential which can be readily obtained from a general expression for the considered particular case of GLE, see in Ref. [32], using the Laplace transform of generalized Mittag-Leffler function from Ref. [38]. The result reads (in the original nonscaled units),

$$\langle \delta x^2(t)\rangle = 2D_0 t E_{1-\alpha,2}[-(t/\tau_0)^{1-\alpha}], \tag{8}$$

where $E_{a,b}(z) := \sum_0^\infty z^n/\Gamma(an+b)$ is a generalized Mittag-Leffler function, $D_0 = k_B T/\eta_0$ is a normal diffusion coefficient, and $\tau_0 = (\eta_0/\eta_\alpha)^{1/(1-\alpha)}$ is a transient time constant. For a small argument, $E_{a,b}(z \ll 1) \approx 1$, and for a large argument, $E_{a,b}(z \gg 1) \sim -1/[\Gamma(b-a)z]$, in the leading order of z^{-1}. Hence, $E_{1-\alpha,2}(-z^{1-\alpha}) \sim z^{\alpha-1}/\Gamma(1+\alpha)$, for $z \gg 1$. This result shows that at small times, $t \ll \tau_0$, the diffusion is normal, $\langle \delta x^2(t)\rangle \approx 2D_0 t$, whereas at large times, $t \gg \tau_0$, it becomes anomalously slow, $\langle \delta x^2(t)\rangle \approx 2D_\alpha t^\alpha/\Gamma(1+\alpha)$, with $D_\alpha = k_B T/\eta_\alpha$. τ_0 defines a characteristic time separating these two different regimes. Notice that it scales as $\eta_0^{1/(1-\alpha)}$ with η_0. For $\alpha = 1/2$, this general result can be expressed in terms of complementary error function and a power-law dependence,

$$\langle \delta x^2(t)\rangle = 2D_\alpha\left\{2\sqrt{\frac{t}{\pi}} + \sqrt{\tau_0}\left[e^{t/\tau_0}\mathrm{erfc}\left(\sqrt{\frac{t}{\tau_0}}\right) - 1\right]\right\}. \tag{9}$$

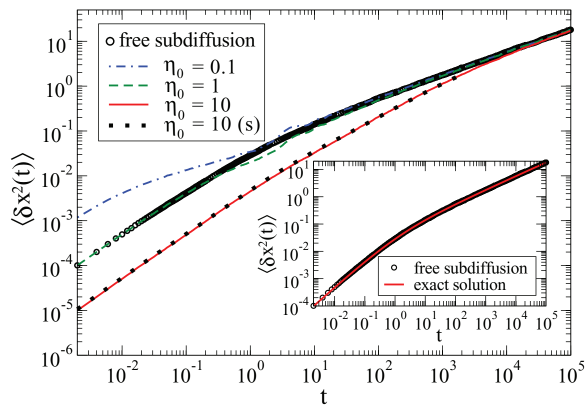

FIG. 1. (Color online) Subdiffusion in driven ratchet potential for various η_0, $\Omega = 1.0$, $A = 1.0$, and $T = 1.0$: solid, dashed, and dash-dotted curves correspond to ensemble averaging; the dotted curve (s) relates to a single trajectory averaging in Eq. (10) and the agreement indicates ergodicity. The insert demonstrates an excellent agreement between the analytical result in Eqs. (8) and (9), and numerics for free diffusion case over eight time decades for $\eta_0 = 1$.

In the scaling used, it is a solution of Eq. (4), with $f \to 0$, $\eta_\alpha \to 1$ and $D_\alpha \to 1/(2\pi)^2$ at $T = 1$. The agreement with numerics is excellent (cf. the insert in Fig. 1).

It has been shown previously for subdiffusive dynamics with inertial effects [8,26,31,32] that a periodic potential does not influence the variance $\langle \delta x^2(t)\rangle$ and the corresponding subdiffusion coefficient D_α in the asymptotical limit. This was named a universality class of viscoelastic subdiffusion in tilted periodic potentials in Refs. [8,32]. This remarkable property is also valid for the overdamped ratchets considered in this work. The explanation is also similar: most sluggish viscoelastic modes of the environment determine the asymptotic character of subdiffusion not being affected by external static fields. Strong time-periodic fields can make some influence on viscoelastic subdiffusion pumping energy into the system at some rate. However, this influence was not strong even in the presence of inertial effects [31], as Figs. 1 and 2 also illustrate for the inertia-free dynamics. For a sufficiently large driving frequency Ω some deviation from the force-free subdiffusion coefficient, $D_\alpha^{(0)} = T/(2\pi)^2$, occurs in Fig. 2. However, it is not appreciably strong.

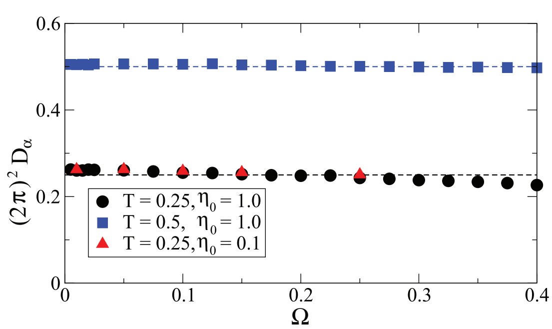

FIG. 2. (Color online) Scaled subdiffusion coefficient D_α as function of driving frequency Ω for several different values of temperature T and friction coefficient η_0.

One can see in Fig. 1 that an increase in viscous friction η_0 leads to extension of the initial normal diffusion regime. In contrast to the case studied earlier [26,27,31,32], there is no ballistic regime here due the absence of inertial effects. Initially diffusion is normal, $\langle \delta x^2(t)\rangle \sim t$, in accordance with above analysis. Finally, on a large time scale, diffusion becomes anomalous, $\langle \delta x^2(t)\rangle \sim t^{1/2}$. Normal friction does not affect this asymptotics, leading merely to increase of the transition time τ_0 (compare the dash-dotted, solid, and dashed curves in Fig. 1).

To check if the considered dynamics is ergodic, in accordance with previous studies of viscoelastic subdiffusion in periodic potentials, we computed a time average $\langle \delta x^2(t)\rangle_{\mathcal{T}}$ of the squared displacement,

$$\langle \delta x^2(t)\rangle_{\mathcal{T}} = \frac{1}{\mathcal{T} - t}\int_0^{\mathcal{T}-t} [x(t+t') - x(t')]^2 dt', \quad (10)$$

over single trajectories [31,39–41]. Here, the total integration time $\mathcal{T}$ is chosen to be much larger than the maximal time $t_{\max}$ for the ensemble-averaged trajectories $\langle \delta x^2(t)\rangle$. We used $\mathcal{T}/t_{\max} = 10^3$ in our calculations. The underlying idea of ergodicity is that the time average of a quantity, here squared particle displacements within a time interval of length t is equal to a corresponding ensemble average. In other words, a moving time average of the squared displacement should coincide with the ensemble average. Comparing solid and dotted curves in Fig. 1 for $\eta_0 = 10$, one can conclude that this indeed is the case. The diffusion is clearly ergodic on the considered time scale.

A. Transport dependence on driving frequency

To compute the frequency dependence $v_\alpha(\Omega)$ we have chosen several different values of temperature T and normal friction coefficient η_0 and varied the driving frequency Ω in the window $(0\ldots2]$ at fixed amplitude $A = 1$. The results for the anomalous current (subvelocity v_α) as a function of driving frequency Ω are shown in Fig. 3.

The occurrence and frequency dependence of the rectification effect is particularly interesting in the considered overdamped dynamics. In the presence of inertial effects, rectification ratchet effect is suppressed in the adiabatic

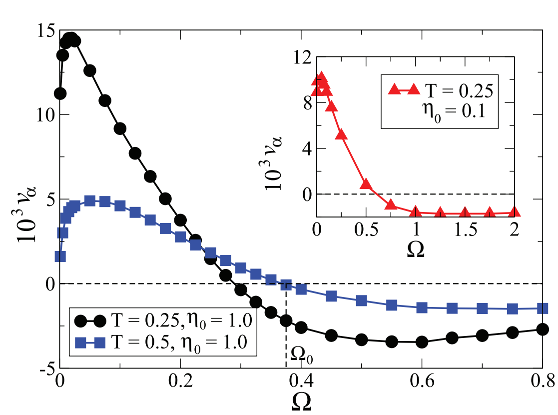

FIG. 3. (Color online) Anomalous current (subvelocity v_α) as function of driving frequency for different temperatures T and friction coefficients η_0.

frequency limit $\Omega \to 0$ [26,27]. This reflects the universality class of anomalous GLE subtransport in washboard potentials and differs markedly from the normal diffusion case, where the rectification effect for the fluctuating tilt ratchets is maximal, namely in the discussed limit [2,3,34]. One expects that this anomalous feature survives also for overdamped dynamics, as Fig. 3 indeed confirms.

Further, let us compare the influence of temperature and driving frequency on the subvelocity $v_\alpha(\Omega)$ at the fixed $\eta_0 = 1$ (see curves with filled circles and squares for small and intermediate temperatures, respectively). For the tested values of temperature the subcurrent indeed optimizes with the driving frequency. An increase in temperature leads to a decrease in the maximal value of subvelocity and to shift of the optimal driving frequency towards larger values. The qualitative explanation of these effects is similar to one for subdiffusive ratchets with inertial effects [27]. With increasing temperature the role of trapping potential diminishes (transport is absent in the absence of ratchet potential), and the mean time to escape out of potential well decreases. We expect that the optimal frequency also corresponds to a stochastic resonance (SR) effect, similar to the one in Ref. [27], when the mean frequency of jumps in the transport direction synchronizes with the corresponding potential tilts. This question was not studied, however, in more detail for the case considered.

Furthermore, it has been shown for the inertial case that an increase in the driving frequency can lead to the subcurrent inversion, where the transport occurs in the counterintuitive direction. The corresponding driving frequency is of the order of magnitude of the inverse anomalous relaxation time constant of the velocity autocorrelation function $\tau_v = (m/\eta_\alpha)^{1/(2-\alpha)}$, in the presence of inertial effects. It depends on the mass of Brownian particle m. We considered here, however, an overdamped limit, $m \to 0$ with $\tau_v \to 0$. For this reason, it is *a priori* not clear if the inversion of transport direction can occur also in the complete absence of inertial effects. Numerics do reveal such an inversion for a sufficiently large frequency. However, the corresponding characteristic time scale is given now not by the velocity relaxation time constant τ_v, but by the time scale of intrawell coordinate relaxation, $\tau_r^{(\mathrm{eff})}$. Such an inversion is similar to one detected for normal diffusion ratchets in Ref. [34]. For small $\eta_0 \to 0$, it is primarily determined by the anomalous relaxation time constant τ_r, i.e., $\tau_r^{(\mathrm{eff})} \sim \tau_r \sim \eta_\alpha^{1/\alpha}$. An increase in η_0 decelerates the intrawell relaxation process and leads to an increase in the effective relaxation time, which becomes proportional to η_0, $\tau_r^{(\mathrm{eff})} \sim \eta_0$, in the normal diffusion limit $\eta_\alpha \to 0$. For this reason, a critical inversion frequency, $\Omega_{\mathrm{cr}} \sim 1/\tau_r^{(\mathrm{eff})}$, should decrease with the increase in η_0 at fixed η_α. Moreover, an increase in temperature should also increase the relaxation rate leading to a larger value of Ω_{cr}. The results in Fig. 3 are consistent with this explanation and show the corresponding tendencies.

B. Temperature dependence of anomalous transport and its dispersion

Given a subthreshold driving, we are dealing with a thermal-noise-assisted ratchet transport. Thermal noise is necessary to overcome the potential barriers and, therefore, subtransport vanishes in the limit of zero temperature, $T \to$

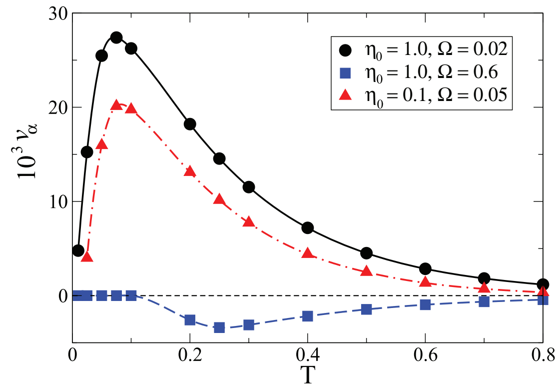

FIG. 4. (Color online) Subvelocity v_α as function of temperature T for several values of frequency Ω and friction coefficient η_0.

0. It vanishes also for a large temperature $T \gg U_0$, when the potential ceases to matter. Therefore, an optimization with temperature is expected also for the inverted transport regime. Indeed, numerics reveal such an optimization clearly; see Fig. 4 for different parameters and different regimes. Dependency on frequency Ω for the same η_0 implies, for sufficiently small Ω (not shown), that the maximum versus temperature corresponds to a stochastic resonance (SR), when the overbarrier jumps synchronize with the potential tilts in the transport direction, like in the presence of inertial effects [27]. A detailed study of such a non-Markovian SR for overdamped dynamics was, however, not done. It is left for a separate study. The coherence quality of the overdamped transport, as measured by the generalized Peclet number $\mathrm{Pe}_\alpha = v_\alpha L/D_\alpha$, can also be rather high, like in the presence of inertial effects. Figure 5 shows this clearly for v_α in Fig. 4. This is an expected result since $D_\alpha \propto T$. Because of this the considered non-Markovian stochastic coherence resonance is shifted to smaller values of optimal T as compare with SR.

The next question we address is whether the subtransport can be further optimized by a variation of η_0 for a maximal value of v_α in Fig. 4. The corresponding results are shown in Fig. 6. They reveal that this dependence on η_0 is rather weak, even if a minor optimization does take place; see the insert in Fig. 6.

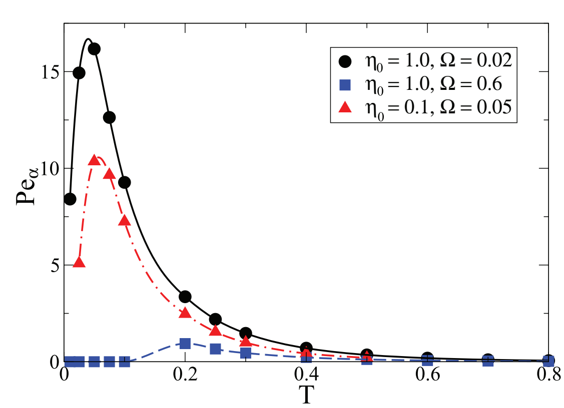

FIG. 5. (Color online) Generalized Peclet number Pe_α as function of temperature T for several values of frequency Ω and friction coefficient η_0.

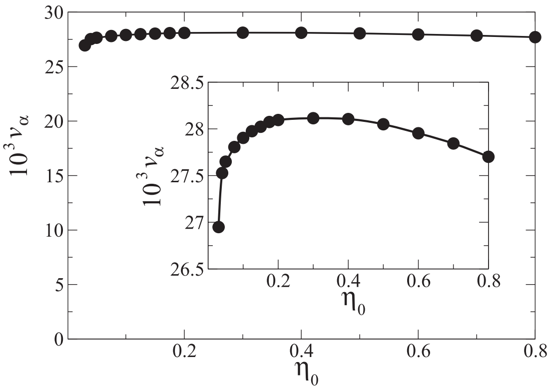

FIG. 6. Dependence of maximal v_α in Fig. 4 on η_0 variations.

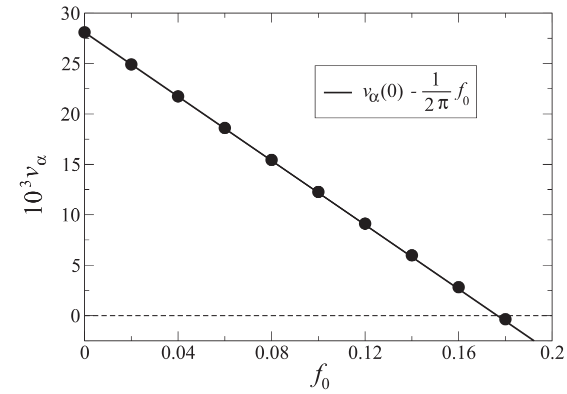

FIG. 7. Dependence of the maximal in Fig. 4 subvelocity v_α on the load f_0: numerics (symbols) *vs.* analytical result (line).

C. Load and efficiency

Finally, we come to clarifying the question of whether the studied anomalous ratchets can do any useful work and how big their thermodynamic efficiency can be. This question is not a trivial issue at all since friction can and does play a useful role, contrary to intuition, which can mislead. Not understanding the essence of the thermal fluctuation-dissipation theorem (FDT) can drive and mislead research into a wrong direction by tempting us to eliminate the dissipative effects overall and concentrating on the limit of the so-called frictionless or Hamiltonian ratchets. Friction is always associated with dissipative losses and one can believe that a complete elimination of friction will result in a most efficient motor. However, the FDT says that at thermal equilibrium the energy dissipated in motion of a Brownian particle by friction is regained due to absorption of energy obtained from thermal random forces, so both processes are balanced at the thermal equilibrium, where the total heat exchange between the particle and its environment is absent. For example, isothermal biological molecular motors can work in spite of a strong friction at thermodynamic efficiency close to 1 (though then infinitely slow at power close to zero) [42–44]. However, subdiffusion introduces new features. Phenomenologically, it can be characterized by a viscous friction that effectively increases in time. Indeed, let us do an *ad hoc* Markovian approximation in GLE (2) by replacing $\dot{x}(t')$ with $\dot{x}(t)$ in the memory friction integral (the explicit form of the term formally written with use of the Caputo fractional derivative). The dissipative part of this equation then is characterized by an effective friction $\eta_{\rm eff}(t) = \int_0^t \eta(t')dt' \propto t^{1-\alpha}$ that infinitely increases in time. This is the simplest way to understand the origin of subdiffusion and subtransport in physical terms (though, generally, one has to be very careful with such an *ad hoc* approximation, especially in the presence of inertial effects). Even if the effective friction increases indefinitely in time, such a ratchet does useful work against a load and is characterized by a finite stopping force; see Fig. 7. Similar results were shown also in the presence of inertial effects; see Fig. 5 in Ref. [27]. The numerics are well described by a simple analytical dependence,

$$v_\alpha(f_0) = v_\alpha(0) - f_0/2\pi, \tag{11}$$

which can be inferred by a linear response argumentation given the asymptotical independence of the subtransport on the presence of periodic potential in the static case, $A \to 0$. The presence of the factor $1/2\pi$ is due to the used scaling of nondimensional variables.

Is the thermodynamic efficiency also finite? We define it in a standard way as a portion of input energy $W_{\rm ext}(t)$ put into useful work $W_{\rm use}(t)$ against the loading force f_0, i.e., $R(t) = W_{\rm use}(t)/W_{\rm ext}(t)$. To find this quantity, we follow the ideas presented in Ref. [45] and first rewrite Eq. (2) as a force balance equation,

$$-f_{\rm int}(x) = u(t) + f_{\rm ext}(t) - f_0, \tag{12}$$

where $f_{\rm int}(x) = -dU(x)/dx$ is the periodic potential force acting on the particle (stator potential) and

$$u(t) = -\eta_0 \frac{dx}{dt} - \eta_\alpha \frac{d^\alpha x}{dt^\alpha} + \xi_0(t) + \xi_\alpha(t) \tag{13}$$

is the total stochastic viscoelastic force acting on the particle from the side of environment. Multiplying the force balance equation (12) by $\dot{x}(t)$, integrating it within the time interval $[0,t)$, and averaging over many trajectories (time averaging is also appropriate since the considered dynamics is ergodic), one obtains the following energy balance equation:

$$\Delta U(t) = \Delta Q(t) + W_{\rm ext}(t) - W_{\rm use}(t), \tag{14}$$

where $\Delta U(t) = U(x(t)) - U(x(0))$ is the change of internal energy of the considered Brownian motor (which is bounded), $\Delta Q(t) = \langle \int_{x(0)}^{x(t)} u(t')dx(t')\rangle$ is the heat exchanged between the motor particle and its environment, and $W_{\rm ext}(t) = \langle \int_{x(0)}^{x(t)} f_{\rm ext}(t')dx(t')\rangle$ is the work done by the external force on the whole system or the input energy provided by it. This input energy is used to do useful work $W_{\rm use}(t) = f_0\langle x(t) - x(0)\rangle$ against a load. For a large t, the fluctuating change of internal energy is negligible and we have

$$W_{\rm use}(t) = W_{\rm ext}(t) - |\Delta Q(t)| \,. \tag{15}$$

Thermodynamic efficiency is $R(t) = W_{\rm use}(t)/W_{\rm ext}(t)$. Notice that this efficiency is zero when an external loading force is absent. All the external work done then is dissipated as heat absorbed by the environment. At thermal equilibrium the total heat exchange is absent, $\Delta Q(t) = 0$. This is expression of FDT, which is guaranteed by the FDT condition for GLE. The efficiency of Brownian motors is therefore maximized when they are operating mostly close to the thermal equilibrium

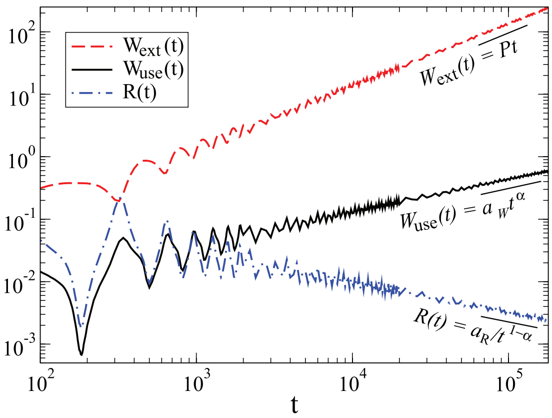

FIG. 8. (Color online) Dependencies of the pumped energy, useful work, and thermodynamic efficiency on time for an optimal load, $\Omega = 0.02$, $A = 1$, $T = 0.075$, and $\eta_0 = 0.2$.

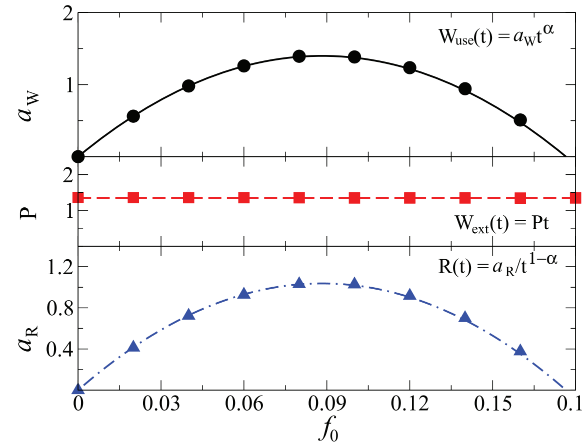

FIG. 9. (Color online) Dependencies of the subpower coefficient a_W (in arbitrary units), input power P (in arbitrary units), and the fractional efficiency a_R on load. Theoretical results (full lines) agree with numerics (symbols). Parameters are same as in Fig. 8.

to minimize heat losses. Theoretically, efficiency can reach a value of 1 (if operating very slowly at almost zero power), and some biological molecular motors can indeed be very efficient by operating in multiple small steps [44,46,47]. This is a classic observation [24].

The efficiency of subtransport presents new features. Since the transport is subdiffusive, the useful work scales sublinearly in time, $W_{\text{use}}(t) = a_W t^{\alpha}$, for a sufficiently large t. We can name the coefficient $P_\alpha = a_W \Gamma(1+\alpha)$ subpower or fractional power. It replaces, for anomalous motors, the notion of power, i.e., the work done per unit of time. However, the averaged energy pumped by a force that is periodically changing in time scales linearly with time, $W_{\text{ext}}(t) = Pt$, i.e., with a number of oscillations done, see Fig. 8. It can be characterized by a pumping power P. For this reason, thermodynamics efficiency decays in time as $R(t) = a_R/t^{1-\alpha}$, with $a_R = a_W/P$. One can name a_R fractional efficiency. Despite this decaying character, (a) the useful work and subpower are always finite in the presence of a load bounded by $0 < f_0 < f_{\text{stop}}$ and (b) the efficiency decays algebraically slowly and compares well with the efficiency of normal fluctuating tilt ratchets on the time scale of simulations. In this respect, the efficiency of normal ratchets estimated for the first time in Ref. [45] was as low as 0.01%. Fluctuating tilt ratchets operate as isothermal Brownian machines rather poor. This is a well-known fact. For larger values of α, say, for $\alpha \sim 0.9$, considered subdiffusive ratchets would operate with a very slow decaying efficiency, $R(t) \propto 1/t^{0.1}$, on a very long time scale. This again confirms that even an infinitely strong effective friction (in a naive Markovian approximation assuming a strict subdiffusion) is not an obstacle for thermodynamical efficiency, paradoxically enough.

Now we are able to provide a very simple theory for subpower coefficient a_W and the fractional efficiency coefficient a_R. Clearly, because of $W_{\text{use}}(t) = f_0\langle x(t)\rangle \sim f_0 v_\alpha(f_0) t^\alpha/\Gamma(1+\alpha)$ and Eq. (11), we have

$$a_W(f_0) = f_0[v_\alpha(0) - f_0/2\pi]/\Gamma(1+\alpha). \qquad (16)$$

This parabolic dependence on f_0 agrees with the numerical results in Fig. 9 very well. Furthermore, since the input power P does not depend on load (the same feature as for normal diffusion ratchets; see Fig. 9), the dependence of fractional efficiency $a_R(f_0)$ on load is qualitatively the same with the only modification $a_R(f_0) = a_W(f_0)/P$.

1. *Other definitions of motor efficiency*

Other definitions of the efficiency of Brownian motors were introduced in order to characterize their performance in the absence of a loading force. Then, all the input energy is dissipated finally as heat. The main idea is to characterize the motor performance against the dissipative force of the environment when a hindering force is absent or its performance against both environment and hindering force while the motor translocates a cargo. Different characterizations have been proposed for normal ratchets [48–50]. Let us consider how they can be modified for anomalous transport and what might be a natural proposal for the efficiency in the absence of load or natural Stokes efficiency. Let us consider the frictional part of the dissipative force of the environment $u(t)$ in Eq. (13). It is $u_{\text{diss}}(t) = f_{\text{mem}}(t) + f_{\text{Stokes}}(t)$ or $u_{\text{diss}}(t) = \langle u(t)\rangle$. The definition of the generalized efficiency given in Ref. [48] for normal diffusion ratchet is

$$R_{\text{DBA}} = \frac{P_{\text{use}} + \eta_0\langle v^2\rangle}{P}, \qquad (17)$$

where $P_{\text{use}} = \dot{W}_{\text{use}}$ is the useful power and $\langle v^2\rangle$ is the steady-state averaged squared motor velocity. $\eta_0\langle v^2\rangle$ is just averaged power of dissipation losses caused by the macroscopic Stokes friction f_{Stokes}. Another related option proposed [49] is to use $\langle v\rangle^2$ instead of $\langle v^2\rangle$, i.e., to neglect the velocity fluctuations $\langle\delta v^2\rangle = \langle v^2\rangle - \langle v\rangle^2$ in calculating work done against the frictional forces. The Stokes efficiency is obtained by setting $P_{\text{use}} = 0$. This corresponds to zero loading force, $f_0 = 0$, although in Ref. [50] it was defined differently as $R_{\text{Stokes}} = \eta_0\langle v\rangle^2/(P + f_0\langle v\rangle)$. A generalization of (17) to the present case should read

$$R_{\text{DBAgen}}(t) = \frac{W_{\text{use}}(t) + W_{\text{diss}}(t)}{W_{\text{ext}}(t)}, \qquad (18)$$

where $W_{\rm diss}(t)=\eta_0\int_0^t\langle v^2(t')\rangle dt'+\int_0^t dt'\int_0^{t'}dt''\eta(t'-t'')\langle v(t')v(t'')\rangle$. A different generalization in the spirit of Ref. [49] would amount to replacing $\langle v^2(t)\rangle$ with $\langle v(t)\rangle^2$ and $\langle v(t')v(t'')\rangle$ with $\langle v(t')\rangle\langle v(t'')\rangle$ in the last expression. We shall not consider these various generalizations further in the present paper but note that the *total* dissipative force acting from the environment on the motor particle is yet $u(t)$. It includes also the randomly fluctuating forces. By Newton's third law the particle exerts on the environment the force $-u(t)$ and the averaged work done by this force on the environment just equals the heat losses $|\Delta Q(t)|$. Therefore, a natural definition for the generalized Stokes efficiency would be just $R_{\rm Stokes}(t)=|\Delta Q(t)|/W_{\rm ext}(t)=1-R(t)$. It equals 100% in the absence of load and approaches this maximal value even in the presence of load asymptotically for the studied anomalous ratchets. In other words, the work on translocation of a cargo in such a highly dissipative viscoelastic environment is done mostly against resistance of this environment. This is a natural conclusion.

IV. SUMMARY AND CONCLUSIONS

We studied a model of overdamped subdiffusive ratchets rocked by a time-periodic force in viscoelastic media characterized by a power-law decaying memory kernel. The Brownian motor particle is subjected to viscous friction and white thermal noise. The viscoelastic medium's degrees of freedom were modeled by auxiliary Brownian particles that are elastically coupled to the central Brownian particle. Some of these auxiliary Brownian particles are extremely slow. They imprint memory about former positions of the central Brownian particle and create a retarding viscoelastic force causing anomalous diffusion and transport. Just a handful of such auxiliary Brownian particles suffices to model subdiffusion and subtransport on practically any experimentally relevant time scale. Our setup is fully equivalent to a generalized Langevin equation, where the memory kernel and random force of environment are related by the fluctuation-dissipation relation at ambient temperature of the environment. The setup of our modeling is ergodic and the Brownian motor subvelocity can be found from a single particle trajectory, though an ensemble averaging over 10^4 particles has been done to obtain most results presented. Subdiffusive current is optimized with the frequency of periodical driving and temperature. It depends also on the viscous friction acting directly on the motor particle. However, the subdiffusion coefficient depends weakly on other parameters being linearly proportional to temperature within the model of temperature-independent friction. Thus, the directed ratchet subtransport can possess at low temperature a very good quality, as characterized by the generalized Peclet number Pe_α.

Furthermore, we have shown that the considered anomalous Brownian motors are able to sustain a substantial load and do useful work which scales sublinearly with time and can be characterized by subpower. Since the energy pumped by the external time-periodic force scales linearly with time the motor efficiency decays algebraically in time. It can, however, favorably agree with the efficiency of the normal Brownian motors of the kind considered on an appreciably long time intervals. We provided a simple theory for thermodynamic efficiency of anomalous Brownian motors which agrees remarkably well with the numerical results obtained.

We expect that nontrivial results obtained in this work will stimulate a further cross-fertilization between the fields of anomalous diffusion and transport and the field of fluctuation-induced transport in the absence of a biasing on average force. A further generalization of the model presented here towards flashing potential ratchets opens a way to treat the operation of molecular motors in such viscoelastic environments as the cytosol of biological cells. The corresponding work is in progress.

ACKNOWLEDGMENT

Support of this research by the Deutsche Forschungsgemeinschaft (German Research Foundation), Grants No. GO 2052/1-1 and No. GO 2052/1-2 is gratefully acknowledged.

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