Exact relativistic expressions for wave refraction in a generally moving fluid

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The law for the refraction of a wave when the two fluids and the interface are moving with relativistic velocities is given in an exact form, at the same time correcting a first order error in a previous paper [Cavalleri and Tonni, Phys. Rev. E **57**, 3478 (1998)]. The treatment is then extended to a generally moving fluid with variable refractive index, ready to be applied to the refraction of acoustic, electromagnetic, or magnetohydrodynamic waves in the atmosphere of rapidly rotating stars. In the particular case of a gas cloud receding because of the universe expansion, our result can be applied to predict observable micro- and mesolensings. The first order approximation of our exact result for the deviation due to refraction of the light coming from a further quasar has a relativistic dependence equal to the one obtained by Einsteins' linearized theory of gravitation.

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I. INTRODUCTION

In 1975, Cavalleri and Spinelli [1] brought to a conclusion an approach to gravity starting from the pseudo-Euclidean space-time. Then, in 1990, Cavalleri and Mauri introduced a new aspect to quantum theory and denoted the resulting construct as "stochastic electrodynamics with spin (SEDS)" [2]. Thereafter, in 1996, Cavalleri sought to merge these approaches to gravitational theory with the concept of the variable speed of light (VSL) [3]. In fact, the VSL theories easily explain the gravitational deviation of light as an effect of refraction, as was originally done by Einstein for the first order deviation of a light beam grazing the Sun's limb. It is difficult, though, to explain gravitation as refraction for matter according to the standard VSL. On the contrary, such an explanation is natural in SEDS, the basis of which is the assumption that all the particles constituting matter are practically point-like, and endowed with a gyration (improperly called spin) at the speed of light along a circular trajectory having the Compton radius. Special relativity and quantum mechanics should not be present at that level [otherwise both relativistic mass and electromagnetic (e.m.) radiation would be infinite], but are a consequence of such a gyration (see the last two papers of Ref. [2]). Special relativity arises because one does not refer to the real velocity of an elementary particle, but to the velocity of its gyration center. The e.m. radiation of all the particles of the universe (including those before the primorial recombination) bring about the zero-point field (ZPF), whose spectrum is proportional to the cube of the frequency, and it is the only one to be invariant under Lorentz transformations. That property completes special relativity and, together with spin, leads to quantum mechanics. The spin (or, better, gyration) motion, being at the speed of light, allows a connection with the refraction of a light beam. Such a theory can overcome the main drawback of Einstein's theory. Actually, all the kinds of stress-energy-momentum tensors have to be inserted into the right-hand side of Einstein equations. However, if we include the ZPF of quantum electrodynamics (QED), an error by a factor of 10^{120} appears. The universe, arisen from a point, would have reached at maximum the size of an atom, and then it would have collapsed. That huge error cannot be eliminated by renormalizing the ZPF, as done in QED in flat space, because the ZPF cannot be renormalized in the Riemannian space-time. Not even the hope of resolving this difficulty by quantizing gravitation is viable, because quantum gravity has been recently disproved by the observation of the phase coherence of light from an active galaxy at a distance of 1.2 Gpc [4].

In order to work out such a new theory, the laws of refraction with media (or fluids) in motion are required. In 1996, Cavalleri, after an inquiry in the scientific literature, saw that such laws were unknown even at the classical level. That is why, together with Ascoli and Bernasconi, he published a paper [5] in which nonrelativistic refraction was given in an exact way. Precisely, the cosine of the refracted angle has been expressed as a function of the cosine of the incident wave beam, of the velocity $\mathbf{V}$ of the interface separating two fluids with velocities $\mathbf{u}_1 = u_1\hat{\mathbf{u}}_1$ and $\mathbf{u}_2 = u_2\hat{\mathbf{u}}_2$, respectively, with both u_1 and u_2 small compared with the speed c of light. Moreover, the treatment was limited to the case of a single refraction. The same limitation was kept in a subsequent paper (1998) with Tonni [6], where the relativistic version was given in an approximate way, claimed to be at the second order while, in 2000, they discovered that a first order error was present. The aim of the present paper is to overcome both limitations, i.e., to obtain the exact relativistic treatment (meanwhile correcting the above error), and to extend it to the case of many successive refractions. Toward that aim, in Sec. II A we revise what was done in our previous paper [6] regarding the first step of our treatment. At the same time, although the first step was already exact, we add some clarifications, and use a more compact formalism. The essential features of the problem are the velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ of the two fluids, the velocity $\mathbf{c}_1$ of the beam of light before refraction, and the equation of the interface σ locally moving with velocity $\mathbf{V}$. All the above quantities are given with respect to the laboratory frame S. The first point is to find the unit normal $\hat{\mathbf{n}}$ to the interface σ

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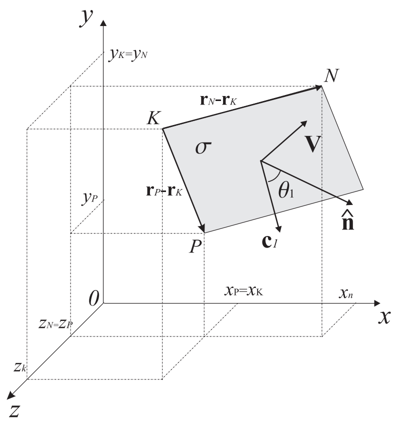

FIG. 1. The local reference system S has been chosen so that the x axis will be parallel to the local velocity $\mathbf{u}_1$ of the first medium. The vectors $\mathbf{r}_K$, $\mathbf{r}_N$, and $\mathbf{r}_P$ denote the positions of three nearby points belonging to the interface σ having local velocity $\mathbf{V}$. The unit vector $\hat{\mathbf{n}}$ is perpendicular to the local element of σ (characterized by $\mathbf{r}_K$, $\mathbf{r}_N$, and $\mathbf{r}_P$). The local velocity of the wave in medium 1 is denoted by $\mathbf{c}_1$ and forms the angle θ_1 with the local normal $\hat{\mathbf{n}}$.

(see Fig. 1) so as to obtain the cosine of the incident angle θ_1 as

$$\cos\theta_1 = \hat{\mathbf{n}}\cdot\hat{\mathbf{c}}_1. \tag{1}$$

If frame S is not at rest with the fluid under consideration, then the wave is generally not transverse; i.e., its velocity $\mathbf{c}_1$ is not locally perpendicular to its equiphase surface. However, as done in Refs. [5,6], and as we do in Sec. II B, we pass to the frame S_0 at rest with the first fluid, where any wave is transverse (i.e., its wave front is perpendicular to the ray $\hat{\mathbf{c}}_{01}$). Such a condition is necessary for the use of the Huygens construction, which is the simplest and most certain method to define refraction, because it is, in essence, just the principle of the superposition of the effects, taking into account the phases of the waves. The Huygens construction is particularly important, or even necessary, when the interface is in motion with respect to both S and S_0. That is why not even a nonrelativistic treatment was available until paper [5].

Afterwards, we transform all the above quantities, but $\hat{\mathbf{n}}$, from S to S_0. The exception for $\hat{\mathbf{n}}$ is due to the fact that both $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_0$ (measured in frame S_0 at rest with the first fluid) are not the transformed unit vectors of $\hat{\mathbf{n}}_\sigma$, perpendicular to the local interface σ in the frame S_σ at rest with σ. This fact does not affect the final result because we are always concerned with the direction of the beam of light in vector form. Consequently, at the end we pass again to S, having found the direction $\hat{\mathbf{c}}_2$ of the beam of light after refraction, so that

$$\cos\theta_2 = \hat{\mathbf{n}}\cdot\hat{\mathbf{c}}_2, \tag{2}$$

where $\hat{\mathbf{n}}$ is the same unit normal as in Eq. (1), measured in S.

The calculations of refraction are performed in Sec. II B taking the second fluid at rest with the first fluid (i.e., as if both fluids were at rest in S_0), but leaving the interface in motion with a velocity $\mathbf{V}_0$ (in S_0). In the third and last step, performed in Sec. II C, we consider that the real velocity of the second fluid in S_0 displaces the wave front parallel to itself. Consequently, the Huygens construction maintains its validity, and it is simply then a matter of relativistically adding the velocity $\mathbf{u}_{02}$ (measured in S_0) of the second fluid. Finally, the transformation from S_0 to S gives $\mathbf{c}_2$, hence the refraction angle through Eq. (2). Notice that we have not taken two different frames for the two different sides of the interface. The frame is only S_0, and in a first stage we suppose that the second fluid is at rest with the first. In a second stage, we consider that the second fluid has a velocity $\mathbf{u}_{02}$ with respect to the first fluid. When the light beam enters the second fluid, there is a process of absorption by part of the electrons, followed by radiation and superposition of the effects, so that the original wave is extinct after a short length λ (denoted as the extinction length), and the new radiated wave propagates in the second fluid with a velocity $\mathbf{c}_{02}$ depending on the refraction index. In other words, after a distance λ there is complete drag. We can therefore add $\mathbf{c}_{02}$ relativistically to the velocity $\mathbf{u}_{02}$ of the second fluid, since the direction of the wave front does not vary, because it is completely dragged by the second fluid. In fact, it was the lack of the above considerations in Ref. [6] that prevented an exact treatment and led to a first order error in $\mathbf{u}_2/c$ (and not to an approximation of the second order as claimed in that paper), as shown in Appendix A . Section II C is therefore radically different from the corresponding one of Ref. [6].

In Sec. III, we check the exact correctness of our procedure, by applying it to a particular case where the result can also be obtained in a standard way. The particular case regards the calculation of the deviation of an e.m. ray traversing a uniform gas cloud that is spherical in its rest frame and recedes with velocity $\mathbf{V}$ because of the expansion of the universe. Surprisingly, we have not found in the literature such a treatment, and that is why we have reported it in full detail. The calculated deviation increases with the recession speed, thus favoring the detection of protostellar atmospheres which might give additional information on the early universe [7]. Our exact result is new even if we only consider the relativistic dependence on β. Its first order approximation is formally equal to the gravitational deviation calculated by the linearized Einstein theory.

In Sec. IV, we extend the exact results to a series of successive refractions. As above, the first refraction is obtained in a frame here denoted by S_{01}, and considering the second fluid at rest with S_{01}, so that it is possible to apply the Huygens construction. When we relativistically add the refracted wave velocity to the velocity of the second fluid measured in S_{01}, the wave propagation is no longer transverse in S_{01}. However, the second refraction is obtained in frame S_{02} at rest with the second fluid, where the wave front and propagation velocity are exactly equal to those observed in S_{01} after the first refraction. As a matter of fact, we have first relativistically added the velocity $\mathbf{u}_{02}$ of the second fluid in S_{01} and then relativistically subtracted the same velocity to pass to S_{02}. The process is repeated at any successive refraction, so that the wave remains always transverse up to the last frame S_{0n}. Only at the end, when we return to the laboratory frame S, is the wave no longer transverse. At that point, however, we are only interested to the velocity direction $\hat{\mathbf{c}}_n$ in order to obtain

$$\cos\theta_n = \hat{\mathbf{n}}\cdot\hat{\mathbf{c}}_n, \tag{3}$$

where θ_n denotes the angle after the nth refraction. The main point of interest is the trajectory of a wave beam, rather than simply the refraction process. Up to S_{0n}, the velocities $\mathbf{c}_{0j}$ do not depend on $\mathbf{u}_j$, whereas the trajectory does. The procedure is applied to a moving fluid with velocity, density, and refraction index that are functions of the position, as is the case of the atmosphere of a rapidly rotating star. The results are put in a recursive way, therefore ready for numerical simulations by computer, so as to obtain the trajectory of a wave beam traversing a rotating stellar atmosphere.

We conclude in Sec. V commenting on our result and emphasizing the importance of the paper for astrophysical applications.

II. DERIVATION OF THE EXACT EXPRESSION FOR RELATIVISTIC REFRACTION

We first summarize the results of Ref. [6], deprived of an additional term that gave an error of first order (see Appendix A). Then, by an exact composition of velocities, we obtain the exact relativistic version of Ref. [5].

We premise a list of symbols. In the laboratory frame S we denote

(i) $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ the unit vectors of the x, y, and z axes, respectively;
(ii) $\mathbf{u}_1$ and $\mathbf{u}_2$ the velocities of fluids 1 and 2, respectively;
(iii) $\mathbf{V}$ the velocity of the interface σ;
(iv) $\mathbf{c}_1$ and $\mathbf{c}_2$ the velocities of the light beam in the two fluids, respectively;
(v) $\hat{\mathbf{n}} = \hat{n}_x\hat{\mathbf{e}}_x + \hat{n}_y\hat{\mathbf{e}}_y + \hat{n}_z\hat{\mathbf{e}}_z$ the unit normal to the interface σ;
(vi) θ_1 and θ_2 the angles of incidence and of refraction, respectively.

In the frame S_0 at rest with fluid 1 we denote

(i) $\mathbf{u}_{01} = 0$ and $\mathbf{u}_{02}$ the velocities of fluids 1 and 2, respectively;
(ii) $\mathbf{V}_0$ the velocity of the interface σ_0;
(iii) $\mathbf{c}_{01}$ the velocity of the light beam in fluid 1;
(iv) $\mathbf{c}_{02}$ the velocity of the light beam in the second fluid assumed at rest with fluid 1;
(v) $\mathbf{c}^*_{02}$ the velocity of the light beam in the second fluid with its real velocity with respect to fluid 1;
(vi) $\hat{\mathbf{n}}_0$ the unit normal to the interface σ_0;
(vii) $\hat{\sigma}_0$ the unit vector along the intersection of the interface σ_0 with the incidence plane;
(viii) θ_{01} and θ_{02} the angles of incidence and of refraction, respectively;
(ix) $\varepsilon = n - 1$, where n denotes the index of refraction;
(x) n the absolute refractive index (i.e., nonrelative), defined as the ratio between the speed of light in vacuum and the speed of the wave in the considered fluid. In our case of intergalactic gas, n depends only on the gas density, being practically independent of frequency in the visible range.

At first sight, it might appear strange to have three independent velocities, namely $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{V}$, since $\mathbf{u}_2 - \mathbf{u}_1$ must be tangential to the interface if the latter remains intact, i.e., if it acts as an impenetrable barrier without sources, wells, and pores. The general case of three independent velocities arose during the elaboration of a new model of elementary particles. The most similar macroscopic example is the one of a cylinder in which a porous piston moves with velocity $\mathbf{V}$ and keeps two different pressures in the two parts containing the same gas. The densities, hence the speeds of sound, of the two parts are different and the diffusion of the gas contained in the part at higher pressure into the other part at a lower pressure implies that $\mathbf{u}_2 - \mathbf{u}_1$ is perpendicular to the piston surface (acting as an interface).

The familiar case of $\mathbf{u}_2 - \mathbf{u}_1$ tangential to the interface is a particular case of the general one treated in this paper.

A still more particular case, where the results of the present paper are applied to the predictions of micro- and mesolensings, and to possible corrections to the gravitational lens effect because of refraction [7], is that of a beam of light traversing the atmospheres either of stars or of dense molecular clouds in a state of star formation, contained in large numbers in all galaxies. In this case the velocity $\mathbf{V}$ of the interface (considered as an isodensity surface of a dense molecular cloud or of a star) is equal to the velocity $\mathbf{u}_2$ of the second fluid (which is the gas of the receding molecular cloud, or of a star, producing the small deviations due to refraction). Another astrophysical case of interest is the motion of waves (acoustic, e.m., or magnetohydrodynamic) in the atmosphere of a rapidly rotating star. In this case we consider two adjacent thin layers with $\mathbf{u}_2 - \mathbf{u}_1$ tangential to the interface, the velocity $\mathbf{V}$ of the interface being equal to the common transversal velocities of the two layers, i.e., $\mathbf{V} = \mathbf{u}_{2\perp} = \mathbf{u}_{1\perp}$. In the astrophysical context we have therefore five scalar components and not nine (three vectors, each with three components).

A. Step one

In this section we find a convenient expression for the unit normal $\hat{\mathbf{n}}$. Afterwards we pass from the laboratory frame S to the frame S_0 at rest with the first fluid. To this aim, we must specify the kind of clock synchronization we use, since, to each kind of synchronization, there is a corresponding transformation [8]. For instance, if we use the internal synchronization for both S and S_0 (obtained either by Einstein's method or by slow clock transport) the corresponding relativistic transformations are those of Lorentz. If we use the external synchronization, the corresponding relativistic transformations are those of Tangherlini [9]. The latter ones have been used in the Appendix of Ref. [6] and in Ref. [10], while here we use the Lorentz transformations, which are more familiar. We express the normal $\hat{\mathbf{n}}_0$ to the interface σ (measured in S_0) pointing out that $\hat{\mathbf{n}}_0$ is *not* the transformed unit vector of $\hat{\mathbf{n}}$ because of the longitudinal contractions (present in both the Lorentz and the Tangherlini transformations) and also of the nonconservation of simultaneity (only present with the Lorentz transformations). On the other hand, the velocity $\mathbf{c}_{01}$ of the beam of light in S_0 is the relativistic transform of $\mathbf{c}_1$ measured in S. Then the incident angle θ_{01} in S_0 is derived from

$$\cos\theta_{01} = \hat{\mathbf{n}}_0 \cdot \hat{\mathbf{c}}_{01}. \tag{4}$$

To find convenient expressions for $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_0$, we choose a frame S of Cartesian axes with the x axis parallel to the velocity $\mathbf{u}_1$ of fluid 1 (through which the incoming wave is propagating before refraction). Let $\mathbf{c}_1$ be the wave velocity in S and $\hat{\mathbf{n}}$ be the unit vector perpendicular to the mobile interface σ and directed from fluid 1 to fluid 2 (see Fig. 1). In frame S, the angle θ_1 of incidence is given by Eq. (1). Notice that

$\hat{\mathbf{n}}$ is *not* the transformed unit vector of $\hat{\mathbf{n}}_\sigma$ perpendicular to the interface σ in the system S_σ at rest with σ. Simply, $\hat{\mathbf{n}}$ is the unit vector locally perpendicular to σ (as seen by S), which is a characteristic of the problem. To characterize $\hat{\mathbf{n}}$ we choose three nearby points $\mathbf{r}_K$, $\mathbf{r}_N$, and $\mathbf{r}_P$ on an element of σ around the point where we consider $\hat{\mathbf{n}}$, which can therefore be expressed as

$$\hat{\mathbf{n}} = \frac{(\mathbf{r}_N - \mathbf{r}_K) \times (\mathbf{r}_P - \mathbf{r}_K)}{|(\mathbf{r}_N - \mathbf{r}_K) \times (\mathbf{r}_P - \mathbf{r}_K)|}. \tag{5}$$

We use Cartesian axes with unit vectors $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_z$, so that

$$\mathbf{r}_i = x_i\,\hat{\mathbf{e}}_x + y_i\,\hat{\mathbf{e}}_y + z_i\,\hat{\mathbf{e}}_z, \quad \text{for} \quad i = K,N,P. \tag{6}$$

The nine scalar components are not free because each point (N, K, P) has to be in an element of σ around $\hat{\mathbf{n}}$, thus defining the plane tangent to σ. It is therefore sufficient to give two components (for instance along x and y) for each point to satisfy the required condition, because the third component (along z) is determined by its belonging to σ, whose equation $f(x,y) = z$ is known. We have, therefore, $3 \times 3 - 2 \times 3 = 3$ free components and we exploit this fact to impose three constraints that simplify the calculations. The three chosen conditions are

$$x_K = x_P, \quad y_P = y_N, \quad z_N = z_K, \tag{7}$$

as shown in Fig. 1.

We now pass from the laboratory frame S to the frame S_0 at rest with fluid 1. We take frame S_0 with the axes parallel to those of frame S and with the x_0 axis of S_0 superimposed and sliding on the x axis of S (the x and x_0 axes are therefore parallel to the velocity $\mathbf{u}_1$ of fluid 1 as observed by S). Consequently, given the stated assumptions (i.e., the internal synchronization that leads to the Lorentz transformations), and denoting by the subscript 0 the quantities measured in S_0, we thus have

$$\begin{cases} x_0 = \gamma_1\,(x - u_1\,t), \\ y_0 = y, \quad z_0 = z, \\ t_0 = \gamma_1\,(t - x\,u_1/c^2), \end{cases} \tag{8}$$

where c is the speed of light in vacuum, and

$$\gamma_1 = \left(1 - u_1^2/c^2\right)^{-1/2} \tag{9}$$

is the usual relativistic factor.

Because of the longitudinal relativistic contractions, the local interface (where the narrow wave beam impinges) observed by S_0 and denoted by σ_0 is bent differently than the σ observed by S. The unit vector $\hat{\mathbf{n}}_0$ (locally perpendicular to the interface σ_0) can be expressed by the same three nearby points used in S to characterize the local interface σ, now measured in S_0, denoted by $\mathbf{r}_{0K}$, $\mathbf{r}_{0N}$, and $\mathbf{r}_{0P}$, and taken simultaneous for S_0. Since $\mathbf{r}_P - \mathbf{r}_K$ lies on a plane perpendicular to the x axis, we derive from Eq. (8)

$$\mathbf{r}_{0P} - \mathbf{r}_{0K} = \mathbf{r}_P - \mathbf{r}_K. \tag{10}$$

On the contrary, $\mathbf{r}_N - \mathbf{r}_K$, hence $\mathbf{r}_{0N} - \mathbf{r}_{0K}$, do not lie on a plane perpendicular to x. If we required that those two space-time events be the transformations of each other, from $t_{0N} = t_{0K}$ we would have $t_N \neq t_K$. Consequently, $\hat{\mathbf{n}}_0$ is not the transformed of $\hat{\mathbf{n}}$, because, to obtain $\hat{\mathbf{n}}$, we take $t_N = t_K$.

The simplest way to obtain $\hat{\mathbf{n}}_0$ is by means of the Tangherlini transformations (that preserve simultaneity), as done in Appendix A of Ref. [6] and in Ref. [10]. By the Lorentz transformations, it is convenient to use auxiliary coordinates x_i', with $i = K,N$, related to x_{0i} and t_{0i} through

$$\begin{cases} x_i' = \gamma_1(x_{0i} + u_1 t_{0i}), \\ y_i' = y_{0i}; \quad z_i' = z_{0i}, \\ t_{0i} = \gamma_1(t_i' - u_1 x_i'/c^2), \end{cases} \tag{11}$$

with $t_{0K} = t_{0N}$, whence

$$\begin{cases} x_N' - x_K' = \gamma_1(x_{0N} - x_{0K}), \\ t_N' - t_K' = u_1(x_N' - x_K')/c^2. \end{cases} \tag{12}$$

During the time interval $t_N' - t_K'$, $\mathbf{r}_N$ moves with the velocity $\mathbf{V}$ of the local interface, thus reaching $\mathbf{r}'_N$ at time t_N', so that

$$\begin{aligned} \mathbf{r}'_N - \mathbf{r}'_K &= \mathbf{r}_N - \mathbf{r}_K + (t_N' - t_K')\mathbf{V} \\ &= \mathbf{r}_N - \mathbf{r}_K + (x_N' - x_K')\mathbf{V}u_1/c^2. \end{aligned} \tag{13}$$

Projecting Eq. (13) on $\mathbf{u}_1$, i.e., on the x axis, gives

$$x_N' - x_K' = x_N - x_K + (x_N' - x_K')V_x u_1/c^2, \tag{14}$$

from which

$$x_N' - x_K' = \frac{x_N - x_K}{1 - V_x\,u_1/c^2}. \tag{15}$$

Then we obtain by Eqs. (12) and (15)

$$x_{0N} - x_{0K} = \gamma_1^{-1}\frac{x_N - x_K}{1 - V_x u_1/c^2}. \tag{16}$$

Projecting Eq. (13) on the y and z axes, and using Eqs. (11) and (15), yields

$$y_{0N} - y_{0K} = y_N' - y_K' = y_N - y_K + \frac{(x_N - x_K)V_x\,u_1/c^2}{1 - V_x\,u_1/c^2}. \tag{17}$$

We now have all the elements to define the unit vector $\hat{\mathbf{n}}_0$ (measured in S_0) by an expression similar to Eq. (5). We obtain

$$\begin{aligned} \hat{\mathbf{n}}_0 &= \frac{(\mathbf{r}_{0N} - \mathbf{r}_{0K}) \times (\mathbf{r}_{0P} - \mathbf{r}_{0K})}{|(\mathbf{r}_{0N} - \mathbf{r}_{0K}) \times (\mathbf{r}_{0P} - \mathbf{r}_{0K})|} \\ &= \frac{n_{0x}\,\hat{\mathbf{e}}_x + n_{0y}\,\hat{\mathbf{e}}_y + n_{0z}\,\hat{\mathbf{e}}_z}{\sqrt{n_{0x}^2 + n_{0y}^2 + n_{0z}^2}}, \end{aligned} \tag{18}$$

with

$$\begin{cases} n_{0x} = +(y_{0N} - y_{0K})(z_{0P} - z_{0K}), \\ n_{0y} = -(x_{0N} - x_{0K})(z_{0P} - z_{0K}), \\ n_{0z} = +(x_{0N} - x_{0K})(y_{0N} - y_{0K}), \end{cases} \tag{19}$$

and $(x_{0N} - x_{0K}), (y_{0N} - y_{0K}), (z_{0P} - z_{0K})$ given by Eqs. (16), (17), and (10), respectively.

Since $\mathbf{u}_1 = u_1\,\hat{\mathbf{e}}_x$, the wave velocity $\mathbf{c}_{01}$ in S_0 is given by

$$\mathbf{c}_{01} = \frac{(c_{1x} - u_1)\,\hat{\mathbf{e}}_x + \gamma_1^{-1}\,(c_{1y}\,\hat{\mathbf{e}}_y + c_{1z}\,\hat{\mathbf{e}}_z)}{1 - \mathbf{u}_1 \cdot \mathbf{c}_1/c^2} = \Psi(\mathbf{c}_1,\mathbf{u}_1), \tag{20}$$

where γ_1 is expressed by Eq. (9), and the vector function Ψ is, in general, defined as

$$\Psi(\mathbf{a},\mathbf{b}) = \frac{(1 - b^2c^{-2})^{1/2}(\mathbf{a} - \mathbf{a}\cdot\hat{\mathbf{b}}\hat{\mathbf{b}}) + \mathbf{a}\cdot\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{b}}{1 + c^{-2}\mathbf{a}\cdot\mathbf{b}}, \tag{21}$$

for every $(\mathbf{a},\mathbf{b}) \in \mathbb{R}^3 \times \mathbb{R}^3$. Notice that the last step of Eq. (20) is independent of the choice of the frame axes.

The incident angle θ_{01} in S_0 is obtained by Eq. (4) with $\hat{\mathbf{n}}_0$ expressed by Eq. (18). The vector differences of $\mathbf{r}_{0K}$, $\mathbf{r}_{0N}$, $\mathbf{r}_{0P}$ and their order in the vector product of Eq. (18) are chosen so that $\hat{\mathbf{n}}_0 \cdot \hat{\mathbf{c}}_{01} = \cos\theta_{01} > 0$.

In the slow velocity limit, $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_0$ coincide, and Eq. (20) reduces to $\mathbf{c}_{01} = \mathbf{c}_1 - \mathbf{u}_1$ (as in Ref. [5]).

B. Step two

In the frame S_0 at rest with the first fluid, the velocity $\mathbf{c}_{01}$ of the beam of light is perpendicular to the equiphase plane of propagation. If the second fluid was also at rest in S_0, it would be possible to use the Huygens construction to find the refraction, even if the interface σ is in motion with a velocity $\mathbf{V}_0$ derivable from $\mathbf{V}$ by means of Eq. (20) with $\mathbf{V}$ for $\mathbf{c}_1$. The situation of two fluids at relative rest, in spite of the fact that their boundary plane moves, can be achieved, in principle, by a thin, porous piston in a cylinder (filled with a fluid) which maintains two different pressures and densities in the two parts of the closed cylinder just by moving. The piston can also be substituted by a shock wave of pressure.

Let us suppose to have found the Huygens construction and the equiphase plane after refraction in S_0 with also the second fluid at rest in S_0. Then let us allow the second fluid to have its velocity $\mathbf{u}_{02}$. If $\mathbf{u}_{02}$ is uniform (at least in the small region of the wave beam), the equiphase plane is dragged, but it remains parallel to the direction it had with $\mathbf{u}_{02} = 0$. On the contrary, $\mathbf{c}_{02}$ changes to $\mathbf{c}^*_{02}$, but it is possible to obtain $\mathbf{c}^*_{02}$ by relativistically adding the drag velocity. That is why in this section we calculate the refraction by the Huygens construction and assuming the second fluid at rest in S_0 as a theoretical, intermediary step that is useful to find the final solution. The interface is in motion with the velocity $\mathbf{V}_0$, and its effective component along the normal $\hat{\mathbf{n}}_0$ to the interface is

$$V_{0\perp} = \mathbf{V}_0 \cdot \hat{\mathbf{n}}_0. \tag{22}$$

The unit vector $\hat{\mathbf{n}}_0$ is drawn so that $\mathbf{c}_{01} \cdot \hat{\mathbf{n}}_0 > 0$. Fluids 1 and 2 contain the incident and refracted wave, respectively. If σ_0 were at rest, there would be no ambiguity about which one is the incident wave. However, if $V_{0\perp} > c_{01}\cos\theta_{01}$, it is the interface σ_0 that reaches the fleeing wave and we have to exchange fluid 1 for fluid 2. Consequently, fluid 1 is the one not containing $\hat{\mathbf{n}}_0$ (drawn starting from the interface) only if

$$s = \frac{(\mathbf{c}_{01} - \mathbf{V}_0)\cdot\hat{\mathbf{n}}_0}{|(\mathbf{c}_{01} - \mathbf{V}_0)\cdot\hat{\mathbf{n}}_0|} \tag{23}$$

is positive. On the contrary, fluid 1 is that containing $\hat{\mathbf{n}}_0$ if s is negative.

We consider the case $s = +1$ in Fig. 2, where the Huygens construction is plotted with respect to observer S_0 (so that this second step is the same as in the nonrelativistic case). We denote AB the trace at time t of the equiphase front in fluid 1 at rest with S_0, so that it is perpendicular to the velocity $\mathbf{c}_{01}$. The wave ray which impinges on the boundary plane at A begins to travel in fluid 2 with the velocity $\mathbf{c}_{02}$ along AA'. At time $t + t_0$, when the phase front reaches A', the same phase started in B reaches the moving boundary plane in B' at time $t + t_0$. Since the ray section BB' has always been in fluid 1 and the ray section AA' in fluid 2, it is

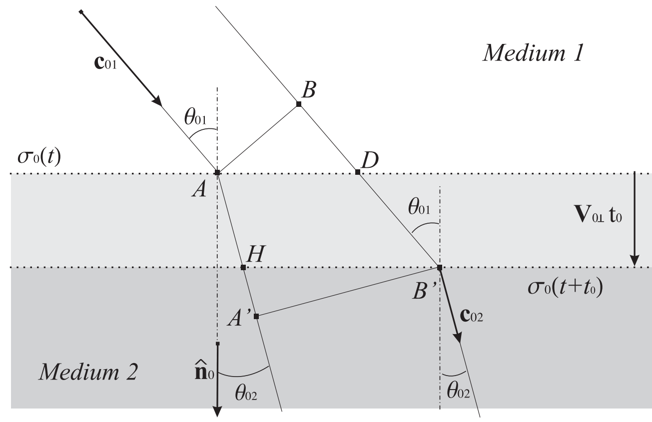

FIG. 2. A wave has velocity $\mathbf{c}_{01}$ in medium 1 and equiphase surface AB perpendicular to $\mathbf{c}_{01}$ if the observer S_0 is at rest with medium 1. An interface having velocity $\mathbf{V}_0$ separates medium 1 from medium 2. $\hat{\mathbf{n}}_0$ is the unit vector perpendicular (for S_0) to the interface and directed from 1 to 2. When a wave ray impinges on the interface at A, the wave is refracted in medium 2 (considered at rest with S_0) with velocity $\mathbf{c}_{02}$. Point B of the wave front reaches the moving interface in B' while point A reaches A' in medium 2 at rest with S_0, so that the equiphase surface $A'B'$ is still perpendicular to the refracted ray AA'. This is the Huygens construction for media at rest but moving interface, $A'B'$ being the envelope of the spherical waves radiated by the points of the interface successively reached by the impinging wave front. In the general case of media moving with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, we add relativistically $\mathbf{u}_1$ to $\mathbf{c}_{01}$ and $\mathbf{u}_2$ to $\mathbf{c}_{02}$. Moreover, H is the intersection of the ray refracted in A at time t with the interface at time $t + t_0$, while D is the geometrical intersection of the ray crossing B with the interface at time t. The medium between $\sigma_0(t)$ and $\sigma_0(t + t_0)$ is fluid 2 for ray AA' and fluid 1 for ray BB'.

$$t_0 = |AA'|/c_{02} = |BB'|/c_{01}. \tag{24}$$

The refracted phase front $A'B'$ is perpendicular to AA' because in this second step of our solution the fluid 2 is still considered at rest with S_0. This phase front is obtained as the envelope of the spherical waves radiated by each point of the boundary plane reached by the incoming wave.

We see from Fig. 2 that $|BB'| = c_{01}\,t_0$ may also be written as

$$c_{01}\,t_0 = |BD| + |DB'| = |AD|\sin\theta_{01} + V_{0\perp}\,t_0/\cos\theta_{01}. \tag{25}$$

Similarly, we may write $|AA'| = c_{02}\,t_0$ as

$$c_{02}\,t_0 = |A'H| + |HA| = |B'H|\sin\theta_{02} + V_{0\perp}t_0/\cos\theta_{02}, \tag{26}$$

where

$$|B'H| = |AD| + V_{0\perp}\,t_0\,(\tan\theta_{01} - \tan\theta_{02}). \tag{27}$$

Obtaining t_0 from Eq. (25), and substituting it in Eq. (26) where Eq. (27) is used, gives, after simplifying the factor $|AD|$ that appears in both sides,

$$c_{02}\,\sin\theta_{01}=\sin\theta_{02}\left(c_{01}-\frac{V_{0\perp}}{\cos\theta_{01}}\right)+V_{0\perp}\,\sin\theta_{01}\,\sin\theta_{02}\times\left(\frac{\sin\theta_{01}}{\cos\theta_{01}}-\frac{\sin\theta_{02}}{\cos\theta_{02}}\right)+V_{0\perp}\,\frac{\sin\theta_{01}}{\cos\theta_{02}}. \quad (28)$$

Simplifying Eq. (28) and denoting

$$m=V_{0\perp}\,\sin\theta_{01},\quad p=c_{01}-V_{0\perp}\,\cos\theta_{01},\quad q=c_{02}\,\sin\theta_{01}, \quad (29)$$

we obtain

$$m\,\cos\theta_{02}+p\,\sin\theta_{02}=q. \quad (30)$$

If $V_{0\perp}=-|V_{0\perp}|$, all the preceding expressions keep their validity.

In the second case ($s=-1$), Eq. (30) is again obtained, as shown in Ref. [5], since this second step is the same in both the relativistic and nonrelativistic treatments. The only difference is given by the connection between $\mathbf{V}$ and $\mathbf{V}_0$ (which in the nonrelativistic case reduces to $\mathbf{V}_0=\mathbf{V}-\mathbf{u}_1$). Taking into account all the cases [$s=\pm 1$, with s given by Eq. (23)], and subcases ($s=-1$, $p>0$ and $p<0$ as examined in Ref. [5]), the solution of Eq. (30) becomes

$$\cos\theta_{02}=\frac{mq+s\,p\,(m^2+p^2-q^2)^{1/2}}{m^2+p^2}, \quad (31)$$

with m, p, and q given by Eq. (29). We derive from Eq. (31)

$$\sin\theta_{02}=\frac{pq-s\,m\,(m^2+p^2-q^2)^{1/2}}{m^2+p^2}. \quad (32)$$

Then

$$\mathbf{c}_{02}=c_{02}\,(\hat{\mathbf{n}}_0\cos\theta_{02}+\hat{\sigma}_0\sin\theta_{02}), \quad (33)$$

where $\cos\theta_{02}$ and $\sin\theta_{02}$ are given by Eqs. (31) and (32), respectively, c_{02} denotes the known speed of the wave in fluid 2 if at rest, $\hat{\mathbf{n}}_0$ is the normal to the local interface [given by Eq. (18)], and $\hat{\sigma}_0$ is the normal to $\hat{\mathbf{n}}_0$ lying in the refraction plane, expressed by

$$\hat{\sigma}_0=\left(\frac{\hat{\mathbf{n}}_0\times\hat{\mathbf{c}}_{01}}{\sin\theta_{01}}\right)\times\hat{\mathbf{n}}_0=\frac{\hat{\mathbf{c}}_{01}-\hat{\mathbf{n}}_0\cos\theta_{01}}{\sin\theta_{01}}, \quad (34)$$

$\mathbf{c}_{01}$ being given by Eq. (20).

C. Step three

We have obtained the velocity $\mathbf{c}_{02}$ [measured in the system S_0 at rest with fluid 1 and given by Eq. (33)] in fluid 2 assumed at rest with fluid 1. We have to take into account that in frame S_0 the second fluid has a velocity $\mathbf{u}_{02}=\Psi(\mathbf{u}_2,\mathbf{u}_1)$, where Ψ is given by Eq. (21). Now comes the most important and delicate step. When the light beam enters the second fluid, there is a process of absorption by part of the electrons, followed by radiation and superposition of the effects, so that the original wave is extinct after a short length λ (denoted as the extinction length), and the new radiated wave propagates in the second fluid with a velocity $\mathbf{c}_{02}$ depending on the refractive index. In other words, after λ there is complete drag. We can therefore add relativistically $\mathbf{c}_{02}$ to the velocity $\mathbf{u}_{02}$ of the second fluid, since the direction of the wave front does not vary, because it is completely dragged by the second fluid. It was just the lack of the above considerations in Ref. [6] that prevented an exact treatment and led to a first order error in $\mathbf{u}_2/c$ (and not to an approximation of the second order as claimed in that paper), as shown in Appendix A.

If the velocity $\mathbf{u}_{02}$ is uniform in the considered region, after λ the wave front is fully dragged by part of the second fluid, but it does not change its direction (always in frame S_0). The velocity $\mathbf{c}^*_{02}$ after refraction in the moving fluid is $\mathbf{c}^*_{02}=\mathbf{c}_{02}+\mathbf{u}_{02}$ in nonrelativistic kinematics, as done in Ref. [5]. In relativistic kinematics we must operate a relativistic composition of the velocities $\mathbf{c}_{02}$ and $\mathbf{u}_{02}$ in S_0. In this way, and without considering the extinction length, the Fizeau drag coefficient can be obtained for monochromatic waves.[1] We therefore operate in the same way, first finding the expression of $\mathbf{u}_{02}$ starting from its known value in the laboratory system S.

The relativistic composition of the velocities $\mathbf{c}_{02}$ and $\mathbf{u}_{02}$ in S_0 is given by $\mathbf{c}^*_{02}=\Psi(\mathbf{c}_{02},\mathbf{u}_{02})$, with Ψ defined by Eq. (21). To obtain $\mathbf{c}_2$, we transform $\mathbf{c}^*_{02}$ from the frame S_0 (at rest with fluid 1) to the laboratory frame S, whence $\mathbf{c}_2=\Psi(\mathbf{c}^*_{02},-\mathbf{u}_1)$.

Finally, the cosine of the refracted angle is given by Eq. (2). The deviation ϑ between $\mathbf{c}_1$ and $\mathbf{c}_2$ can directly be obtained by $\cos\vartheta=\hat{\mathbf{c}}_1\cdot\hat{\mathbf{c}}_2$.

III. RELATIVISTIC CALCULATION OF THE DEVIATION OF AN E.M. RAY TRAVERSING A UNIFORM GASEOUS SPHERE

To test the correctness of the above procedure, we apply it to a particular case where the result can also be obtained in a standard way. The particular case regards the calculation of the deviation, due to refraction, of an e.m. ray whose source is in vacuum, and which traverses a uniform gas cloud that is spherical in its rest frame and recedes with velocity $\mathbf{V}$. The considered simple case has astrophysical interest when the e.m. source is a distant quasar, and when the uniform gas sphere is a dense gas cloud (or a protostar) contained in a spiral galaxy intermediate between the quasar and Earth, which recedes with relativistic velocity $\mathbf{V}$ because of the universe's expansion. We originally wanted to compare our result with the one obtainable in the standard way that, in the considered case, is valid and much simpler. Surprisingly, we have not found it in the scientific literature. We therefore show in the following the standard procedure (the one using the new, above procedure requires five pages).

To assume a uniform spherical gas cloud with a sharp interface is a convenient approximation in order to calculate refraction. The consequent reflected light does not modify the refracted angle. To have neglected the shading connection changes slightly the trajectory of the light beam, but not the

[1] In the case where the wave beam consists of a superposition of monochromatic components, we have to consider their dispersion. For instance, in the case of white light, Lorentz found that there is a correction to the Fizeau result due to dispersion, which was later verified by Zeeman. Fortunately, our applications regard galactic clouds and stellar atmospheres, for which there are no resonances, so that the refraction index is practically insensitive to frequency, with consequent absence of dispersion.

refracted angle. The main application of the present case regards the lens effect due to refraction, where the transverse distance between the nondeviated light beam and the refracting gas cloud is at least of the order of a galaxy radius. Even dispersion can be neglected, because the average composition of the galactic gas is roughly 78% hydrogen and 22% helium, with resonances at ultraviolet frequencies. The refractive index in the visible is practically insensitive to frequency from red to violet light and only depends on the gas density. For our treatment, it is as if light was monochromatic.

We take the origin of the Cartesian axes at the center of the gas cloud for both the frame S_0 at rest with the cloud and the frame S at rest with Earth. With respect to the latter, the gas cloud recedes with a velocity parallel to the x axis,

$$\mathbf{V} = \beta c \hat{\mathbf{e}}_x, \tag{35}$$

and it is measured by S as an ellipsoid with the minor axis along $\mathbf{V}$. Its section with a plane containing the x axis is therefore an ellipse expressed by

$$p(x,y) = \left(\frac{x}{\gamma^{-1}R}\right)^2 + \left(\frac{y}{R}\right)^2 - 1 = 0, \tag{36}$$

where R and $\gamma^{-1}R$ are the major and the minor semiaxes, respectively, with

$$\gamma^{-1} = [1-(V/c)^2]^{1/2} = (1-\beta^2)^{1/2}. \tag{37}$$

Let $\mathbf{c}_1$ be the initial velocity of the ray of light emitted by the quasar Q. Its speed c_1 is practically equal to the one of light in vacuum, i.e., to c. Its direction $\hat{\mathbf{c}}_1$ forms a small angle ξ with the line joining Earth with the considered quasar (see Fig. 3). Since $\xi < 10^{-5}$ rad (it is at maximum of the order of 1″ arc), we can write $|\hat{\mathbf{c}}_1 \times \hat{\mathbf{e}}_x| = \sin\xi \simeq \xi$, whence

$$\mathbf{c}_1 = c(-\cos\xi\,\hat{\mathbf{e}}_x + \sin\xi\,\hat{\mathbf{e}}_y) \simeq c(-\hat{\mathbf{e}}_x + \xi\,\hat{\mathbf{e}}_y). \tag{38}$$

When the beam of light impinges at A on the gas cloud, it undergoes refraction and then traverses the gas cloud. In order to find the point B where the beam of light exits, it is more convenient to use frame S_0 for which the gas cloud is at rest and spherical. Consequently, it is much simpler to start at the beginning with S_0, for which the velocity of the same beam of light becomes

$$\mathbf{c}_1^{(0)} = c\left(-\hat{\mathbf{e}}_x\frac{\cos\xi+\beta}{1+\beta\cos\xi} + \hat{\mathbf{e}}_y\frac{\gamma^{-1}\sin\xi}{1+\beta\cos\xi}\right)$$
$$\simeq c\left(-\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y\xi\sqrt{\frac{1-\beta}{1+\beta}}\right), \tag{39}$$

the third side having the same approximation as the third side of Eq. (38).

The sine of the incident angle is expressed by

$$\sin\theta_{1i}^{(0)} = \left|\hat{\mathbf{c}}_1^{(0)} \times \hat{\mathbf{n}}^{(0)}\right|, \tag{40}$$

where $\hat{\mathbf{n}}^{(0)}$ denotes the unit vector normal to the gas cloud observed in S_0, derivable from the gas cloud profile $p(x,y)$ given by Eq. (36) with $\gamma = 1$, as a plane scalar field whose gradient (in the xy plane) gives the outward normal. Since we need the inward normal, as shown in Fig. 3, we have

$$\hat{\mathbf{n}}^{(0)} = \hat{\mathbf{n}}^{(0)}(x,y) = -\frac{\nabla p(x,y)}{|\nabla p(x,y)|} = -\frac{x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y}{R}. \tag{41}$$

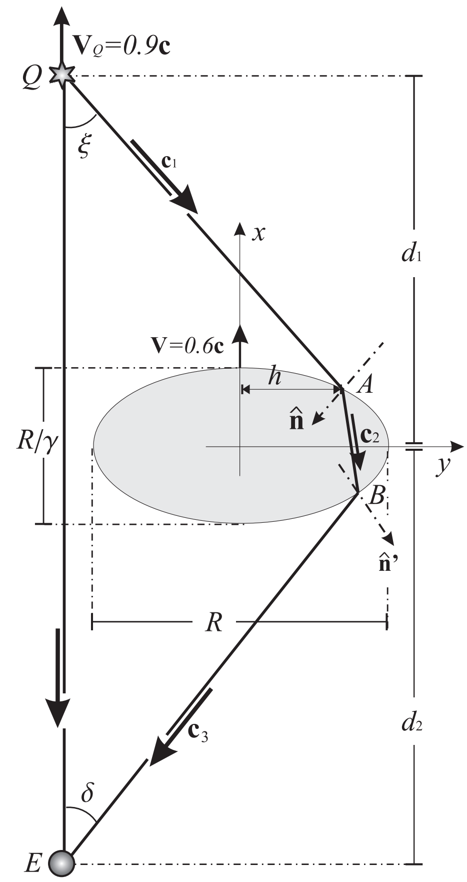

FIG. 3. A light beam coming from a quasar Q, for instance receding with $\beta = 0.9$, impinges at A on an uniform gaseous sphere (spherical in its rest system), for instance receding with $\beta = 0.6$. The light beam is refracted in A and comes out at B. If ξ and δ are the angles of the light beam with the straight line x joining Earth E with Q, the deviation of the light beam is given by $|-\delta-\xi| = |-\delta| + \xi$. The proportions and the deviations are strongly altered for clarity, since ξ and δ are of the order of 0.1″ of an arc.

At the incidence point A, characterized by $y_A = h$, we derive from Eqs. (36) with $\gamma = 1$ and (41)

$$\hat{\mathbf{n}}_A^{(0)} = -\sqrt{1-\frac{h^2}{R^2}}\,\hat{\mathbf{e}}_x - \frac{h}{R}\,\hat{\mathbf{e}}_y, \tag{42}$$

and from Eqs. (39), (40), and (42)

$$\sin\theta_{1i}^{(0)} = \frac{h}{R} + \xi\sqrt{1-\frac{h^2}{R^2}}\sqrt{\frac{1-\beta}{1+\beta}}. \tag{43}$$

The speed of light c_{02} in fluid 2 has a value a bit smaller than in fluid 1 (vacuum); i.e.,

$$c_{02} = \frac{c}{n} = \frac{c}{1+\varepsilon} \simeq c(1-\varepsilon), \quad \text{with} \quad \varepsilon \ll 1. \tag{44}$$

The sine of the refracted angle is obtainable by Snell's law,

$$\sin\theta_{1r}^{(0)} = \frac{\sin\theta_{1i}^{(0)}}{1+\varepsilon}, \tag{45}$$

from which

$$\cos\theta_{1r}^{(0)} \simeq \cos\theta_{1i}^{(0)} + \varepsilon\frac{\sin^2\theta_{1i}^{(0)}}{\cos\theta_{1i}^{(0)}}. \tag{46}$$

In the last two equations, the first order approximation in $\varepsilon \ll 1$ has been used.

With the aid of Eq. (44), the velocity $\mathbf{c}_{02}$ in fluid 2 can be written as

$$\mathbf{c}_{02} \simeq c(1-\varepsilon)\big[\hat{\mathbf{n}}_A^{(0)} \cos\theta_{1r}^{(0)} + \hat{\sigma}_A^{(0)} \sin\theta_{1r}^{(0)}\big], \tag{47}$$

where $\hat{\sigma}_A^{(0)}$ denotes the normal to $\hat{\mathbf{n}}_A^{(0)}$ lying in the refraction plane, expressed, similarly to Eq. (34), by

$$\hat{\sigma}_A^{(0)} = \left[\frac{\hat{\mathbf{n}}_A^{(0)} \times \hat{\mathbf{c}}_1^{(0)}}{\sin\theta_{1i}^{(0)}}\right] \times \hat{\mathbf{n}}_A^{(0)} = \frac{\hat{\mathbf{c}}_1^{(0)} - \hat{\mathbf{n}}_A^{(0)}\cos\theta_{1i}^{(0)}}{\sin\theta_{1i}^{(0)}}. \tag{48}$$

We derive from Eqs. (39), (42), and (45)–(48)

$$\begin{aligned}\hat{\mathbf{c}}_{02} &= \frac{\mathbf{c}_{02}}{c}(1+\varepsilon) \simeq \hat{\mathbf{c}}_1^{(0)}(1+\varepsilon) + \frac{\varepsilon\hat{\mathbf{n}}_A^{(0)}}{\cos\theta_{1i}^{(0)}} - 2\varepsilon\hat{\mathbf{e}}_x \\ &= \hat{\mathbf{c}}_1^{(0)} - \frac{\varepsilon\, h\, \hat{\mathbf{e}}_y}{\sqrt{R^2-h^2}},\end{aligned} \tag{49}$$

where, in the last side, we have inserted the explicit expression of $\cos\theta_{1i}^{(0)}$ derivable from

$$\cos\theta_{1i}^{(0)} = \hat{\mathbf{c}}_1^{(0)} \cdot \hat{\mathbf{n}}_A^{(0)} = \sqrt{1-\frac{h^2}{R^2}} - \xi\frac{h}{R}\sqrt{\frac{1-\beta}{1+\beta}}, \tag{50}$$

having used Eqs. (39) and (42) in the last side. We now find the point B of Fig. 3 where the light beam impinges again on the cloud profile with a second refraction, passing from fluid 2 to fluid 3, which is again the vacuum. Point B is the intersection between the cloud profile defined by Eq. (36) with $\gamma = 1$, and the straight line parallel to $\mathbf{c}_2$, expressed by

$$\frac{x-x_A}{y-y_A} = \frac{\mathbf{c}_{02}\cdot\hat{\mathbf{e}}_x}{\mathbf{c}_{02}\cdot\hat{\mathbf{e}}_y}. \tag{51}$$

Solving the system of two equations (ellipse and straight line), we obtain

$$x_B = -\sqrt{R^2-h^2} + 2h\left[\xi\sqrt{\frac{1-\beta}{1+\beta}} - \frac{\varepsilon\, h}{\sqrt{R^2-h^2}}\right], \tag{52}$$

$$y_B = h + 2\sqrt{R^2-h^2}\left[\xi\sqrt{\frac{1-\beta}{1+\beta}} - \frac{\varepsilon\, h}{\sqrt{R^2-h^2}}\right]. \tag{53}$$

The unit normal at B to the gas sphere is expressed by Eq. (41) with the opposite sign (it is an outward normal), thus yielding

$$\hat{\mathbf{n}}_B^{(0)} = \frac{x_B\,\hat{\mathbf{e}}_x + y_B\,\hat{\mathbf{e}}_y}{R}, \tag{54}$$

where x_B and y_B are given by Eqs. (52) and (53), respectively. Similarly to Eq. (48), we obtain

$$\hat{\sigma}_B^{(0)} = \frac{\hat{\mathbf{c}}_{02} - \hat{\mathbf{n}}_B^{(0)}\cos\theta_{2i}^{(0)}}{\sin\theta_{2i}^{(0)}}, \tag{55}$$

where, as derivable from Eqs. (40), (49), and (52)–(54),

$$\begin{aligned}\sin\theta_{2i}^{(0)} &= \big|\hat{\mathbf{c}}_{02} \times \hat{\mathbf{n}}_B^{(0)}\big| = \frac{h}{R}(1-\varepsilon) \\ &\quad + \sqrt{1-\frac{h^2}{R^2}}\,\xi\sqrt{\frac{1-\beta}{1+\beta}},\end{aligned} \tag{56}$$

and

$$\begin{aligned}\cos\theta_{2i}^{(0)} &= \hat{\mathbf{c}}_{02}\cdot\hat{\mathbf{n}}_B^{(0)} = \sqrt{1-\frac{h^2}{R^2}} - \frac{h}{R} \\ &\quad \times \left[\xi\sqrt{\frac{1-\beta}{1+\beta}} - \frac{\varepsilon\, h}{\sqrt{R^2-h^2}}\right].\end{aligned} \tag{57}$$

After the second refraction, again because of Snell's law, it is

$$\sin\theta_{2r}^{(0)} = (1+\varepsilon)\sin\theta_{2i}^{(0)}, \tag{58}$$

whence

$$\cos\theta_{2r}^{(0)} \simeq \cos\theta_{2i}^{(0)} - \varepsilon\frac{\sin^2\theta_{2i}^{(0)}}{\cos\theta_{2i}^{(0)}}. \tag{59}$$

The refracted velocity, being now in vacuum, is expressed, with the use of Eqs. (49) and (52)–(59), by

$$\begin{aligned}\hat{\mathbf{c}}_{02r} &= \frac{\mathbf{c}_{02r}}{c} \simeq \hat{\mathbf{n}}_B^{(0)}\cos\theta_{2r}^{(0)} + \hat{\sigma}_B^{(0)}\sin\theta_{2r}^{(0)} \\ &= -\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y\left[\xi\sqrt{\frac{1-\beta}{1+\beta}} - \frac{2\varepsilon\, h}{\sqrt{R^2-h^2}}\right],\end{aligned} \tag{60}$$

where the divergence for h tending to R is due to the expansions, performed in Eqs. (59) and (60), which are no longer valid when the denominator of the last term vanish, thus rendering such term no longer infinitesimal. On the contrary, the above expansions are excellent when $h \ll R$, so that the denominator can be expanded as

$$(R^2-h^2)^{-1/2} \simeq R^{-1}(1-h^2/2R^2). \tag{61}$$

In the following, we keep the above expansion that, together with the approximation used in Eqs. (59) and (60), leads to a result for the deviation that is very close to the exact result even for $h \to R$.

The direction $\hat{\mathbf{c}}_3$ of the emerging velocity $\mathbf{c}_3$ measured in Earth's frame can be obtained by a relativistic composition of velocities, and reads, with the use of Eq. (61),

$$\begin{aligned}\hat{\mathbf{c}}_3 = -\hat{\mathbf{e}}_x + \sqrt{\frac{1+\beta}{1-\beta}}\Bigg[&\xi\sqrt{\frac{1-\beta}{1+\beta}} \\ &- \frac{2\varepsilon\, h}{R}\left(1-\frac{h^2}{2R^2}\right)\hat{\mathbf{e}}_y\Bigg].\end{aligned} \tag{62}$$

The deviation Δ is obtained by

$$\Delta \simeq \sin\Delta = |\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_3| = \Delta_{\beta=0}\sqrt{\frac{1+\beta}{1-\beta}}, \tag{63}$$

where

$$\Delta_{\beta=0} = \frac{2\varepsilon\, h}{R}\left(1-\frac{h^2}{2R^2}\right). \tag{64}$$

The divergence of the last side of Eq. (63) for $\beta \to 1$ is due to the first order approximation in ξ. Had we taken the exact expression for the ξ dependence, given by the second side of Eq. (39), we would have obtained

$$\Delta_{\text{ex}} = \Delta_{\beta=0}\frac{\sqrt{1-\beta^2}}{1-\beta\cos\xi}, \tag{65}$$

which vanishes for $\beta \to 1$ and reduces to Eq. (64) for $\xi \to 0$. At the same time, when β is very close to unity, the inequality

$$1 - \beta \ll \xi \ll 1 \tag{66}$$

has to be satisfied in order to obtain the approximate Eq. (64).

Equating the derivative with respect to β of Δ_{ex} to zero leads to the value

$$\beta_M = \cos\xi \simeq 1 - \tfrac{1}{2}\xi^2, \tag{67}$$

at which the exact deviation is maximum,

$$\Delta_{\text{ex}}(\beta_M) = \frac{\Delta_{\beta=0}}{\sin\xi} \simeq \frac{\Delta_{\beta=0}}{\xi}. \tag{68}$$

The corresponding approximate expression then becomes

$$\Delta(\beta_M) = \Delta_{\beta=0}\frac{1+\cos\xi}{\sin\xi} \simeq 2\frac{\Delta_{\beta=0}}{\xi}. \tag{69}$$

Since ξ is of the order of 10^{-5}, β_M is far beyond any possible observation with telescopes. The earliest dense clouds, protostars, and quasars will have been brought into existence at about $\sim 10^8$ yr after the big bang, corresponding to

$$1 - \beta_l \simeq \frac{10^8}{1.5\times 10^{10}} = 6.67\times 10^{-3}. \tag{70}$$

The relevant increase of the deviation with respect to the static case ($\beta = 0$) is by a factor 17.3 as given by Eq. (65), which differs from the corresponding value given by Eq. (64) after the sixth decimal figure.

The fact that the calculated deviation increases with the recession speed favors the observation of the atmosphere of protostars that can give additional information on the early universe. The new result can also give some small corrections, due to refraction, to the gravitational lens effect. With regard to the latter, we have also found a surprising analogy, with interesting theoretical suggestions. Our exact result (65) is new even if we only consider the relativistic dependence on β. Under the same limitation, its first order approximation (64) is formally equal to the gravitational deviation calculated by Wucknitz and Sperhake [11] using the linearized Einstein theory. Obviously, the static expression is completely different, since the gravitational deviation due to a spherical body of mass M is the classical expression

$$\Delta^{\text{grav}}_{\beta=0} = \frac{4GM}{Rc^2}, \tag{71}$$

where G denotes the gravitational constant, and R the minimum distance of the light beam from the center of the spherical body. Already Einstein showed that the static result can be obtained by refraction, using the Huygens' principle, and treating the appropriate ratio of the metric coefficients as an index of refraction whose value turned out to be

$$n = 1 + \varepsilon = 1 + \frac{2GM}{rc^2}, \tag{72}$$

where r denotes the distance of a generic point of the light beam from the center of the spherical body. In Appendix B, we derive the following general expression that gives the deviation when n depends on r (and not of $\mathbf{r}$),

$$\Delta_{\beta=0} = 2\int_1^{+\infty}\frac{dx}{x}\left[\left(x\frac{n(x)}{n(1)}\right)^2 - 1\right]^{-1/2} - \pi. \tag{73}$$

where $x = r/R$. Using x in Eq. (72), and substituting the result into Eq. (73), we obtain Eq. (71).

In a future paper, we will apply Eq. (73) to the case of the dense clouds with a King' profile [7], where n depends on r in a way more complicated than the simple r^{-1}. Another future development takes its basis in the two above results, namely, (i) the equality of the relativistic first order deviation obtained either by refraction or by the first order approximation of Einstein theory; and (ii) the equality of the static deviations (71) provided n be given by Eq. (72). The relevant application will be a new gravitational theory starting from the flat pseudo-Euclidean space-time, of the kind of a previous paper [1], but with the local speed of light depending on the local scalar gravitational potential and also with the aid of stochastic electrodynamics with spin [2].

IV. APPLICATION TO INHOMOGENEOUS, GENERALLY MOVING FLUIDS

Let us extend our exact results relevant to a single refraction (found in Sec. II) to a regularly moving inhomogeneous fluid in a steady-state condition, such as the atmosphere of a rapidly rotating star. We know the velocity $\mathbf{c}_{\text{en}}$ of the impinging wave at the entrance point and the velocity $\mathbf{v}_{\text{star}}$ of the center of mass of the considered star, both with respect to the laboratory system S. Moreover, the velocity $\mathbf{u}_M(\mathbf{r}_M)$ of the fluid with respect to the inertial system S_M at rest with the center-of-mass of the rotating star is known. If in the considered region the star atmosphere behaves as a rigid body rotating with angular velocity Ω, the local fluid velocity is given by

$$\mathbf{u}_M(\mathbf{r}_M) = \Omega \times \mathbf{r}_M. \tag{74}$$

We consider as the entrance point $\mathbf{r}_{M\_\text{en}}$ (of the ray of light in the star atmosphere) the one where the refraction index differs from unity by a value half that leading to an appreciable deviation for the wave beam.

The entrance velocity with respect to the system S at rest with Earth is given by $\mathbf{u}_{\text{en}} = \Psi(\mathbf{u}_{M\_\text{en}}, \mathbf{v}_{\text{star}})$, where Ψ is defined by Eq. (21), and $\mathbf{u}_{M\_\text{en}}$ is the fluid velocity at the entrance point $\mathbf{r}_{M\_\text{en}}$, with respect to the star's center of mass. At $\mathbf{r}_{M\_\text{en}}$ the velocity of the wave ray in the system S_0 at rest with the local fluid is given by $\mathbf{c}_{0\_\text{en}} = \Psi(\mathbf{c}_{\text{en}}, \mathbf{u}_{\text{en}})$. Consequently, using the last two expressions, we obtain

$$\mathbf{c}_{0\_\text{en}} = \Psi[\mathbf{c}_{\text{en}}, \Psi(\mathbf{u}_{M\_\text{en}}, \mathbf{v}_{\text{star}})]. \tag{75}$$

Once we are inside the star, we can translate into a recursive way that done in Sec. II, in order to find the trajectory of the beam of light in S_M. We make some preliminary clarifications. Our aim is to find relativistic refraction, so that we could consider monochromatic waves in order to avoid dispersion. There is no need of that limitation, though, because the refractive index is practically independent of frequency in the case of stellar atmospheres, as already said in footnote 1, in the last entry of the list of symbols, and, in a more detailed way, four paragraphs after the beginning of Sec. III [and before Eq. (35)]. Another clarification regards our subdivision in many successive layers, so that, at first sight, we should take into account the reflected beam after each layer, with consequent interference, a well-known phenomenon in the theory of thin films. However, even with modest computation, the fractional variation of the refractive index from one layer to the successive one is of the order of 10^{-7}. The Fresnel reflection coefficient is therefore $\sim 10^{-7}$ times the transmission coefficient. The intensity of the second reflection, after the first backreflection, is $\sim 10^{-14}$ times the transmitted beam, thus has a negligible effect. Moreover, the problem is purely instrumental, because the thickness of each layer can be reduced as needed. Physically, in a regularly varying fluid the internal reflection is simply zero.

We denote $\mathbf{r}_{M\_\mathrm{en}}$ the initial (or entrance) position, and then $\mathbf{r}$ a generic point, as convenient notations to obtain expressions to be used in a recursive computation. Notice that we want the trajectory of the wave beam in S_M, so that $\mathbf{r}$ is referred to the center of the star. However, the calculation of refraction is performed, for each point, in the frame S_0 at rest with the local fluid, as done in Sec. II B. That is why we denote $\mathbf{c}_0(\mathbf{r})$ the velocity of light in S_0. What matters is the trajectory of the wave ray and, in particular, its exit direction $\hat{\mathbf{c}}_{0\_\mathrm{ex}}$. Then, by a relativistic transformation, we obtain $\hat{\mathbf{c}}_{\mathrm{ex}}$ and therefore the angle of deviation $\Delta\vartheta = \arccos(\hat{\mathbf{c}}_{\mathrm{en}} \cdot \hat{\mathbf{c}}_{\mathrm{ex}})$. The trajectory can be obtained numerically, calculating small displacements starting from the entrance point $\mathbf{r}_{\mathrm{en}}$ (hereafter denoted $\mathbf{r}$ because it can be a generic, subsequent point), where we know $\mathbf{c}_0(\mathbf{r})$. We first perform a virtual (or extrapolated) displacement in S_M,

$$\delta\mathbf{r}^* = \mathbf{c}_M \delta t, \tag{76}$$

with

$$\mathbf{c}_M = \Psi(\mathbf{c}_0, \mathbf{u}). \tag{77}$$

We denote fluid 1 the local one at $\mathbf{r}$, fluid 2 the local one at

$$\mathbf{r}^*_\delta = \mathbf{r} + \delta\mathbf{r}^*, \tag{78}$$

and the interface σ as the isodensity surface passing through

$$\mathbf{r}^*_{\delta/2} = \mathbf{r} + \delta\mathbf{r}^*/2. \tag{79}$$

The unit normal $\hat{\mathbf{n}}_0$ to σ at $\mathbf{r}_{\delta/2}$, such that $\mathbf{c}_0(\mathbf{r}) \cdot \hat{\mathbf{n}}_0 > 0$, is given by

$$\hat{\mathbf{n}}_0 = \frac{\nabla c_0(\mathbf{r}^*_{\delta/2})}{|\nabla c_0(\mathbf{r}^*_{\delta/2})|} \frac{\mathbf{c}_0(\mathbf{r}) \cdot \nabla c_0(\mathbf{r}^*_{\delta/2})}{|\mathbf{c}_0(\mathbf{r}) \cdot \nabla c_0(\mathbf{r}^*_{\delta/2})|}. \tag{80}$$

To obtain the real displacement $\delta\mathbf{r}$, we should know $c_0(\mathbf{r} + \delta\mathbf{r})$, i.e., the wave speed of the previously termed "fluid 2," in order to calculate the refraction. Since $\delta\mathbf{r}$ is considered as a first order infinitesimal, so is the variation of c_0. Consequently, the deviation $\delta\vartheta$ of $\hat{\mathbf{c}}_0$ is a first order infinitesimal, so that the displacement $|\delta\mathbf{r} - \delta\mathbf{r}^*| \simeq \delta r \delta\vartheta$ is a second order infinitesimal. The direction $\hat{\mathbf{c}}_0(\mathbf{r} + \delta\mathbf{r})$ is thus derived by the law of refraction. Precisely, it is

$$\mathbf{c}_0(\mathbf{r}^*_\delta) = c_0(\mathbf{r}^*_\delta)(\hat{\mathbf{n}}_0 \cos\theta_{02} + \hat{\sigma}_0 \sin\theta_{02}), \tag{81}$$

$\cos\theta_{02}$, $\hat{\sigma}_0$, and $\hat{\mathbf{n}}_0$ being given by Eqs. (31), (34), and (80), respectively.

A simplification can be done in the considered case of a steady-state star rotating around its center-of-mass axis, with angular velocity Ω. In the frame S_M (at rest with the star center of mass) the velocity of the fluid in $\mathbf{r}_M = \mathbf{r}$ is given by Eq. (74). The velocity $\mathbf{u}_M$ can be high, but not relativistic, so that we can limit to first order approximation in $\mathbf{u}_M/c$. Within this approximation, in the system S_0 (locally at rest with the fluid in $\mathbf{r}$) the interface has an infinitesimal velocity $\mathbf{V}_0 = \Omega \times \delta\mathbf{r}^*/2$ [however, we cannot reduce Eqs. (20) and (21) to their corresponding Galilean expression, because their denominators differ from unity by first order terms]. Fortunately, $\mathbf{V}_0$ is tangential to σ_0, so that $V_{0\perp} = 0$. Consequently, the coefficients m, p, and q, appearing in Eq. (29), simplify drastically. They become, translated for the continuum,

$$m = 0; \quad p = c_0(\mathbf{r}); \quad q = c_0(\mathbf{r}^*_\delta)\sqrt{1 - [\hat{\mathbf{c}}_0(\mathbf{r}) \cdot \hat{\mathbf{n}}_0(\mathbf{r}^*_{\delta/2})]^2}, \tag{82}$$

where $\mathbf{r}^*_\delta$ and $\mathbf{r}^*_{\delta/2}$ are given by Eqs. (78) and (79), respectively.

Using the first order expansion

$$c_0(\mathbf{r}^*_\delta) \simeq c_0(\mathbf{r}) + \delta\mathbf{r}^* \cdot \nabla c_0(\mathbf{r}), \tag{83}$$

the cosine of the refracted angle, expressed by Eq. (31), now becomes, with $s = 1$,

$$\cos\theta_{02} \simeq \hat{\mathbf{c}}_0(\mathbf{r}) \cdot \hat{\mathbf{n}}_0(\mathbf{r}^*_{\delta/2}) - \frac{\delta\mathbf{r}^* \cdot \nabla c_0(\mathbf{r})}{\mathbf{c}_0(\mathbf{r}) \cdot \hat{\mathbf{n}}_0(\mathbf{r}^*_{\delta/2})} \times [1 - \hat{\mathbf{c}}_0(\mathbf{r}) \cdot \hat{\mathbf{n}}_0(\mathbf{r}^*_{\delta/2})]^{1/2}. \tag{84}$$

We have now all the elements to calculate the real displacement $\delta\mathbf{r}$. The best approximation for $\delta\mathbf{r}$ is given by a half sum of the initial and final (i.e., after refraction) velocities,

$$\delta\mathbf{r} = [\mathbf{c}_M(\mathbf{r}) + \mathbf{c}_M(\mathbf{r}^*_\delta)]\delta t/2. \tag{85}$$

In fact, we obtain two segments tangent to the real trajectory, the first at the beginning of each step and the second at the end of each step.

One can therefore obtain the trajectory of a wave beam by computation, since $\mathbf{u}_{\mathrm{star}}(\mathbf{r} + \delta\mathbf{r})$ and $\mathbf{c}_0(\mathbf{r} + \delta\mathbf{r})$ become the initial known values of the subsequent step. As noted before Eq. (3), $\mathbf{c}_0(\mathbf{r} + \delta\mathbf{r})$ remains the same in the new $S_0(\mathbf{r} + \delta\mathbf{r})$ at rest with the local fluid at $\mathbf{r} + \delta\mathbf{r}$, because we first add, and then subtract, the velocity of the fluid at $\mathbf{r} + \delta\mathbf{r}$ with respect to the fluid at $\mathbf{r}$. Finally, step by step, the computations yield the whole trajectory with respect to S_M, that can easily be visualized.

What matters is the value of $\hat{\mathbf{c}}_M(\mathbf{r}_{\mathrm{ex}})$ at the point $\mathbf{r}_{\mathrm{ex}}$ where the beam of light emerges from the star atmosphere. The deviation of the light ray in the star system S_M is given by

$$\cos\vartheta_M = \hat{\mathbf{c}}_M(\mathbf{r}_{\mathrm{en}}) \cdot \hat{\mathbf{c}}_M(\mathbf{r}_{\mathrm{ex}}), \tag{86}$$

where $\hat{\mathbf{c}}_M(\mathbf{r}_{\mathrm{en}})$ and $\hat{\mathbf{c}}_M(\mathbf{r}_{\mathrm{ex}})$ are derivable from Eq. (81), with the entrance $\mathbf{r}_{\mathrm{en}}$ and exit position $\mathbf{r}_{\mathrm{ex}}$, respectively, for $\mathbf{r}$. The total deviation for Earth observer S is finally given by

$$\cos\vartheta = \hat{\mathbf{c}}_{\mathrm{en}} \cdot \hat{\mathbf{c}}_{\mathrm{ex}}, \tag{87}$$

with $\hat{\mathbf{c}}_{\mathrm{en}}$ being the known initial (or entrance) datum, and $\hat{\mathbf{c}}_{\mathrm{ex}} = \mathbf{c}_{\mathrm{ex}}/c_{\mathrm{ex}}$, where $\mathbf{c}_{\mathrm{ex}} = \Psi(\mathbf{c}_{M\_\mathrm{ex}}, \mathbf{u}_{M\_\mathrm{ex}})$, always with Ψ given by Eq. (21).

By our recursive procedure, we can also find the trajectory of a wave beam, which is often more interesting than the angle of refraction.

V. CONCLUSIONS

The exact expressions for wave refraction have been found in the case where the two fluids and the interface have three different relativistic velocities. That result is the relativistic extension of a previous, nonrelativistic paper [5]. The procedure is then extended to many successive refractions, so as to be applied also to a continuum, where the refraction index n and the local fluid velocity $\mathbf{u}$ are functions of the position $\mathbf{r}$. The result is put in a recursive way, ready to be implemented by a numerical computation. Each refraction is obtained by the Huygens construction in frame $S_0(\mathbf{r})$ at rest with the local fluid, so as to have a transverse wave, i.e., a wave whose local $\mathbf{c}_0(\mathbf{r})$ is perpendicular to the local equiphase surface. The interface has a velocity $\mathbf{V}_0(\mathbf{r})$ with respect to frame $S_0(\mathbf{r})$, while the fluid in $\mathbf{r}+\delta\mathbf{r}$ is considered at rest with $S_0(\mathbf{r})$, so that the wave after refraction remains transverse. Then we relativistically add $\mathbf{c}_0(\mathbf{r}+\delta\mathbf{r})$ to $\mathbf{u}_0(\mathbf{r}+\delta\mathbf{r})$, while $\mathbf{u}_0(\mathbf{r})=0$. At this point, the wave is no longer transverse in $S_0(\mathbf{r})$, although it becomes such again in $S_0(\mathbf{r}+\delta\mathbf{r})$. It was just the lack of those considerations that implied a first order error in our previous paper [6] (see Appendix A). For nonrelativistic velocities, the present exact expressions reduce exactly to those of Ref. [5].

In the case of a generally moving fluid, it is less convenient to obtain the trajectory of a beam of light in moving fluids by a standard method, for instance by Hamilton's equations [12]. Actually, at any step we must perform a (relativistic) composition of velocities, which is equivalent to changing the frame after each step. That is why our treatment is useful since it gives the possibility to obtain the trajectory of a light beam by means of a recursive computational method. Our results are interesting for the refraction of a quasar light by part of the atmosphere of a star in rapid differential rotation belonging to a relativistic receding galaxy. They are also important for the refraction of waves (e.m., magnetohydrodynamic, or acoustic) in the atmosphere of a rapidly rotating star whose center of mass is at rest with the observer, so that a nonrelativistic treatment is sufficient. As a matter of fact, such treatment has not been performed in Ref. [5].

The most important result of the present paper is the expression of the deviation of light coming from a distant quasar, and refracted by a gas cloud in a galaxy receding at relativistic speed. The result is given in exact form by Eq. (65) of Sec. III, which reduces to Eq. (64) when the angle ξ formed by the considered light beam with the x axis is much smaller than unity, but much larger than $1-\beta$ [see Eq. (66)]. Notice that relativity increases the deviation in an inversely proportional way to the decrease of the frequencies, thus enhancing the possibility of observing deviations due to refraction in receding gas clouds. Equation (64) will be applied in Ref. [7] to inquire into possible additional contributions to the gravitational lens effect. Another application (performed in Ref. [7]) regards the prediction of micro- and mesolensings due to the refraction of the light coming from a quasar and traversing dense clouds in the stage of star formation (DCSF). Such lensings, observable in the infrared by large telescopes with active and adaptive optics, can give information on the star formation and even on the early universe. In fact, yellow light ($\lambda \simeq 0.6\ \mu$m), refracted by a DCSF at a redshift $z \geqslant 3$, is observed at $\lambda \simeq 2.4\ \mu$m. If observed in a mesolensing, it implies the absence of dust produced by supernovas. The DCSFs detected through microlensing are therefore a window on the production of the first generation of stars.

The relativistic dependence on β of the first order Eq. (64) is made identical to the one obtained by Wucknitz and Sperhake [11] by applying the first order (linearized) Einstein equation to the gravitational lens effect. Actually, those authors obtained the inverse relativistic correction because they considered a galaxy moving towards the observer. The equality of the results is another step in favor of the new gravitational theory mentioned at the beginning of the Introduction. Such a theory, which will be developed in the near future, much depends on the results of the present paper, in particular on the recursive method of Sec. IV and on the result of Sec. III. As a hint, it starts from the flat, pseudo-Euclidean space-time of special relativity [1], but with the local speed of light depending on the local scalar gravitational potential [3] and also with the aid of stochastic electrodynamics with spin [2]. As said at the start of the Introduction, it will avoid the main drawback of Einstein theory, i.e., an error by a factor of 10^{120} if the ZPF is introduced, as it must be, into the source term (the stress-energy-momentum tensor) of Einstein's gravitational theory.

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APPENDIX A: THE ORIGIN OF THE FIRST ORDER ERROR IN A PREVIOUS PAPER

It is instructive to see how the first order error in Ref. [6] arose. The method used in Ref. [6] is shown in Fig. 4, which is very different from the present one shown in Fig. 2. The Huygens construction is performed considering the spherical wave emitted by point F, which denotes the position reached by point A if dragged along by the second fluid, and at time t_0 when point B reaches the second fluid at B'. What is done is equivalent to performing the Huygens construction in the frame S_2 at rest with the second fluid, thus avoiding the double composition of velocities done in Sec. II C of the present paper. The error is due to the fact that Huygens construction is valid only in frame S_0 at rest with the fluid, where the wave front is perpendicular to the wave rays. In frame S_2, the front of the wave moving in the first fluid is seen to be oblique with respect to the propagation velocity.

The procedure used in Ref. [6] was considered as being nonexact in that paper. However, the approximation, or error, was thought to be of the second order in u_{02}/c. To show that the error is of the first order, consider the case of $u_{02}=0$ for S_0, find the direction of the refracted ray and the corresponding wave front. Then consider another observer S for which $\mathbf{u}_{02}$ is along the direction of the refracted ray for S_0. It is obvious that the direction does *not* change, whereas, with the construction of Fig. 4, both for this direction and the one of the wave front, it varies depending on $|\mathbf{u}_{02}|$.

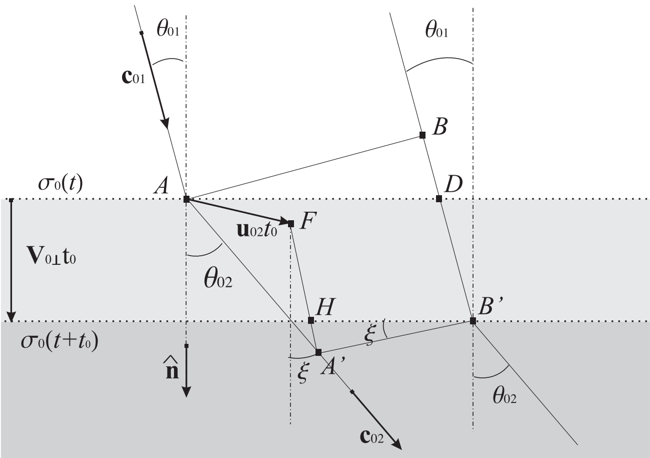

FIG. 4. Huygens construction used in Ref. [6]. A wave has velocity $\mathbf{c}_{01}$ in medium 1 and equiphase surface AB perpendicular to $\mathbf{c}_{01}$ if the observer S_0 is at rest with medium 1. An interface σ, having local velocity $\mathbf{V}_0$, separates medium 1 from medium 2. At time t medium 2 includes both the less shaded region and the shaded region. At time $t+t_0$, when B reaches B', medium 2 is represented by the shaded area only. The less shaded region is the one swept by the interface σ from time t to time $t+t_0$. Consequently, during the time interval between t and $t+t_0$, the ray AA' travels only in medium 2 while ray BB' travels only in medium 1. The unit vector $\hat{\mathbf{n}}$ is perpendicular to the interface, chosen so that $\mathbf{c}_{01}\cdot\hat{\mathbf{n}}>0$. When a wave ray impinges on the interface in A the wave is refracted in medium 2 with velocity $\mathbf{c}_{02}$. Point B of the wave front reaches the moving interface in B'. The wave front $A'B'$ of the refracted wave is obtained as the envelope of the spherical waves radiated by the points of the interface consecutively reached by the impinging wave front. The spherical wave radiated by A has its center moving with velocity $\mathbf{u}_{02}$ and reaching F at time t. Its radius at time t is FA'. Since $\mathbf{u}_{02}$ does not lie, in general, in the incidence plane ζ (that is the plane of the figure), points B, D, B', and A' lie in a plane parallel to ζ and containing F.

APPENDIX B: CALCULATION OF THE DEVIATION DUE TO THE REFRACTION OF A WAVE BEAM TRAVERSING A GAS CLOUD WITH SPHERICAL SYMMETRY

In a frame at rest with a cloud of gas having refraction index $n(\mathbf{r})$, function of $r=|\mathbf{r}|$, the deviation of a wave beam can be obtained by an integral, as shown by Maréchal [13]. For the reader's convenience, we simplify Maréchal's cumbersome procedure, and use less notation. We denote $\mathbf{r}$ the radius vector starting from the center of the distribution,

$$ds = |d\mathbf{r}| \tag{B1}$$

the absolute value of the displacement $d\mathbf{r}$ (which is different from dr), and

$$\hat{\mathbf{v}} = d\mathbf{r}/ds. \tag{B2}$$

the unit vector tangent to the trajectory of the wave ray.

We first prove that the Bouguer vector

$$\mathbf{G} = n(\mathbf{r})\mathbf{r}\times\hat{\mathbf{v}} \tag{B3}$$

along the same trajectory is invariant when $n_r(\mathbf{r}) = n_r(r)$.

Differentiating Eq. (B3), we obtain

$$\frac{d\mathbf{G}}{ds} = \frac{dn}{ds}\mathbf{r}\times\hat{\mathbf{v}} + n\hat{\mathbf{v}}\times\hat{\mathbf{v}} + n\mathbf{r}\times\frac{d\hat{\mathbf{v}}}{ds} = \mathbf{r}\times\left(\frac{dn}{ds}\hat{\mathbf{v}} + n\frac{d\hat{\mathbf{v}}}{ds}\right). \tag{B4}$$

We now use the Fermat principle, which is valid when the refractive fluid is at rest, as in the case here considered. We elaborate it in order to obtain an equivalent expression for the term inside the round bracket in the third side of Eq. (B4). For any wave ray passing through two fixed points $\mathbf{A}$ and $\mathbf{B}$ (so that $\delta\mathbf{B} = \delta\mathbf{A} = \mathbf{0}$), we have

$$0 = \delta\int_{\mathbf{A}}^{\mathbf{B}} ds\, n = \int_{\mathbf{A}}^{\mathbf{B}} (n\delta ds + ds\delta n). \tag{B5}$$

Since $\delta n = \delta\mathbf{r}\cdot\nabla n$, and

$$\delta ds = d\delta s = d(\hat{\mathbf{v}}\cdot\delta\mathbf{r}) = d\hat{\mathbf{v}}\cdot\delta\mathbf{r} + \hat{\mathbf{v}}\cdot d\delta\mathbf{r} = \hat{\mathbf{v}}\cdot d\delta\mathbf{r}, \tag{B6}$$

because $d\hat{\mathbf{v}}\cdot\delta\mathbf{r} \propto d\hat{\mathbf{v}}\cdot\mathbf{v} = 0$, Eq. (B5) becomes

$$\begin{aligned} 0 &= \int_{\mathbf{A}}^{\mathbf{B}} [n\hat{\mathbf{v}}\cdot d\delta\mathbf{r} + ds\delta\mathbf{r}\cdot\nabla n] = n\hat{\mathbf{v}}\cdot\delta\mathbf{B} \\ &\quad - n\hat{\mathbf{v}}\cdot\delta\mathbf{A} + \int_{\mathbf{A}}^{\mathbf{B}} ds\left[-\delta\mathbf{r}\cdot\frac{d}{ds}(n\hat{\mathbf{v}}) + \delta\mathbf{r}\cdot\nabla n\right] \\ &= \int_{\mathbf{A}}^{\mathbf{B}} ds\delta\mathbf{r}\cdot\left(-\frac{dn}{ds}\hat{\mathbf{v}} - n\frac{d}{ds}\hat{\mathbf{v}} + \nabla n\right). \end{aligned} \tag{B7}$$

Equation (B7) is valid for every couple of points $\mathbf{A}$, $\mathbf{B}$ in our space, whence

$$\nabla n = \frac{dn}{ds}\hat{\mathbf{v}} + n\frac{d\hat{\mathbf{v}}}{ds}. \tag{B8}$$

We derive from Eqs. (B4) and (B8)

$$d\mathbf{G}/ds = \mathbf{r}\times\nabla n. \tag{B9}$$

In the case the $n(\mathbf{r})$ distribution is spherically symmetric, i.e., $n(\mathbf{r}) = n(r)$, the n gradient is directed as $\mathbf{r}$, so that $d\mathbf{G}/ds = \mathbf{0}$. Consequently, the invariance of the Bouguer vector $\mathbf{G}$ of Eq. (B3) is proven.

$\mathbf{G}$ invariance implies that the whole trajectory of a wave beam lies on a plane σ containing the symmetry center. If we use in σ the polar coordinates r and φ, the incident angle ϑ_i, formed by the wave ray with the normal to the local isodensity surface, can be expressed as

$$\begin{aligned} |\sin\vartheta_i| = |\hat{\mathbf{r}}\times\hat{\mathbf{v}}| = \frac{rd\varphi}{ds} &= \frac{rd\varphi}{(r^2d\varphi^2 + dr^2)^{1/2}} \\ &= \frac{r}{[r^2 + (dr/d\varphi)^2]^{1/2}}. \end{aligned} \tag{B10}$$

We derive from Eqs. (B10) and (B3)

$$G^2 = \frac{n^2r^4}{r^2 + (drd\varphi)^2}, \tag{B11}$$

whence

$$\varphi_{\mathbf{B}} - \varphi_{\mathbf{A}} = \int_{\mathbf{A}}^{\mathbf{B}} \frac{dr}{r}\left(\frac{n^2r^2}{G^2} - 1\right)^{-1/2}. \tag{B12}$$

If we want the total variation of the φ angle when the wave ray starts from infinity, and then goes again to infinity in a practically opposite direction, it is convenient to double the result starting from the distance of minimum approach R, corresponding to $\sin\vartheta_i = 1$, so that $G = n(R)\,R$. We therefore obtain

$$\delta\varphi_{\text{tot}} = 2\int_R^{+\infty}\frac{dr}{r}\left[\left(\frac{n(r)r}{n(R)R}\right)^2 - 1\right]^{-1/2} = 2\int_1^{+\infty}\frac{dx}{x}\left[\left(x\frac{n(x)}{n(1)}\right)^2 - 1\right]^{-1/2}, \quad \text{(B13)}$$

having set $x = r/R$ in the last integral. When $n(x) = 1$ (negligible refraction), it is

$$\delta\varphi_{\text{tot}}(n = 1) = 2\int_1^{+\infty}\frac{dx}{x(x^2-1)^{1/2}} = \pi. \quad \text{(B14)}$$

The total deviation $\Delta_{\beta=0}$ of the wave ray coming from the quasar, passing at a minimum distance r_0 from the center of a gas cloud, and arriving at Earth is

$$\Delta_{\beta=0} = \delta\varphi_{\text{tot}} - \delta\varphi_{\text{tot}}(n = 1) = 2\int_1^{+\infty}\frac{dx}{x} \times\left[\left(x\frac{n(x)}{n(1)}\right)^2 - 1\right]^{-1/2} - \pi, \quad \text{(B15)}$$

where the subscript "$\beta = 0$" denotes nonrelativistic calculus, i.e., the choice of a frame at rest with the gas cloud.

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