Magnetic loop generation by collisionless gravitationally bound plasmas in axisymmetric tori

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Current-carrying string loops are adopted in astrophysics to model the dynamics of isolated flux tubes of magnetized plasma expected to arise in the gravitational field of compact objects, such as black holes. Recent studies suggest that they could provide a framework for the acceleration and collimation of jets of plasma observed in these systems. However, the problem remains of the search of physical mechanisms which can consistently explain the occurrence of such plasma toroidal structures characterized by nonvanishing charge currents and are able to self-generate magnetic loops. In this paper, the problem is addressed in the context of Vlasov-Maxwell theory for nonrelativistic collisionless plasmas subject to both gravitational and electromagnetic fields. A kinetic treatment of quasistationary axisymmetric configurations of charged particles exhibiting epicyclic motion is obtained. Explicit solutions for the species equilibrium phase-space distribution function are provided. These are shown to have generally a non-Maxwellian character and to be characterized by nonuniform fluid fields and temperature anisotropy. Calculation of the relevant fluid fields and analysis of the Ampere equation then show the existence of nonvanishing current densities. As a consequence, the occurrence of a kinetic dynamo is proved, which can explain the self-generation of both azimuthal and poloidal magnetic fields by the plasma itself. This mechanism can operate in the absence of instabilities, turbulence, or accretion phenomena and is intrinsically kinetic in character. In particular, several kinetic effects contribute to it, identified here with finite Larmor radius, diamagnetic and energy-correction effects together with temperature anisotropy, and non-Maxwellian features of the equilibrium distribution function.

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I. INTRODUCTION

This paper provides a theoretical treatment concerning the physical mechanisms responsible for the generation of magnetic loops in astrophysical context by collisionless plasmas belonging to axisymmetric tori and subject to external gravitational and electromagnetic (EM) fields. The research is based on a kinetic theory developed in the framework of the Vlasov-Maxwell description for nonrelativistic plasmas.

The dynamical description of current-carrying relativistic string loops, which can arise in the background gravitational field of a compact object (such as black holes), has been addressed for a long time (see, for example, Refs. [1,2]). The string-loop model is usually introduced to investigate the dynamics of astrophysical plasmas that can exhibit stringlike behavior through the dynamics of magnetic fields in the plasma itself or due to thin isolated flux tubes of plasma that could be described by a one-dimensional string [3,4]. In this framework, the string-loop model is therefore a convenient mathematical tool which allows one to build up a representative configuration for the underlying physical system of interest. Several string-loop models have been developed, according to the geometry of the problem and of the phenomena to be studied (see Refs. [3–6] and the references reported there). In these works, the string is described as a physical quantity which is characterized by an energy density and a tension, possibly carrying mass and/or charge currents, and with several model generalizations in terms of different system equations of state [7]. The relativistic dynamics of a string loop is usually determined in the framework of a curved space-time for a given background metric tensor based on a Lagrangian formalism. The string motion can then be expressed in terms of a properly chosen action by assigning the scalar Lagrangian function in terms of which the corresponding covariant equations of motion can be obtained [3,4].

Recent studies prove that string loops of this type, which are initially confined in a finite subset of configuration space by the action of tension and angular momentum barriers, can ultimately be accelerated enough to overcome the gravitational attraction and escape to infinity [3–6]. This results as a consequence of a scatter process of the string loop near the black hole horizon and the efficient conversion of string oscillation energy into translational kinetic energy. From the astrophysical point of view, this effect is relevant because it can potentially represent a plausible mechanism for acceleration and collimation of jets of plasmas in neutron stars and black holes systems as well as for supermassive black holes in active galactic nuclei and microquasars [8]. More precisely, string loops of magnetized plasma could arise near equatorial plane of accretion disks and, due to transmutation process of converting the internal string energy to the kinetic energy of their translational motion, a stream of string loops moving along the symmetry axis forms, representing a well collimated jet [3,5]. String loops of this type moving with ultrarelativistic velocities (Lorentz factor $\gamma \gtrsim 100$) can be created if the transmutation process occurs in sufficiently deep gravitational fields [6]. The string-loop model is then adopted as a simplified description for the behavior of relatively thin and isolated flux tubes of plasmas orbiting around compact objects and coupled to magnetic fields. In connection with this, another key application of the string-loop model would be to coronal loops and flares associated with disks, including the illumination of the disk from off-equatorial coronal plasma, which is thought to be involved in the production of the observed fluorescent iron lines.

However, the problem remains of the proof of the existence of a physical mechanism which can explain the generation of magnetic loops by orbiting plasma in toroidal structures and the determination of the physical requirements for its occurrence. For consistency with the string-loop theory, a mechanism of this type should operate in stationary configurations and should predict the possibility of having confined plasmas carrying a nonvanishing current density giving rise to axisymmetric azimuthal magnetic fields. Aside from its conceptual importance and its own relevance for plasma physics, such a result would also necessarily give support to the string-loop model. The configuration explained in this way in fact could serve as an initial condition for the dynamical studies of these systems according to the approach of Refs. [3–6]. As it will be proved in the following, a theory of this type can only be developed in the framework of kinetic theory for magnetized plasmas.

A further motivation for this kind of investigation is represented by a series of works concerning the motion of test particles in strong gravitational and EM fields related to black holes and neutron stars. We refer here in particular to the case of charged particles moving in stable circular orbits determined by minima of effective potential in axisymmetric systems [9]. In Refs. [10–17], it has been recently proved that configurations of this type can arise both in the equatorial plane and also in regions of the configuration space out of it (off-equatorial orbits), while in Ref. [18] similar conclusions have been extended also to charged fluid tori encircling black holes. A notable aspect of the dynamics of these particles is the possibility of exhibiting epicyclic motion corresponding to oscillations around minima of the potential in both radial and vertical directions with corresponding epicyclic frequencies [19–23]. The question arises therefore as to whether systems of oscillating charged particles confined near minima of effective potential can represent a plausible model for plasma tori either in or off the equatorial plane and can provide the conditions for the validity of the string-loop approach. This requires the construction of a specific statistical treatment of the system formed by these particles and an investigation of the role of epicyclic oscillations for the characterization of the corresponding equilibrium configurations and the generation of magnetic loops.

The problem posed by the magnetic loop generation and its connection with epicyclic motion of charged particles belong to the general framework of the theoretical description of magnetized astrophysical plasmas in gravitating systems and the search of dynamo mechanisms responsible for the self-generation and amplification of magnetic fields by the plasma currents. In this regard, a pioneering contribution is represented by the kinetic theory developed for collisionless magnetized and gravitationally bound accretion disk plasmas in quasistationary configurations presented in Refs. [24–29] and their stability properties with respect to axisymmetric perturbations given in Ref. [30]. Specifically, in these works, an approach based on the nonrelativistic Vlasov-Maxwell description was adopted. It was shown that consistent solutions of the Vlasov equation can be determined for the kinetic distribution function (KDF) associated with collisionless plasmas, based on the identification of the relevant single-particle invariants. In particular, it was proved that the species KDFs admit a representation in terms of generalized Maxwellian and bi-Maxwellian distributions, characterized by temperature anisotropy, nonuniform fluid fields, and local plasma flows. The perturbative kinetic theory developed therein made possible the analytical calculation of the relevant fluid fields carried by the equilibrium KDF and the identification of the relevant kinetic effects included in the corresponding magnetohydrodynamic (MHD) description. These were identified with diamagnetic, energy-correction, and finite Larmor-radius (FLR) kinetic effects. As a basic consequence, it was shown that these equilibria can exhibit nonvanishing current densities which can also support a kinetic dynamo mechanism for the self-generation of EM fields in which the plasma is immersed and operating in the absence of instabilities or turbulence phenomena [24,25,27,28]. Remarkably, similar conclusions were shown to apply also to laboratory plasmas in tokamak devices [31], while in Ref. [32] the theory was extended to collisionless plasmas in spatially nonsymmetric configurations, showing that also in this case the relevant phenomenology characterizing axisymmetric systems can be recovered.

A. Goals and scheme of the paper

Putting all the previous issues in perspective, in this paper we pursue the following target: to construct a theoretical formulation based on kinetic theory for equilibrium configurations of collisionless magnetized plasmas composed of charged particles in epicyclic motion in toroidal systems. The theory is based on the results previously obtained in the case of accretion disk plasmas in such a way to warrant the relevant kinetic effects discovered for axisymmetric and nonsymmetric systems to be recovered. The ultimate aim consists of proving that the new theoretical development can effectively provide a physical mechanism for the generation of magnetic loops by these toroidal plasma structures, showing that this represents a specifically kinetic effect which can only be consistently dealt with in the framework of kinetic theory.

Notice that here the treatment deals with collisionless plasmas. The collisionless regime is realized when binary Coulomb interactions become negligible at the microscopic level as far as the particle dynamics is concerned. In this case, the statistical behavior of such plasmas can be described in terms of a mean-field interaction within the Vlasov description. The proper definition of the collisionless regime requires, in principle, comparing the characteristic time and length scales of the system (e.g., the Larmor frequency and Larmor radius, the mean free path, and the collision time). We refer to Refs. [24,25,31] for a detailed discussion on the issue. On the other hand, it is possible to give an indication as to which real astrophysical systems associated with the existence of compact objects can be consistent with the collisionless assumption. Among accretion disks, a notable example of this type is represented by the hydrodynamic model known in the astrophysics literature as radiatively inefficient accretion flows (RIAFs, see Ref. [33]). These are expected to arise in geometrically thick disks around black holes and consisting of a two-temperature plasma, with the ion temperature being much higher than the electron one, and the time scale of the Coulomb collision frequency being much longer than the inflow time.

Other interesting applications of the model include the possible statistical description of hot and low-density plasmas in magnetized coronas expected to arise above accretion disks, for which collisionless kinetic effects should be important [34,35]. Finally, it is worth mentioning the case of plasmas in strong magnetic field around compact objects, such as for neutron stars. In fact, in this framework, when the magnetic field of the central object becomes dominant, ions and electrons can be collisionally decoupled and sustain different temperatures. This happens, in particular, if the radiative cooling time scale of the plasma is shorter than the electron-ion collision time scale [36,37]. Aside from this discussion, it is nevertheless necessary to stress, however, that this paper has a theoretical setting and is focused on presenting the fundamentals of kinetic theory for the problem of interest. Concrete astrophysical applications of the results obtained here in physically realizable systems will be considered in future works.

In detail, the main goals of the investigation are as follows: (1) The construction of a statistical treatment of a plasma system composed by charged particles in epicyclic motion in the framework of the Vlasov-Maxwell description: This requires determination of the general form of the equilibrium KDF and its functional dependencies to be expressed in terms of the relevant particle integrals of motion and adiabatic invariants. (2) The determination of an explicit solution for the equilibrium KDF and the proof that a kinetic solution exists for the posed problem: As a consequence, this implies the simultaneous existence of the corresponding fluid-MHD equilibria (kinetic MHD). (3) The identification of the relevant plasma regimes in which the solution admits a Chapman-Enskog representation and the development of a suitable perturbative theory for the treatment of implicit phase-space dependencies carried by the equilibrium KDF. (4) The proof that the solution obtained in this way is generally non-Gaussian and the identification of the physical reason for this. (5) The determination of the conditions for analytical calculation of fluid fields: This calculation has the main scope of proving that the equilibrium KDF carries nonuniform fluid fields and can generate the sources for the magnetic-field creation, i.e., azimuthal and poloidal current densities. (6) The investigation of the physical features which characterize the present kinetic theory: This includes the display of the role of the adiabatic invariants in determining the kinetic solution and the corresponding fluid fields, and in particular the constraints posed on them by the particle epicyclic oscillations and non-Maxwellian features of the KDF.

It must be stressed at this point that the kinetic approach followed in this work becomes appropriate for collisionless magnetized plasmas and is necessary in order to include the new relevant effects which are typically missing in "standalone" fluid approaches but are necessary to explain the occurrence of a stationary kinetic dynamo. These include diamagnetic FLR effects, energy-correction contributions, as well as phase-space anisotropies leading to non-Maxwellian features in the KDF. As it is shown below, the kinetic solution obtained here for toroidal plasmas of epicyclic charged particles is new and intrinsically different from previous solutions obtained for accretion disks. Hence, this represents the first consistent statistical treatment of these systems based on nonrelativistic Vlasov-Maxwell kinetic theory.

The scheme of the paper is as follows. In Sec. II, the basic assumptions and definitions are first introduced. In Sec. III, a brief summary of the main features of epicyclic motion for single particles in combined external gravitational and EM fields is recalled. Section IV deals with the solution method adopted here for the construction of stationary solutions for the species KDF, while in Sec. V the identification of the relevant plasma regime to which the present treatment applies is determined. In Sec. VI, the general features of the equilibrium KDF are discussed and an explicit solution is then given in Sec. VII. In Sec. VIII, a perturbative theory is developed and a Chapman-Enskog representation of the equilibrium KDF is obtained. In Sec. IX, the relevant fluid fields associated with the equilibrium KDF are evaluated, corresponding to the number density, the flow velocity, and the plasma temperature. These results are then applied in Sec. X to investigate the constraints posed by the Ampere equation and prove the existence of a kinetic dynamo mechanism. Finally, concluding remarks are summarized in Sec. XI.

II. BASIC ASSUMPTIONS AND DEFINITIONS

Ignoring possible weakly dissipative effects (e.g., Coulomb collisions and turbulence) and EM radiation effects [38–42], we shall assume that the KDF and the EM fields associated with the plasma obey the system of Vlasov-Maxwell equations, with Maxwell's equations being considered in the quasistatic approximation. For definiteness, we shall consider here a plasma consisting of s species of charged particles which are characterized by proper mass M_s and total charge $Z_s e$.

We shall take the plasma to be as follows: (a) *nonrelativistic*, in the sense that it has nonrelativistic species flow velocities, that the gravitational field can be treated within the classical Newtonian theory, and that the nonrelativistic Vlasov kinetic equation is used as the dynamical equation for the KDF; (b) *collisionless*, so that the mean free path of the plasma particles is much longer than the largest characteristic scale length of the plasma; (c) *axisymmetric*, so that the relevant dynamical variables characterizing the plasma (e.g., the fluid fields) are independent of the azimuthal angle φ, when referred to a set of either cylindrical coordinates (R,φ,z) or spherical coordinates (r,θ,φ); (d) acted on by both gravitational and EM fields.

In the following, we will focus on solutions for the equilibrium magnetic field $\mathbf{B}$ which admit, at least locally, a family of nested and open axisymmetric toroidal magnetic surfaces $\{\psi(\mathbf{x})\} \equiv \{\psi(\mathbf{x}) = \text{const}\}$, where ψ denotes the poloidal magnetic flux of $\mathbf{B}$ and the vector $\mathbf{x}$ denotes either $\mathbf{x} = (R,z)$ or $\mathbf{x} = (r,\theta)$. A set of magnetic coordinates (ψ,φ,ϑ) can be defined locally, where ϑ is a curvilinear anglelike coordinate on the magnetic surfaces $\psi(\mathbf{x}) = \text{const}$. Each relevant physical quantity $G(\mathbf{x},t)$ can then be conveniently expressed either in terms of the cylindrical or spherical coordinates or as a function of the magnetic coordinates, i.e., $G(\mathbf{x},t) = \overline{G}(\psi,\vartheta,t)$, where the φ dependence has been suppressed due to the axisymmetry.

We require the EM field to be slowly varying in time, i.e., of the form

$$[\mathbf{E}(\mathbf{x},\lambda^k t),\mathbf{B}(\mathbf{x},\lambda^k t)], \tag{1}$$

where $\lambda \ll 1$ denotes a small dimensionless parameter to be properly identified (see below), with $k \geqslant 1$ being a suitable integer. This kind of time dependence can be due to either external sources, boundary conditions for the KDF, or intrinsic time evolution associated with adiabatic invariants. In particular, we assume the magnetic field to be represented as

$$\mathbf{B} \equiv \nabla \times \mathbf{A} = \mathbf{B}^{self}(\mathbf{x},\lambda^k t) + \mathbf{B}^{ext}(\mathbf{x},\lambda^k t), \quad (2)$$

where $\mathbf{B}^{self}$ and $\mathbf{B}^{ext}$ denote the self-generated magnetic field produced by the toroidal plasma and a finite external magnetic field (vacuum field) produced by a rotating compact object with mass M_+, radius R_+, and spin Ω_+ oriented in the positive z direction. The external field is assumed to be purely poloidal, namely,

$$\mathbf{B}^{ext} = \nabla\psi_D(\mathbf{x},\lambda^k t) \times \nabla\varphi, \quad (3)$$

where ψ_D represents the magnetic flux function of a dipolar field. In terms of the spherical coordinates (r,θ), this is given by

$$\psi_D = \mathcal{M}_0 \frac{\sin^2\theta}{r}, \quad (4)$$

with $\mathcal{M}_0$ being the magnitude of the dipole magnetic moment. The self-magnetic field instead is taken to be of the general form

$$\mathbf{B}^{self} = I(\mathbf{x},\lambda^k t)\nabla\varphi + \nabla\psi_p(\mathbf{x},\lambda^k t) \times \nabla\varphi, \quad (5)$$

where $\mathbf{B}_T \equiv I(\mathbf{x},\lambda^k t)\nabla\varphi$ and $\mathbf{B}_P \equiv \nabla\psi_p(\mathbf{x},\lambda^k t) \times \nabla\varphi$ are the corresponding toroidal (i.e., azimuthal) and poloidal components, respectively. As a result, the total magnetic field can be written in the equivalent form

$$\mathbf{B} = I(\mathbf{x},\lambda^k t)\nabla\varphi + \nabla\psi(\mathbf{x},\lambda^k t) \times \nabla\varphi, \quad (6)$$

where the function $\psi(\mathbf{x},\lambda^k t)$ is defined as $\psi(\mathbf{x},\lambda^k t) \equiv \psi_p(\mathbf{x},\lambda^k t) + \psi_D(\mathbf{x},\lambda^k t)$, with $k \geqslant 1$ and (ψ,φ,ϑ) defining a set of local (nonorthogonal) magnetic coordinates. In the following treatment, we consider the case in which the self-component of the magnetic field is weak with respect to the external one, in the sense that the relative ordering

$$\frac{|\mathbf{B}^{self}|}{|\mathbf{B}^{ext}|} \sim O(\lambda^k) \quad (7)$$

applies, with $k \geqslant 1$.

Because of the central object rotation, the plasma is subject to an induced corotating electric field $\mathbf{E}_{cor}$. This can be estimated in terms of the dominant external magnetic field (see Ref. [12]), obtaining for the corresponding potential the expression

$$\Phi_{cor}(\mathbf{x},\lambda^k t) = \frac{\mathcal{M}_0}{c}\Omega_+ \frac{\sin^2\theta}{r}. \quad (8)$$

Charged particles of the plasma are then assumed to be subject to the action of the effective potential $\Phi_s^{eff}(\mathbf{x},\lambda^k t)$ defined as

$$\Phi_s^{eff}(\mathbf{x},\lambda^k t) = \Phi_{cor}(\mathbf{x},\lambda^k t) + \Phi(\mathbf{x},\lambda^k t) + \frac{M_s}{Z_s e}\Phi_G(\mathbf{x},\lambda^k t), \quad (9)$$

with $\Phi(\mathbf{x},\lambda^k t)$ and $\Phi_G(\mathbf{x},\lambda^k t)$ denoting the electrostatic (ES) potential generated by the plasma charge density and the gravitational potential, which in principle is produced both by the central object and the plasma. In the following, we shall neglect the contribution of the plasma to Φ_G, with the latter being therefore assumed as stationary and expressed in terms of either the Newtonian potential or a pseudo-Newtonian potential, such as the Paczyński-Wiita potential [43]

$$\Phi_G(\mathbf{x}) = -\frac{G_N M_+}{r - r_{sc}}, \quad (10)$$

where G_N is the Newton gravitational constant and $r_{sc} \equiv 2G_N M_+/c^2$ is the Schwarzschild radius. A comment is in order here. In fact, it is well known that, while pseudo-Newtonian potentials are consistent with a nonrelativistic treatment, they nevertheless allow for the inclusion in the treatment of some relevant features which are characteristic of a fully relativistic solution in curved space-time [44]. The use of pseudo-Newtonian potentials is therefore motivated in the present context for the description of collisionless plasmas arising around compact objects, such as black holes, and the generation of magnetic loops in these environments. Although a general relativistic treatment is in principle required, as far as the plasma kinetic description is concerned, for the purpose of this investigation a nonrelativistic formulation in terms of pseudo-Newtonian potentials is satisfactory. As discussed in the Introduction, the goal here is to prove the existence of a plausible physical mechanism for the generation of magnetic loops, the dynamics of which can then be investigated in a general relativistic framework by means of string-loop models.

III. EFFECTIVE POTENTIAL AND EPICYCLIC FREQUENCIES

In Refs. [10,11], the possible existence of equatorial as well as of off-equatorial particle orbits in combined strong gravitational and EM fields near compact objects was addressed. The case of isolated single-particle dynamics was considered in the absence of self-generated EM fields. Thanks to the ordering assumptions introduced in the previous section, the self-EM fields are considered as being of higher order with respect to the external EM field. Hence, in the nonrelativistic regime, to leading order the configuration considered here is consistent with the one studied in Refs. [10,11], so that the same qualitative features pointed out there still apply. We recall here briefly the main points of this analysis, which are needed later on. As far as the single-particle motion is concerned, this is described by the Hamiltonian function

$$\mathcal{H}_s = \frac{1}{2M_s}\left(\mathbf{P}_s - \frac{Z_s e}{c}\mathbf{A}\right)^2 + \mathcal{U}_s, \quad (11)$$

where $\mathbf{P}_s$ is the particle canonical momentum and $\mathcal{U}_s$ is the potential energy given by $\mathcal{U}_s = Z_s e \Phi_s^{eff}$. The motion of single particles in the $R - z$ plane can be studied by means of the effective potential $\mathcal{U}_{eff,s}$ defined as

$$\mathcal{U}_{eff,s} = \frac{1}{2M_s R^2}\left(P_{\varphi s} - \frac{Z_s e}{c}\psi\right)^2 + \mathcal{U}_s, \quad (12)$$

where $P_{\varphi s}$ is the particle toroidal canonical momentum given by

$$P_{\varphi s} = M_s R\mathbf{v}\cdot\mathbf{e}_\varphi + \frac{Z_s e}{c}\psi \equiv \frac{Z_s e}{c}\psi_{*s}, \quad (13)$$

with $v_\varphi = \mathbf{v}\cdot\mathbf{e}_\varphi$ and $\mathbf{e}_\varphi$ being the unit vector along the azimuthal direction φ. We notice that, to leading order, i.e., neglecting the contribution of the EM self-fields, the expression of the effective potential coincides with the one given for example in Ref. [12]. Minima of $\mathcal{U}_{eff,s}$ determine the location of bound orbits. In the following we restrict to effective potential profiles with minima in the equatorial plane only. For particles which are found close to the minimum, the motion in the radial and vertical directions can be treated in the linear regime for small excursions δR and δz (epicyclic motion). In the following, we shall refer to these particles as oscillating particles. Here, $\delta R \equiv R - R_{\min}$ and $\delta z \equiv z - z_{\min}$ are the spatial deviations from positions of the minima in radial ($R_{\min}$) and vertical ($z_{\min} = 0$) directions, respectively. Given validity of the ordering assumption (7), the epicyclic dynamics in the R-z plane can be treated as being decoupled from the azimuthal dynamics and is described in terms of the two Hamiltonian functions

$$H_R \equiv \frac{p_R^2}{2M_s} + \frac{M_s}{2}\omega_R^2 \delta R^2, \tag{14}$$

$$H_z \equiv \frac{p_z^2}{2M_s} + \frac{M_s}{2}\omega_z^2 \delta z^2, \tag{15}$$

where $p_R \equiv M_s v_R$ and $p_z \equiv M_s v_z$ are the linear momenta in the R and z directions, respectively, corresponding to the displacements δR and δz. In this approximation, one correspondingly obtains dynamical equations which recover the form of simple decoupled harmonic oscillators, namely,

$$\frac{d^2}{dt^2}\delta R + \omega_R^2 \delta R = 0, \tag{16}$$

$$\frac{d^2}{dt^2}\delta z + \omega_z^2 \delta z = 0. \tag{17}$$

Here, the characteristic epicyclic frequencies are, respectively,

$$\omega_R^2 = \frac{1}{M_s}\left[\frac{\partial^2 \mathcal{U}_{eff,s}}{\partial R^2}\right]_{\min}, \tag{18}$$

$$\omega_z^2 = \frac{1}{M_s}\left[\frac{\partial^2 \mathcal{U}_{eff,s}}{\partial z^2}\right]_{\min}, \tag{19}$$

which are to be evaluated at the minimum of the potential and are assumed to be nonvanishing. Given the potential $\mathcal{U}_s$, one finds that in general $\omega_R^2 \neq \omega_z^2$, while the precise expressions of the two epicyclic frequencies critically depend on the form of the effective potential as well as on the spatial position where they are evaluated [19–23].

IV. SOLUTION METHOD

The technique adopted here for the construction of Vlasov-Maxwell equilibria is based on the method developed in Refs. [24–26,31,32]. This consists in the construction of equilibrium KDFs f_s for collisionless plasma species such that, in a suitable subset of phase space, each of them can be realized in terms of appropriate generalized Gaussian distributions. In the case of stationary or quasistationary configurations, two possible approaches are available. The first one is the Chapman-Enskog solution method of the drift-kinetic Vlasov equation (see, for example, Refs. [45–49] in the case of astrophysical symmetric systems). This is achieved by seeking a perturbative solution of the form $f_s = f_{Ms} + \lambda f_{1s} + \ldots$, where $0 < \lambda \ll 1$ is an appropriate small dimensionless parameter to be properly identified and f_{Ms} a suitable equilibrium KDF. Usually, the latter is identified with a drifted Maxwellian KDF. The Chapman-Enskog method thus requires solving a hierarchy of PDEs which follow from the Vlasov equation as a consequence of the series representation for the equilibrium KDF.

A second solution technique for the Vlasov equation is adopted here. It consists in the determination of particular solutions of the KDF of the form $f_s = f_{*s}$, where f_{*s} is prescribed as a function of a suitable set of particle invariant phase functions $\{K_j,\ j = 1,n\}$. Here, we follow the definitions given in Ref. [32]. Thus, in particular, K_j is regarded as a first integral if it does not depend explicitly on time and satisfies the equation

$$\frac{d}{dt}K_j(\mathbf{z}) = 0 \tag{20}$$

for a properly defined state $\mathbf{z}$ in the phase space Υ'. Instead, K_j represents an adiabatic invariant of order $k \geqslant 1$ when it depends at most slowly on time, in the sense $K_j = K_j(\mathbf{z},\lambda^k t)$, and satisfies the asymptotic equation

$$\frac{d}{dt}K_j(\mathbf{z},\lambda^k t) = 0 + O(\lambda^k) \tag{21}$$

for the state $\mathbf{z}$.

From these definitions, it follows that f_{*s} is a first integral when it is expressed as a function of first integrals only. This identifies a stationary KDF. On the other hand, if f_{*s} depends also on adiabatic invariants of prescribed order, it is necessarily quasistationary, in the sense that it is itself an adiabatic invariant too. In the following, we shall consider the second case, i.e., f_{*s} as an adiabatic invariant. As a short way, this will be also equivalently referred to as an equilibrium KDF (or kinetic equilibrium). It is important to stress that a basic requirement of this second type of solution method is the possibility of determining *a posteriori* a perturbative representation of the KDF equivalent to the Chapman-Enskog expansion, according to the perturbative technique developed in Refs. [24–27,31,32]. This feature requires identifying the relevant characteristic dimensionless parameters of the system and the classification of the plasma according to the scheme presented in Ref. [26] (see Sec. V). It will be proved in the following that this feature can be realized also in the present case, determining an asymptotic expression for the equilibrium KDF which affords also evaluation of the relevant fluid fields associated with it.

A necessary prerequisite is therefore the identification of the relevant first integrals of motion or more generally adiabatic invariants which characterize single-particle dynamics. Because of axisymmetry, the toroidal canonical momentum $P_{\varphi s}$ defined by Eq. (13) is a first integral of motion. Thanks to the assumption introduced for the EM fields in Eqs. (6) and (9), which are slowly time varying, the total particle energy

$$E_s = \frac{M_s}{2}v^2 + Z_s e \Phi_s^{eff} \equiv Z_s e \Phi_{*s} \tag{22}$$

is an adiabatic invariant of prescribed order. Additional adiabatic invariants can be determined for magnetized plasmas by using gyrokinetic (GK) theory. A variational nonperturbative formulation of GK theory can be found in Ref. [32] for nonrelativistic charged particles in the presence of both EM and gravitational fields (for a formulation holding in the case of relativistic particles, see Ref. [50]). It is proved that, when EM radiation-reaction effects are neglected [38–42], the particle magnetic moment m_s' associated with the Larmor rotation of charges around magnetic-field lines is an adiabatic invariant. According to standard notation, here and in the rest of the paper, quantities labeled with a prime refer to dynamical variables which are evaluated at the particle guiding-center position. In the framework of an asymptotic formulation of GK theory carried out by means of a Larmor-radius expansion, the magnetic moment can be in principle determined with arbitrary accuracy. In Ref. [32], both nonperturbative expression as well as second-order Larmor-radius expansion have been determined for m_s'.

We further point out that the subset of oscillating particles admits additional adiabatic invariants which characterize the two-dimensional epicyclic harmonic motion in the R-z plane. In this case, when assumptions of Sec. II apply, the Hamilton-Jacobi equation corresponding to the problem is completely separable, so that the adiabatic invariants are identified with the action variables $\mathcal{J}_i$ of classical dynamics, with $i = R,z$ (see, for example, Refs. [51–53]). These are defined in terms of line integrals over complete periods of the orbit in the (p_i,q_i) plane as

$$\mathcal{J}_i \equiv \frac{1}{2\pi}\oint p_i dq_i. \tag{23}$$

Direct calculation for oscillating particles gives the two invariants

$$\mathcal{J}_R = \frac{H_R}{\omega_R}, \tag{24}$$

$$\mathcal{J}_z = \frac{H_z}{\omega_z}. \tag{25}$$

To conclude this section, we notice that in order to assure the adiabatic conservation of the canonical momenta and the action variables, we must exclude the possibility of having phase-space resonance phenomena. Thus, in the case of slowly varying EM and gravitational fields, to avoid occurrence of resonances we restrict the analysis to the subset of particles for which the azimuthal frequency $\omega_\varphi = \frac{v_\varphi}{R}$ and the Larmor frequency $\Omega_{cs} = \frac{Z_s eB}{M_s c}$ are different and aliquant with the two epicyclic frequencies ω_R and ω_z.

V. DIMENSIONLESS PARAMETERS AND PLASMA REGIME

In order to identify the appropriate plasma regime for the case of interest and the conditions providing the corresponding quasistationary kinetic solution, we follow the regime classification determined in Ref. [26] . Thus, we first introduce the dimensionless species parameters $\varepsilon_{M,s}$, ε_s, and σ_s. These are prescribed in such a way to be all independent of single-particle velocity and at the same time to be related to the characteristic species thermal velocities. Both perpendicular and parallel thermal velocities (defined with respect to the magnetic-field direction) must be considered. These are defined, respectively, by $v_{\perp ths} = \{T_{\perp s}/M_s\}^{1/2}$ and $v_{\parallel ths} = \{T_{\parallel s}/M_s\}^{1/2}$, with $T_{\perp s}$ and $T_{\parallel s}$ denoting here the species perpendicular and parallel temperatures.

In detail, the first parameter is defined as $\varepsilon_{M,s} \equiv \frac{r_{Ls}}{L}$, where $r_{Ls} = v_{\perp ths}/\Omega_{cs}$ is the species average Larmor radius, with L being the minimum scale length characterizing the spatial variations of all of the fluid fields associated with the KDF and of the EM fields. The second parameter ε_s is related to the particle canonical momentum $P_{\varphi s}$. Denoting by $v_{ths} \equiv \sup\{v_{\parallel ths}, v_{\perp ths}\}$, ε_s is identified with $\varepsilon_s \equiv |\frac{M_s R v_{ths}}{\frac{Z_s e}{c}\psi}|$. Hence, ε_s effectively measures the ratio between the toroidal angular momentum $L_{\varphi s} \equiv M_s R v_\varphi$ and the magnetic contribution to the toroidal canonical momentum, for all particles in which v_φ is of the order $v_\varphi \sim v_{ths}$ while ψ is assumed as being nonvanishing. Finally, σ_s is related to the particle total energy E_s and is prescribed as $\sigma_s \equiv |\frac{\frac{M_s}{2} v_{ths}^2}{Z_s e \Phi_s^{eff}}|$. It follows that σ_s measures the ratio between particle kinetic and potential energies, for all particles having velocity v of the order $v \sim v_{ths}$, with Φ_s^{eff} being assumed as nonvanishing. In the following, we shall denote as thermal subset of velocity space the subset of the Euclidean velocity space in which the asymptotic conditions $\frac{v}{v_{ths}} \sim \frac{v_\varphi}{v_{ths}} \sim O(1)$ holds.

Then, following the classification scheme proposed in Ref. [26], in this work the equilibrium plasma is assumed to belong to the *strongly magnetized and strong effective potential energy (SEPE) regime* for which the asymptotic orderings

$$\varepsilon_{M,s} \ll 1, \tag{26}$$

$$\varepsilon_s \ll 1, \tag{27}$$

$$\sigma_s \ll 1 \tag{28}$$

hold. These imply the following asymptotic expansions for ψ_{*s} and Φ_{*s}:

$$\psi_{*s} = \psi[1 + O(\varepsilon_s)], \tag{29}$$

$$\Phi_{*s} = \Phi_s^{eff}[1 + O(\sigma_s)]. \tag{30}$$

We finally notice that one can consistently identify the small parameter λ introduced above with one of the parameters given here, and in particular with $\varepsilon_{M,s}$.

Before concluding this section, it is worth commenting on the choice of the strongly magnetized SEPE regime and its physical realization in astrophysical systems. We consider first the magnetized orderings determined by $\varepsilon_{M,s}$ and ε_s. We notice that the magnetic field enters the two parameters in a different way. In fact, ε_s contains the poloidal flux ψ which contributes to the toroidal canonical momentum $P_{\varphi s}$, while $\varepsilon_{M,s}$ depends on the magnitude of the total magnetic field. Indeed, the parameter ε_s determines the particle spatial excursion from a magnetic flux surface $\psi(\mathbf{x}) = \text{const}$, while $\varepsilon_{M,s}$ measures the amplitude of the Larmor radius with respect to the inhomogeneities of the background fluid and EM fields. These two effects correspond to different physical magnetic-related processes due, respectively, to the Larmor-radius and magnetic-flux surface confinement mechanisms. As pointed out in Ref. [26], the two ordering conditions (26) and (27) are expected to be easily verified in accretion disk systems for a wide range of magnetic-field magnitudes that can be present in these scenarios. Hence, recalling the assumptions introduced in Sec. II, these conditions properly apply also in the present context.

For what concerns the validity of the ordering (28), this can result from the action of some energy nonconserving mechanisms. Plausible physical mechanisms that can be responsible for the decrease of the single-particle kinetic energy, in both collisionless and collisional accretion disk plasmas, are EM interactions (e.g., binary Coulomb collisions among particles and particle-wave interactions, such as Landau damping) and/or radiation emission (radiation reaction). These can in principle be ascribed also to the occurrence of EM instabilities and EM turbulence. For single particles, these processes can be dissipative, i.e., they can involve the loss of kinetic energy. As a consequence, these particles tend to move towards regions with higher gravitational and/or ES potential energy (in absolute value). After multiple interactions of this type, the process can ultimately reach an equilibrium state which corresponds to the SEPE regime. In turn, this condition requires the presence of a strong gravitational field, as in the surrounding of compact objects, and of strong ES fields. As shown, for example, in Refs. [24,31], the latter are characteristic of neutral or more generally quasineutral magnetized axisymmetric collisionless plasmas with azimuthal flow velocity. All these conditions are applicable to the astrophysical scenarios mentioned in the Introduction in connection with the problem of magnetic loop generation.

VI. EQUILIBRIUM KDF: GENERAL FEATURES

Following the prescriptions of Sec. IV, here we determine the general features of the equilibrium KDF consistent with the plasma regime determined above. It is understood that, in the general situation, a plasma consisting of both particles with epicyclic and nonepicyclic motion is likely to occur. In the framework of kinetic theory for collisionless plasmas, the coexistence of these two populations is allowed, with each of them being treated independently and interacting only through the mean EM field. In a realistic situation of this type, one should expect that charged particles which are not in epicyclic motion should represent the major component. For them, the kinetic solution obtained in Refs. [24,25] applies. Instead, plasma particles in epicyclic motion could represent a minority. However, the separate study of this population is of fundamental importance. In fact, as proved in the following, particles which exhibit epicyclic motion are characterized by a number of distinctive statistical features which are missing in previous results. This has consequences for what concerns the generation of localized magnetic loops and spectral features observed in disk systems, for which the present theory can provide a contribution toward their theoretical investigation in the framework of a kinetic description.

In view of these considerations, in the following we restrict the analysis only to the subset of charged particles exhibiting epicyclic motion. For this subspecies, the quasistationary KDF is expressed in terms of the set of adiabatic invariants listed above, and is of the form

$$f_{*s} = f_{*s}(E_s, \psi_{*s}, m'_s, \mathcal{J}_R, \mathcal{J}_z, \Lambda_{*s}, \lambda^k t), \tag{31}$$

where Λ_{*s} denotes the so-called structure functions, i.e., functions which depend implicitly on the particle state $\mathbf{x}$ and must be properly prescribed according to the specific form of the solution (see following). For definiteness, both f_{*s} and Λ_{*s} are assumed to be analytic functions. In order for f_{*s} to be an adiabatic invariant, Λ_{*s} must also be a function of the adiabatic invariants. This restriction is referred to as a kinetic constraint. More precisely, following Ref. [26], here Λ_{*s} is taken to be of the type

$$\Lambda_{*s} = \Lambda_s(\Phi_{*s}, \psi_{*s}). \tag{32}$$

Under these assumptions, invoking Eqs. (29) and (30), the structure functions can be Taylor expanded to give

$$\Lambda_{*s} = \Lambda_s\left(\Phi_s^{eff}, \psi\right)[1 + O(\varepsilon_s) + O(\sigma_s)]. \tag{33}$$

The following remarks are in order:

(1) Because of the conservation of the two adiabatic invariants $\mathcal{J}_R$ and $\mathcal{J}_z$, the functional dependence of f_{*s} in Eq. (31) differs from the type of solution previously obtained for collisionless accretion disk plasmas given in Refs. [24–27]. This means that the subspecies of oscillating particles in epicyclic motion is characterized by an intrinsically different equilibrium KDF.

(2) The choice of the kinetic constraints, namely, the functional form to be prescribed on the structure functions Λ_{*s}, is determined by the possibility of allowing for an analytical treatment of the kinetic solution and the calculation of the corresponding fluid fields. In particular, the validity of the expansion (33) is a prerequisite for obtaining a solution which is consistent with the Chapman-Enskog series representation. In fact, Eq. (32) is consistent with the requirement of adiabatic invariance and permits at the same time an explicit treatment of the implicit phase-space dependencies carried by the same structure functions.

(3) The equilibrium KDF is generally non-Maxwellian. This feature arises because of both the kinetic constraints as well as the existence of the established adiabatic invariants, and in particular the magnetic moment m'_s and the invariants $\mathcal{J}_R$ and $\mathcal{J}_z$.

(4) In order to make possible a physical description of a toroidal plasma, the equilibrium KDF must be characterized by nonuniform fluid fields, to be properly specified according to the case of equatorial or off-equatorial tori. These must include in particular nonuniform number density, flow velocity, as well as parallel and perpendicular temperatures arising from temperature anisotropy. As shown in Refs. [24,25,31,32], the latter should be mainly ascribed to the adiabatic conservation of the particle magnetic moment. It is also expected that the constraints imposed by the existence of the two adiabatic invariants $\mathcal{J}_R$ and $\mathcal{J}_z$ should affect the functional form of the equilibrium fluid fields (including the temperature anisotropy) with respect to the case of generalized bi-Maxwellian KDF treated in Refs. [24,25,31,32].

VII. CONSTRUCTION OF THE EQUILIBRIUM KDF

In this section, we proceed with the determination of an explicit solution for the equilibrium KDF which is characterized by the general features outlined above. Here, we consider the case of a collisionless plasma composed by oscillating particles in epicyclic motion in axisymmetric configurations and look for a quasistationary KDF with the following properties:

(i) The KDF is an adiabatic invariant of the form expressed by Eq. (31), which in general is different from a Maxwellian distribution.

(ii) Structure functions: a suitable set of structure functions Λ_{*s} is identified, which are subject to the kinetic constraints according to Eq. (32).

(iii) Temperature anisotropy: it is assumed that different parallel and perpendicular temperatures are allowed. In the case of temperature anisotropy associated with the magnetic moment conservation, the components are defined with respect to the local direction of the magnetic field.

(iv) Azimuthal rotation: a nonvanishing species-dependent azimuthal flow velocity is prescribed.

(v) Open, locally nested magnetic flux surfaces: the magnetic field is taken to allow quasistationary solutions with magnetic flux lines belonging to open and locally nested magnetic surfaces.

(vi) The solution is required to admit an asymptotic representation consistent with the Chapman-Enskog expansion which can allow for the treatment of implicit phase-space dependencies for the calculation of the relevant fluid fields.

In order to determine the equilibrium KDF, we generalize the solution obtained in Refs. [24–26] for accretion disk plasmas with the inclusion of the two adiabatic invariants $\mathcal{J}_R$ and $\mathcal{J}_z$. We impose the following requirements:

(a) The equilibrium KDF must recover the solution of Refs. [24–26] in the limit of vanishing $\mathcal{J}_R$ and $\mathcal{J}_z$, namely, for charges which are not characterized by epicyclic motion.

(b) The KDF f_{*s} must be a strictly positive real function and it must be summable, in the sense that the velocity moments of the form

$$\Xi_s(\mathbf{x}) = \int_W d^3v\, X_s(\mathbf{x}) f_{*s} \tag{34}$$

must exist for a suitable ensemble of weight functions $\{X_s(\mathbf{x})\}$, to be prescribed in terms of polynomials of arbitrary degree defined with respect to components of the velocity vector field $\mathbf{v}$, and where W denotes the volume of integration in velocity space.

Starting from the generalized bi-Maxwellian KDF introduced in Refs. [24–26] and in line with all of the previous requirements, a particular solution for the equilibrium distribution is therefore given by

$$f_{*s} = \frac{\beta_{*s}}{(2\pi/M_s)^{3/2}\, T_{\|*s}^{1/2}} \exp\left\{-\frac{\mathcal{J}_R \Omega_{*s}^R}{T_{\|*s}} - \frac{\mathcal{J}_z \Omega_{*s}^z}{T_{\|*s}}\right\} \times \exp\left\{-\frac{E_s - \frac{Z_s e}{c}\psi_{*s}\Omega_{*s}}{T_{\|*s}} - m_s' \alpha_{*s}\right\}, \tag{35}$$

which we refer to as the *generalized harmonic bi-Maxwellian KDF*. Here, the name "harmonic" refers to the fact that the KDF depends explicitly on the two invariants $\mathcal{J}_R$ and $\mathcal{J}_z$ which are characteristic of particles undergoing harmonic epicyclic oscillations. We also notice that the representation (35) is consistent with the requirements (a) and (b) specified above. In particular, the choice of introducing the exponential factor $\exp\{-\frac{\mathcal{J}_R \Omega_{*s}^R}{T_{\|*s}} - \frac{\mathcal{J}_z \Omega_{*s}^z}{T_{\|*s}}\}$ implies both the consistency with the previous results for nonoscillating particles [condition (a)] and the convergence of the solution in velocity space [i.e., its integrability according to condition (b)]. Following the notation of Refs. [24–26,32], here

$$\alpha_{*s} \equiv \frac{B}{\Delta_{T_{s*}}}, \tag{36}$$

$$\frac{1}{\Delta_{T_s *}} \equiv \frac{1}{T_{\perp s}} - \frac{1}{T_{\|*s}}, \tag{37}$$

where the quantities $\Delta_{T_s *}$, $T_{\perp s}$, and $T_{\|*s}$ are denoted, respectively, as generalized species temperature anisotropy, perpendicular and parallel temperatures, while

$$\beta_{*s} \equiv \frac{\eta_s}{T_{\perp s}}, \tag{38}$$

with η_s to be referred to as the generalized species pseudodensity. In Eq. (35), the structure functions are identified with the set

$$\{\Lambda_{*s}\} \equiv \left\{\beta_{*s}, \alpha_{*s}, T_{\|*s}, \Omega_{*s}, \Omega_{*s}^R, \Omega_{*s}^z\right\}, \tag{39}$$

where the quantities Ω_{*s}, Ω_{*s}^R, and Ω_{*s}^z are generalized frequencies to be related, respectively, to the fluid azimuthal rotation frequency and the radial and vertical oscillation frequencies. Notice that the functional form of the structure functions is prescribed according to the kinetic constraints (32). Hence, at this stage, the set $\{\Lambda_{*s}\}$ can not be directly identified with particular fluid fields, i.e., velocity moments of the KDF. In addition, it must be stressed that f_{*s} depends on the magnetic moment m_s' which is evaluated at the guiding-center position. The evaluation of the fluid fields requires preliminarily a back-transformation to the actual particle position (see Ref. [32]).

We remark that by construction, f_{*s} in Eq. (35) is an adiabatic invariant and therefore an asymptotic solution of the Vlasov equation. It follows that all of the velocity-moment equations obtained from the Vlasov equation (e.g., the continuity and linear momentum fluid equations) are identically satisfied in an asymptotic sense [25,31]. In addition, it is immediate to verify that Eq. (35) recovers the solution obtained in Refs. [24–26] in the limit in which the adiabatic invariants $\mathcal{J}_R$ and $\mathcal{J}_z$ vanish identically. Finally, invoking the definitions (13) and (22), Eq. (35) can be equivalently written as

$$f_{*s} = \frac{\beta_{*s} \exp\left[\frac{X_{*s}}{T_{\|*s}}\right]}{(2\pi/M_s)^{3/2}\, T_{\|*s}^{1/2}} \exp\left\{-\frac{\mathcal{J}_R \Omega_{*s}^R}{T_{\|*s}} - \frac{\mathcal{J}_z \Omega_{*s}^z}{T_{\|*s}}\right\} \times \exp\left\{-\frac{M_s(\mathbf{v} - \mathbf{V}_{*s})^2}{2T_{\|*s}} - m_s' \alpha_{*s}\right\}, \tag{40}$$

where $\mathbf{V}_{*s} = \mathbf{e}_\varphi R \Omega_{*s}$ and

$$X_{*s} \equiv M_s \frac{|\mathbf{V}_{*s}|^2}{2} + \frac{Z_s e}{c}\psi\Omega_{*s} - Z_s e \Phi_s^{eff}. \tag{41}$$

VIII. PERTURBATIVE THEORY

In this section, we proceed determining an asymptotic representation of the solution (35) which permits the treatment of the implicit velocity dependencies contained in the structure functions. We do this by adopting the perturbative technique developed in Refs. [25,26,31] and appropriate for the plasma belonging to the strongly magnetized SEPE regime defined in Sec. V. The perturbative expansion is first carried out in terms of the dimensionless parameters ε_s and σ_s introduced in Sec. V. Here, for greater generality we treat ε_s and σ_s as infinitesimals of the same order, with $\varepsilon_s \sim \sigma_s \ll 1$. Upon invoking Eqs. (29) and (30), the structure functions can be Taylor expanded accordingly. Correct to first order, i.e., neglecting corrections of $O(\varepsilon_s\sigma_s)$, as well as of $O(\varepsilon_s^k)$ and $O(\sigma_s^k)$, with $k \geqslant 2$, one obtains

$$\Lambda_{*s} \cong \Lambda_s + (\psi_{*s} - \psi)\left[\frac{\partial \Lambda_{*s}}{\partial \psi_{*s}}\right]_{\substack{\psi_{*s}=\psi \\ E_s = Z_s e \Phi_s^{eff}}} + (\Phi_{*s} - \Phi_s^{eff})\left[\frac{\partial \Lambda_{*s}}{\partial \Phi_{*s}}\right]_{\substack{\psi_{*s}=\psi \\ E_s = Z_s e \Phi_s^{eff}}}, \tag{42}$$

where $\Lambda_s = \Lambda_s(\Phi_s^{eff},\psi)$ are the leading-order structure functions. When Eq. (42) is applied to the equilibrium KDF given in Eq. (35), a Chapman-Enskog representation can be recovered for f_{*s}. To first order in the expansion parameters, this gives

$$f_{*s} = f_s'\left[1 + \varepsilon_s h_s^{(1)} + \sigma_s h_s^{(2)}\right], \tag{43}$$

where $f_s' \equiv f_s'(E_s,\psi_{*s},m_s',\mathcal{J}_R,\mathcal{J}_z,\Lambda_s,\lambda^k t)$. Here, $h_s^{(1)}$ and $h_s^{(2)}$ represent, respectively, the diamagnetic and energy-correction contributions carried by the ε_s and σ_s expansions of the structure functions Λ_{*s}. In particular, the leading-order KDF f_s' is found to be

$$f_s' = \frac{\eta_s \exp\left[\frac{X_s}{T_{\parallel s}}\right]}{(2\pi/M_s)^{3/2}\, T_{\parallel s}^{1/2} T_{\perp s}} \exp\left\{-\frac{\mathcal{J}_R \Omega_s^R}{T_{\parallel s}} - \frac{\mathcal{J}_z \Omega_s^z}{T_{\parallel s}}\right\} \times \exp\left\{-\frac{M_s(\mathbf{v}-\mathbf{V}_s)^2}{2T_{\parallel s}} - m_s'\frac{B}{\Delta_{T_s}}\right\}, \tag{44}$$

which is referred to as the *harmonic bi-Maxwellian KDF*. Here, $\frac{1}{\Delta_{T_s}} \equiv \frac{1}{T_{\perp s}} - \frac{1}{T_{\parallel s}}$ is related to the temperature anisotropy carried by m_s', η_s denotes the pseudodensity, $\mathbf{V}_s = \mathbf{e}_\varphi R\Omega_s$, and

$$X_s \equiv M_s\frac{R^2\Omega_s^2}{2} + \frac{Z_s e}{c}\psi\Omega_s - Z_s e\Phi_s^{eff}. \tag{45}$$

The leading-order structure functions are therefore identified with the set $\{\Lambda_s\} = \{\beta_s,\alpha_s,T_{\parallel s},\Omega_s,\Omega_s^R,\Omega_s^z\}$, $\alpha_s \equiv \frac{B}{\Delta_{T_s}}$, and $\beta_s \equiv \frac{\eta_s}{T_{\perp s}}$. By construction, the functional form of the set $\{\Lambda_s\}$ is uniquely prescribed in terms of ψ and Φ_s^{eff}. It is important to notice here that, although $T_{\parallel s}$ and $T_{\perp s}$ have the dimensions of temperatures, they are not the parallel and perpendicular temperatures of the plasma. These in fact must be computed as velocity moments of the KDF. Since the distribution f_s' is non-Maxwellian (due to m_s', $\mathcal{J}_R$, and $\mathcal{J}_z$), its leading-order fluid moments do not necessarily coincide with the structure functions $\{\Lambda_s\}$. Furthermore, we notice that Eq. (44) contains more structure functions with respect to the solution obtained in Ref. [25] for accretion disk plasmas. This is a consequence again of the fact that the KDF is non-Maxwellian. It is precisely the conservation of $\mathcal{J}_R,\mathcal{J}_z$ which allows for the inclusion of the two fluid frequencies Ω_s^R and Ω_s^z. One can therefore interpret Eq. (44) as the equilibrium KDF which describes a collisionless plasma species characterized by macroscopic oscillations having frequencies Ω_s^R and Ω_s^z and exhibiting at the same time azimuthal flow velocity (to leading order) and temperature anisotropy. These features can have important consequences for the investigation of the kinetic stability of these systems.

Concerning the first-order terms in Eq. (43), explicit calculation gives

$$h_s^{(1)} = [Y_1 + Y_2]\frac{c}{Z_s e}M_s R(\mathbf{v}\cdot\mathbf{e}_\varphi), \tag{46}$$

$$h_s^{(2)} = [Y_3 + Y_4]\frac{M_s}{2Z_s e}v^2. \tag{47}$$

Here, Y_i, $i = 1,4$, are defined as

$$Y_1 \equiv A_{1s} + A_{2s}\left(\frac{H_s}{T_{\parallel s}} - \frac{1}{2}\right) + \frac{\frac{Z_s e}{c}\psi_*\Omega_s}{T_{\parallel s}}A_{3s} - \alpha_s m_s' A_{4s}, \tag{48}$$

$$Y_2 \equiv A_{2s}\left(\frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}} + \frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}}\right) - \frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}}A_{5s} - \frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}}A_{6s} \tag{49}$$

and

$$Y_3 \equiv C_{1s} + C_{2s}\left(\frac{H_s}{T_{\parallel s}} - \frac{1}{2}\right) + \frac{\frac{Z_s e}{c}\psi_*\Omega_s}{T_{\parallel s}}C_{3s} - \alpha_s m_s' C_{4s}, \tag{50}$$

$$Y_4 \equiv C_{2s}\left(\frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}} + \frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}}\right) - \frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}}C_{5s} - \frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}}C_{6s}, \tag{51}$$

where

$$H_s \equiv E_s - \frac{Z_s e}{c}\psi_s\Omega_s, \tag{52}$$

and the following definitions have been introduced:

$$A_{1s} \equiv \frac{\partial \ln \beta_s}{\partial \psi},\quad A_{2s} \equiv \frac{\partial \ln T_{\parallel s}}{\partial \psi},\quad A_{3s} \equiv \frac{\partial \ln \Omega_s}{\partial \psi},$$
$$A_{4s} \equiv \frac{\partial \ln \alpha_s}{\partial \psi},\quad A_{5s} \equiv \frac{\partial \ln \Omega_s^R}{\partial \psi},\quad A_{6s} \equiv \frac{\partial \ln \Omega_s^z}{\partial \psi}$$

and

$$C_{1s} \equiv \frac{\partial \ln \beta_s}{\partial \Phi_s^{eff}},\quad C_{2s} \equiv \frac{\partial \ln T_{\parallel s}}{\partial \Phi_s^{eff}},\quad C_{3s} \equiv \frac{\partial \ln \Omega_s}{\partial \Phi_s^{eff}},$$
$$C_{4s} \equiv \frac{\partial \ln \alpha_s}{\partial \Phi_s^{eff}},\quad C_{5s} \equiv \frac{\partial \ln \Omega_s^R}{\partial \Phi_s^{eff}},\quad C_{6s} \equiv \frac{\partial \ln \Omega_s^z}{\partial \Phi_s^{eff}}.$$

The quantities A_{is} and C_{is}, $i = 1,6$, represent the gradients of the structure functions across equipotential surfaces and are here referred to as generalized thermodynamic forces (see also Refs. [25,26]). Therefore, consistent with the Chapman-Enskog solution, the first-order terms $h_s^{(1)}$ and $h_s^{(2)}$ are proportional to the gradients of Λ_s and permit the treatment of collisionless plasmas characterized by nonuniform fluid fields. We also notice that consistency of Eqs. (46) and (47) requires that $\varepsilon_s \frac{c}{Z_s e}M_s R(\mathbf{v}\cdot\mathbf{e}_\varphi)(\frac{H_s}{T_{\parallel s}} - \frac{1}{2})\frac{\partial \ln T_{\parallel s}}{\partial \psi} \lesssim O(\varepsilon_s)$ and $\sigma_s \frac{M_s}{2Z_s e}v^2(\frac{H_s}{T_{\parallel s}} - \frac{1}{2})\frac{\partial \ln T_{\parallel s}}{\partial \Phi_s^{eff}} \lesssim O(\sigma_s)$, which implies that $T_{\parallel s}$ must be of the form $T_{\parallel s} = T_{\parallel s}(\varepsilon_s^l\psi,\sigma_s^l\Phi_s^{eff})$, with $l \geqslant 1$, i.e., slowly dependent on both ψ and Φ_s^{eff}.

IX. FLUID MOMENTS

Given the equilibrium KDF (35), in this section we consider the possibility of evaluating the relevant fluid fields associated with it, which are defined as velocity integrals according to Eq. (34). In principle, an exact calculation of the fluid moments could be carried out (e.g., numerically) for f_{*s} after prescription of the structure functions Λ_{*s}. Here, however, the calculation is based on analytical inspection by considering suitable asymptotic estimations of the equilibrium KDF in terms of the expansion pointed out in the previous section. We do this because the analytical approach allows one to display the basic physical properties of the solution and to understand how the kinetic features affect the corresponding fluid configuration. In particular, in the following we are concerned with the expressions of the species number density, the flow velocity, and the generally nonisotropic plasma temperature.

We start by considering plasma species satisfying the strongly magnetized SEPE regime orderings, for which the equilibrium KDF f_{*s} is expressed according to Eq. (43). We notice that, with respect to the solution obtained for accretion disk plasmas in Refs. [24–26], here the perturbative theory performed in terms of the ε_s and σ_s expansions as well as the corresponding analytical calculation of the fluid fields are modified by the simultaneous occurrence of the following two features: (1) The inclusion of the two adiabatic invariants $\mathcal{J}_R$ and $\mathcal{J}_z$, which carry additional velocity dependencies. (2) The existence of a temperature anisotropy associated with the adiabatic conservation of the magnetic moment m'_s.

In order to proceed, it is necessary here to develop further the perturbative treatment of the equilibrium KDF by considering additional asymptotic limits. We first assume that in Eq. (44)

$$\exp\left\{-\frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}}\right\} \sim \exp\left\{-\frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}}\right\} \sim O\left(1\right), \tag{53}$$

so that these exponential functions can be both expanded to first order to get

$$\exp\left\{-\frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}} - \frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}}\right\} \simeq 1 - \frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}} - \frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}}. \tag{54}$$

From the physical point of view, this means that, given the frequencies Ω_s^R and Ω_s^z, we consider a configuration in which the vertical and radial energies associated with the harmonic oscillations are small compared to the plasma temperature-related parameter $T_{\parallel s}$. We then introduce the parameter ς_s to compare the relative magnitude of the diamagnetic and energy-correction terms with respect to $\frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}}$ and $\frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}}$. Thus, we set

$$\left|\frac{h_s^{(i)}}{\frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}}}\right| \sim \left|\frac{h_s^{(i)}}{\frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}}}\right| \sim O\left(\varsigma_s\right), \tag{55}$$

with $i = 1,2$ according to Eq. (43), where we recall that here ε_s and σ_s are treated as infinitesimals of the same order. Given these premises, we can now proceed by considering two separate cases which permit us to display the role of the three invariants $\mathcal{J}_R$, $\mathcal{J}_z$, and m'_s in determining the functional form of the fluid fields and the relevant physical properties of the solution.

A. Case 1: Dominant $\mathcal{J}_R$ and $\mathcal{J}_z$

In this limit, we single out the contribution due to the invariants $\mathcal{J}_R$ and $\mathcal{J}_z$ with respect to the diamagnetic and energy-correction terms. This requires setting $0 < (\varepsilon_s \sim \sigma_s) < \varsigma_s < 1$. In addition, we also assume that the plasma is characterized by a small temperature anisotropy associated with m'_s, which implies in Eq. (43) the ordering $|m'_s \frac{B'}{\Delta_{T_s}}| \sim O(\varepsilon_{Ms}^k)$, with $k \geqslant 1$. Therefore, to leading order we can set $T_{\parallel s} \sim T_{\perp s}$. Then, neglecting corrections of $O(\varepsilon_s)$, $O(\sigma_s)$, and $O(\varepsilon_{Ms}^k)$, the equilibrium KDF in Eq. (43) reduces to

$$f_{*s} = f_s^{(asym-1)}\left[1 + \varsigma_s h_s^{(3)}\right], \tag{56}$$

where, respectively,

$$f_s^{(asym-1)} = \frac{\eta_s \exp\left[\frac{X_s}{T_{\parallel s}}\right]}{(2\pi/M_s)^{3/2}\, T_{\parallel s}^{3/2}} \exp\left\{-\frac{M_s\left(\mathbf{v}-\mathbf{V}_s\right)^2}{2T_{\parallel s}}\right\} \tag{57}$$

is the Maxwellian contribution, and

$$h_s^{(3)} = -\frac{\mathcal{J}_R\Omega_s^R}{T_{\parallel s}} - \frac{\mathcal{J}_z\Omega_s^z}{T_{\parallel s}} \tag{58}$$

represents the non-Maxwellian correction term, while the corresponding leading-order structure functions now become $\{\Lambda_s\} = \{\eta_s, T_s, \Omega_s, \Omega_s^R, \Omega_s^z\}$, with X_s retaining its expression given by Eq. (45). For the sake of illustration, we restrict the calculation of the species number density and temperature correct up to $O(\varsigma_s)$ in terms of f_{*s} given by Eq. (56), thus neglecting the higher-order contributions, while the flow velocity is calculated to its leading-order expression.

Let us start from the species number density. In validity of the previous orderings, the total number density n_s^{tot} is approximated as $n_s^{tot} \cong n_s$ and is given by the velocity integral

$$n_s = \int_W d^3v\, f_s^{(asym-1)}\left[1 + \varsigma_s h_s^{(3)}\right]. \tag{59}$$

Three contributions arise, giving

$$n_s = n_s^{(asym-1)}[1 + \Delta^R n_s + \Delta^z n_s], \tag{60}$$

where

$$n_s^{(asym-1)} = \eta_s \exp\left[\frac{X_s}{T_{\parallel s}}\right] \tag{61}$$

is the Maxwellian contribution to the number density, while $\Delta^R n_s$ and $\Delta^z n_s$ are the corrections arising from the non-Maxwellian features associated with $\mathcal{J}_R$ and $\mathcal{J}_z$, respectively. These are found to be

$$\Delta^R n_s = -\frac{\Omega_s^R}{T_{\parallel s}} \int_W d^3v\, \mathcal{J}_R f_s^{(asym-1)}, \tag{62}$$

$$\Delta^z n_s = -\frac{\Omega_s^z}{T_{\parallel s}} \int_W d^3v\, \mathcal{J}_z f_s^{(asym-1)}, \tag{63}$$

where the integrals can be explicitly computed (analytically or numerically) on the appropriate volume of integration W once the representation of the corresponding epicyclic frequencies is assigned (see Sec. III).

The calculation of the leading-order flow velocity is immediate and gives $\mathbf{V}_s = \mathbf{e}_\varphi R\Omega_s$, which is purely azimuthal and coincides with the drift velocity carried by the exponential factor in Eq. (57). We notice that, in this approximation, the leading-order flow velocity is therefore not affected by the non-Maxwellian contributions. It is nevertheless possible to show that, since $h_s^{(3)}$ is an even function in the radial and vertical velocity components, it can not be responsible for the occurrence of additional components of the flow velocity along the same spatial directions.

Finally, we can proceed computing the species temperature T_s^{tot}. Given validity of the previous orderings, it is found that to leading order the temperature T_s^{tot} is approximated as $T_s^{tot} \cong T_s$ and defined as

$$T_s = \frac{M_s}{n_s}\int_W d^3v \frac{(\mathbf{v}-\mathbf{V}_s)^2}{3} f_{*s}, \tag{64}$$

in terms of Eq. (56). The calculation yields

$$T_s = T_s^{(asym-1)}[1+\Delta^R T_s + \Delta^z T_s], \tag{65}$$

where here $T_s^{(asym-1)} \equiv T_{\parallel s}$ is the leading-order term, which coincides with the temperature carried by the Gaussian exponent in Eq. (57) (Maxwellian term). The non-Maxwellian contributions $\Delta^R T_s$ and $\Delta^z T_s$ are defined, respectively, by

$$\Delta^R T_s = -\frac{\Omega_s^R}{T_{\parallel s}}\frac{M_s}{n_s}\int_W d^3v \frac{(\mathbf{v}-\mathbf{V}_s)^2}{3}\mathcal{J}_R f_s^{(asym-1)}, \tag{66}$$

$$\Delta^z T_s = -\frac{\Omega_s^z}{T_{\parallel s}}\frac{M_s}{n_s}\int_W d^3v \frac{(\mathbf{v}-\mathbf{V}_s)^2}{3}\mathcal{J}_z f_s^{(asym-1)}. \tag{67}$$

These results show how the non-Maxwellian character of the equilibrium KDF affects the number density and the temperature of the plasma with respect to the Maxwellian solution. In this approximation, the flow velocity is found to have only azimuthal direction. Additional corrections in fact can only arise when the contributions $h_s^{(1)}$ and $h_s^{(2)}$ of the KDF are taken into account. When the magnetic moment is retained, the latter are expected to be also responsible for the occurrence of nonvanishing velocity components along R and z directions (see next section).

B. Case 2: Temperature anisotropy

The main purpose of this section is to prove that the temperature anisotropy associated with the magnetic moment conservation can be responsible for the occurrence of a nonvanishing poloidal flow velocity associated with the equilibrium plasma configuration. We then proceed by examining the physical reasons behind this and the role of the adiabatic invariants $\mathcal{J}_R$ and $\mathcal{J}_z$ in affecting this feature.

We start by assuming that the temperature anisotropy term Δ_{T_s} enters the KDF in its leading-order expression. Concerning the first-order contributions, we impose the validity of the ordering $0 < \varepsilon_s \sim \sigma_s \sim \varsigma_s < 1$, namely, we consider the diamagnetic and energy-correction terms of the same order of the contributions associated with the conservation of $\mathcal{J}_R$ and $\mathcal{J}_z$. The latter are treated according to Eq. (54). Before calculating the relevant fluid fields, it is necessary to express the guiding-center magnetic moment m_s' at the actual particle position. This can be done by invoking a back-transformation from the gyrokinetic state defined in terms of the Larmor-radius expansion parameter $\varepsilon_{M,s}$, as illustrated in detail in Ref. [32]. For this purpose, the particle velocity is represented as $\mathbf{v} = u\mathbf{b} + \mathbf{w} + \mathbf{V}_D$, where u denotes the parallel velocity with respect to the magnetic-field line having unit tangent vector $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$, $\mathbf{w} = w[\mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi]$ is the component of the velocity perpendicular to $\mathbf{b}$ of magnitude w, and $\mathbf{V}_D$ is a suitably defined drift velocity (see, for example, Refs. [25,32]). Here, $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{b})$ are orthogonal unit vectors, while ϕ denotes the gyrophase angle associated with the Larmor rotation around magnetic-field lines (see Ref. [32]). Then, correct through first order in $\varepsilon_{M,s}$, the back-transformation gives for m_s' the representation

$$m_s' = \mu_{0s}[1+\varepsilon_{M,s}\Delta\mu_s], \tag{68}$$

where $\mu_{0s} = \frac{M_s w^2}{2B}$ is its leading-order expression, while $\Delta\mu$ represents the correction of $O(\varepsilon_{M,s})$. Without loss of clarity, we omit here to report explicitly the full expression of $\Delta\mu_s$. This can be found in Ref. [32]. For the purpose of this section, it is sufficient to notice that $\Delta\mu_s$ is generally a function of the type $\Delta\mu_s = \Delta\mu_s(w^2,u,\mathbf{w})$, so that it contains both even as well as odd contributions in terms of the particle velocity components.

Given validity of these assumptions, for this example case the equilibrium KDF (43) becomes

$$f_{*s} = f_s^{(asym-2)}\left[1+\varepsilon_s h_s^{(1)} + \sigma_s h_s^{(2)} + \varsigma_s h_s^{(3)} + \varepsilon_{M,s}\Delta\mu_s\right], \tag{69}$$

where

$$f_s^{(asym-2)} = \frac{\eta_s \exp\left[\frac{X_s}{T_{\parallel s}}\right]}{(2\pi/M_s)^{3/2}T_{\parallel s}^{1/2}T_{\perp s}} \times \exp\left\{-\frac{M_s(\mathbf{v}-\mathbf{V}_s)^2}{2T_{\parallel s}} - \mu_{0s}\frac{B}{\Delta_{T_s}}\right\}. \tag{70}$$

Here, the leading-order term $f_s^{(asym-2)}$ coincides with a drifted bi-Maxwellian KDF, $h_s^{(1)}$ and $h_s^{(2)}$ are given by Eqs. (46) and (47), while $h_s^{(3)}$ is defined according to Eq. (58).

Let us now consider the calculation of the dominant contributions of the fluid fields associated with the solution (69). Regarding the number density, one obtains

$$n_s = n_s^{(asym-2)}[1+\Delta n_s], \tag{71}$$

where for the leading-order contribution $n_s^{(asym-2)}$ carried by $f_s^{(asym-2)}$ one finds that formally $n_s^{(asym-2)} = n_s^{(asym-1)}$, with the latter being given by Eq. (61). Instead, Δn_s is the whole contribution originating by the first-order terms in Eq. (69) and whose explicit calculation is omitted because it is beyond the scope of this section. For what concerns the plasma temperature, a straightforward calculation proves that this is nonisotropic, with the leading-order parallel and perpendicular components carried by $f_s^{(asym-2)}$ coinciding, respectively, with $T_{\parallel s}$ and $T_{\perp s}$. These components of the temperature enter the definition of the corresponding leading-order species pressure tensor $\underline{\underline{\Pi}}_s$:

$$\underline{\underline{\Pi}}_s = \int_W d^3v\, M_s(\mathbf{v}-\mathbf{V}_s)(\mathbf{v}-\mathbf{V}_s) f_s^{(asym-2)}. \tag{72}$$

Explicit calculation proves that this is nonisotropic too and is given by

$$\underline{\underline{\Pi}}_s = p_{\perp s}\underline{\underline{\mathbf{I}}} + (p_{\parallel s} - p_{\perp s})\mathbf{bb}, \tag{73}$$

where $p_{\perp s} \equiv n_s^{(asym-2)} T_{\perp s}$ and $p_{\parallel s} \equiv n_s^{(asym-2)} T_{\parallel s}$ represent the leading-order perpendicular and parallel pressures, with $\underline{\underline{\mathbf{I}}}$ being the unitary tensor. Finally, the leading-order contribution of the flow velocity is found to be purely azimuthal and coincides with the velocity $\mathbf{V}_s = \mathbf{e}_\varphi R\Omega_s$ carried by $f_s^{(asym-2)}$. Concerning the first-order contribution, this is given by the velocity integral

$$\Delta\mathbf{V}_s = \int_W d^3v\,\mathbf{v} f_s^{(asym-2)} \times\left[\varepsilon_s h_s^{(1)} + \sigma_s h_s^{(2)} + \varsigma_s h_s^{(3)} + \varepsilon_{M,s}\Delta\mu_s\right]. \tag{74}$$

This integral can be computed either in terms of the cylindrical velocity components (v_R, v_φ, v_z) or, for example, in terms of the set (u,w,ϕ). In both cases, the calculation consists of integrals of a Gaussian distribution multiplied by a polynomial function of the particle velocity. The detailed expression of $\Delta\mathbf{V}_s$ is out of the scope of this work and we do not report it here. Instead, we are interested in showing that $\Delta\mathbf{V}_s$ has generally nonvanishing components in all the three spatial directions. In terms of the magnetic coordinates (ψ,φ,ϑ), this is therefore of the type

$$\Delta\mathbf{V}_s = \Delta V_{\varphi s}\mathbf{e}_\varphi + \Delta V_{\vartheta s}\mathbf{e}_\vartheta + \Delta V_{\psi s}\mathbf{e}_\psi, \tag{75}$$

where $\Delta V_{\varphi s}$ is the azimuthal contribution, $\Delta V_{\vartheta s}$ is the component along $\mathbf{e}_\vartheta \equiv \frac{\nabla\psi\times\nabla\varphi}{|\nabla\psi\times\nabla\varphi|}$ and $\Delta V_{\psi s}$ is the component along the direction $\mathbf{e}_\psi \equiv \frac{\nabla\vartheta\times\nabla\varphi}{|\nabla\vartheta\times\nabla\varphi|}$. The proof follows immediately by noting that, once expressed for example in the variables (u,w,ϕ), letting $\mathbf{v} = u\mathbf{b} + \mathbf{w} + \mathbf{V}_s$, the first-order polynomial contributions $(h_s^{(1)}, h_s^{(2)}, h_s^{(3)}, \Delta\mu_s)$ contain both odd and even dependencies on the particle velocity. Odd terms give null contribution to the integral (74), which is therefore only determined by the even terms. The latter are independent and generally nonvanishing in the presence of temperature anisotropy and nonuniform structure functions. It is also immediate to recognize that in general all three components in Eq. (75) are present. Consider, for example, odd integrals in the parallel velocity u. Since, from Eq. (6) $\mathbf{b} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{I\nabla\varphi + \nabla\psi\times\nabla\varphi}{|\mathbf{B}|}$, these terms decompose in contributions along both azimuthal and $\mathbf{e}_\vartheta$ directions. Similar considerations hold for odd integrals in the perpendicular velocity $\mathbf{w}$. In general, all terms in Eq. (74) contribute to generate $\Delta V_{\varphi s}$ and $\Delta V_{\vartheta s}$, which therefore arise due to diamagnetic, energy-correction, FLR effects, and non-Maxwellian features. On the other hand, the drift component $\Delta V_{\psi s}$ is uniquely associated with FLR effects and is due only to the contribution $\Delta\mu_s$ coming from the back-transformation of the guiding-center magnetic moment. Equation (75) represents the main conceptual result of this section and can have notable consequences on the self-generation of magnetic fields by the torus plasma (see discussion in the next section).

Before concluding, the following comments are in order about the physical interpretation of these results and the role of the conservation laws of $\mathcal{J}_R$ and $\mathcal{J}_z$ in this. In particular, note the following:

(1) According to the description of collisionless epicyclic plasma particles developed here, the contributions $\Delta V_{\vartheta s}$ and $\Delta V_{\psi s}$ arise only due to the existence of the temperature anisotropy which is included in the equilibrium KDF by the condition of adiabatic conservation of the particle magnetic moment m'_s.

(2) If such a temperature anisotropy is negligible, the adiabatic invariants $\mathcal{J}_R$ and $\mathcal{J}_z$ alone can not provide a physical mechanism for the occurrence of $\Delta V_{\vartheta s}$ and $\Delta V_{\psi s}$. In fact, as shown in the previous section (example case 1), they carry only even dependencies on the particle velocity which do not contribute to the fluid velocity.

(3) The flow velocity $\Delta\mathbf{V}_s$ is effectively modified by the existence of the adiabatic invariants $\mathcal{J}_R$ and $\mathcal{J}_z$. The latter contribute in two different ways. First, in an indirect way through the terms $h_s^{(1)}$ and $h_s^{(2)}$ by means of the nonuniform structure functions Ω_s^R and Ω_s^z to which they are associated with in the equilibrium KDF. Second, in a direct way through the term $h_s^{(3)}$. A similar type of contribution holds also for the rest of fluid fields (e.g., number density and temperature).

(4) The results presented here are applicable to the subset of the collisionless plasma characterized by the existence of oscillating epicyclic particles. Hence, the nonvanishing flow velocity components $\Delta V_{\vartheta s}$ and $\Delta V_{\psi s}$ can not be here interpreted as generating an accretion velocity of matter away from the neighborhood of the minimum of the effective potential. While preserving the conservation of m'_s, particles also oscillate keeping $\mathcal{J}_R$ and $\mathcal{J}_z$ asymptotically conserved. It follows that the poloidal velocity $\Delta V_{\vartheta s}\mathbf{e}_\vartheta + \Delta V_{\psi s}\mathbf{e}_\psi$ can be consistently interpreted as a drift velocity of the plasma on a bounded configuration domain.

X. KINETIC DYNAMO

In this section, we consider the solubility constraints imposed by the Ampere equation on the equilibrium kinetic solution obtained above. This will allow us to point out the existence of a kinetic dynamo, namely, the possibility of self-generation of a quasistationary magnetic field by the plasma currents.

In detail, for a quasistationary and nonrelativistic configuration, the Ampere equation for the self-magnetic field is

$$\nabla\times\mathbf{B}^{self} = \frac{4\pi}{c}\mathbf{J}, \tag{76}$$

where $\mathbf{B}^{self}$ is defined by Eq. (5) while $\mathbf{J}$ denotes the total plasma current density. The latter is defined as

$$\mathbf{J} = \sum_s \mathbf{J}_s = \sum_s Z_s e n_s^{tot}\mathbf{V}_s^{tot} = \sum_s Z_s e\int_W d^3v\,\mathbf{v} f_{*s}. \tag{77}$$

Invoking the representation of f_{*s} given in the previous section by Eq. (69) in the case of both nonvanishing temperature anisotropy and non-Maxwellian contributions give the following result:

$$\mathbf{J} = \sum_s [J_{\varphi s}\mathbf{e}_\varphi + J_{\vartheta s}\mathbf{e}_\vartheta + J_{\psi s}\mathbf{e}_\psi], \tag{78}$$

where, respectively, correct to first order in the relevant expansion parameters

$$J_{\varphi s} = Z_s e n_s^{(asym-2)}[R\Omega_s + \Delta V_{\varphi s}] + Z_s e n_s^{(asym-2)} \Delta n_s R\Omega_s, \tag{79}$$

$$J_{\vartheta s} = Z_s e n_s^{(asym-2)} \Delta V_{\vartheta s}, \tag{80}$$

$$J_{\psi s} = Z_s e n_s^{(asym-2)} \Delta V_{\psi s}. \tag{81}$$

Hence, in this case $\mathbf{J}$ has nonvanishing components along all of the three directions identified by the set of magnetic coordinates (ψ,φ,ϑ). We notice that in general $\mathbf{J}$ is nonvanishing for a single-species plasma or for a multispecies collisionless plasma in which each species has distinctive structure functions.

Let us now consider the azimuthal component of Eq. (76). This yields a generalized Grad-Shafranov equation for the poloidal flux function ψ_p of the self-field:

$$\Delta^* \psi_p = -\frac{4\pi}{c} J_\varphi, \tag{82}$$

where Δ^* is the Grad-Shafranov differential operator and $J_\varphi \equiv \sum_s J_{\varphi s}$. The remaining terms of Eq. (76) along the directions $\mathbf{e}_\vartheta$ and $\mathbf{e}_\psi$ yield two differential equations for the azimuthal component of the self-magnetic field I/R. These are, respectively,

$$\frac{\partial I}{\partial \psi} = \frac{4\pi}{c} J_\vartheta, \tag{83}$$

$$\frac{\partial I}{\partial \vartheta} = \frac{4\pi}{c} J_\psi, \tag{84}$$

which are subject to the constraint of solenoid current $\nabla \cdot \mathbf{J} = 0$, namely,

$$\frac{\partial J_\psi}{\partial \psi} = \frac{\partial J_\vartheta}{\partial \vartheta}. \tag{85}$$

This equation represents a solubility condition arising from the Ampere equation and must be intended as a constraint for the structure functions. Since generally I is of the form $I = I(\psi,\vartheta,\lambda^k t)$, the solubility condition (85) can always be satisfied by an appropriate choice of one of the structure functions. As a consequence, Eqs. (82)–(85) prove that the collisionless plasma considered here is able to generate a nonvanishing poloidal and azimuthal magnetic field in a quasistationary configuration. This mechanism is referred to here as a kinetic dynamo, in analogy with the mechanism found in Refs. [24,25,27,28,31,32] in the case of accretion disk plasmas. It must be stressed that this dynamo effect can occur in the absence of possible instabilities or turbulence phenomena. For the validity of the theoretical model developed here, the resulting self-magnetic field must be checked *a posteriori* to be consistent with the ordering assumption (7) introduced in Sec. II. Several notable features characterize the present result:

(1) The poloidal magnetic field is generated by azimuthal currents. To leading order, the azimuthal flow velocity $R\Omega_s$ is determined by the combined presence of gravitational and magnetic fields, mainly generated by external sources. Kinetic effects enter as first-order contributions in both the expressions of number density and azimuthal flow velocity.

(2) The azimuthal magnetic field is generated by poloidal currents. These arise only due to specifically kinetic effects. In the asymptotic approximation considered above, these contributions enter as first-order terms (with respect to the corresponding expansion parameters).

(3) Among the kinetic effects responsible for the kinetic dynamo, we distinguish the diamagnetic contributions originating from the ε_s expansion (poloidal magnetic flux expansion), the energy-correction contributions from the σ_s expansion, the non-Maxwellian contributions carried by the adiabatic invariants $\mathcal{J}_R$ and $\mathcal{J}_z$, and finally the FLR effects due to the conservation of the guiding-center magnetic moment.

(4) The kinetic dynamo for the toroidal field can be present only if the plasma exhibits a temperature anisotropy associated with m_s', which gives rise to poloidal drift currents in the presence of nonuniform structure functions (and therefore nonuniform fluid fields).

We conclude this section by noting that the result obtained here proves the main goal of this investigation, namely, the existence of a physically based dynamo mechanism that can operate in quasistationary configurations for axisymmetric collisionless plasmas in toroidal structures. For plasmas composed of epicyclic particles, such a dynamo effect is consistent with the result of Refs. [24,25,27,28,31,32] and can represent a promising phenomenon for the generation of magnetic loops in the presence of external EM and gravitational fields for astrophysical plasmas around compact objects.

XI. CONCLUSIONS

In this paper, the problem concerning the existence of astrophysical systems composed by magnetized plasmas able to generate magnetic loops in the gravitational field of compact objects has been addressed. To this aim, a suitable kinetic theory appropriate for the description of collisionless nonrelativistic and gravitationally bound plasmas has been developed in the framework of Vlasov-Maxwell description. The case of charged particles in toroidal axisymmetric configurations exhibiting epicyclic motion around minima of the effective potential in the equatorial plane has been treated.

Quasistationary solutions for the species equilibrium kinetic distribution function describing these systems have been determined, which are expressed in terms of the relevant particle adiabatic invariants characterizing particles oscillating with epicyclic frequencies. It has been shown that such solutions are generally non-Maxwellian, while, as proved here, they can always be conveniently expressed in terms of generalized harmonic bi-Maxwellian distributions. The main physical properties as well as the relevant phase-space anisotropies characteristic of the equilibrium distribution functions have been discussed. A suitable perturbative theory has been developed in order to allow for the treatment of the implicit phase-space dependencies carried by the quasistationary solution. As a consequence, the analytical calculation of the fluid fields associated with the kinetic distribution function has been investigated, proving that these are generally nonuniform, with the plasma exhibiting both azimuthal and poloidal nonvanishing flow velocities. Then, analysis of the Ampere equation has shown the existence of a kinetic dynamo effect, responsible for the self-generation of both poloidal and azimuthal quasistationary magnetic fields by the plasma currents in the absence of instabilities, turbulence, or accretion phenomena.

This conclusion provides a plausible physical mechanism that can explain the occurrence of magnetic loops around compact objects originating by magnetized plasmas confined in toroidal configurations and consisting of charged particles in epicyclic motion. The main properties of the dynamo effect have been analyzed. In particular, it has been proved that this represents a specifically kinetic phenomenon that can only be dealt with consistently in the framework of kinetic theory. Several notable features contribute to this mechanism, which have been identified here with diamagnetic, energy-correction, and finite Larmor-radius effects together with non-Maxwellian features of the equilibrium distribution function. Finally, concerning the self-generation of the azimuthal field, it has been pointed out that this can only take place in nonuniform plasmas characterized by temperature anisotropy associated with conservation of particle magnetic moment.

The theoretical outcomes of this work contribute to a better understanding of the dynamics of astrophysical gravitationally bound plasmas in connection with magnetic-field generation. In particular, they can provide a valuable background for possible future investigations on the subject of plasma kinetic equilibria and corresponding stability analyses, including studies concerning accretion disk plasmas and current-carrying string-loop models around compact objects.

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[1] A. L. Larsen, Classical Quantum Gravity **11**, 1201 (1994).
[2] A. V. Frolov and A. L. Larsen, Classical Quantum Gravity **16**, 3717 (1999).
[3] T. Jacobson and T. P. Sotiriou, Phys. Rev. D **79**, 065029 (2009).
[4] M. Kološ and Z. Stuchlík, Phys. Rev. D **82**, 125012 (2010).
[5] Z. Stuchlík and M. Kološ, Phys. Rev. D **85**, 065022 (2012).
[6] Z. Stuchlík and M. Kološ, J. Cosmology Astroparticle Phys. 10 (2012) 008.
[7] B. Carter, arXiv:hep-th/9705172.
[8] M. Livio, Phys. Rep. **311**, 225 (1999).
[9] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[10] J. E. Howard, H. R. Dullin, and M. Horányi, Phys. Rev. Lett. **84**, 3244 (2000).
[11] H. R. Dullin, M. Horányi, and J. E. Howard, Phys. D (Amsterdam) **171**, 178 (2002).
[12] J. Kovář, Z. Stuchlík, and V. Karas, Classical Quantum Gravity **25**, 095011 (2008).
[13] Z. Stuchlík, J. Kovář, and V. Karas, Symposium–Int. Astron. Union **259**, 125 (2009).
[14] J. Kovář, O. Kopáček, V. Karas and Z. Stuchlík, Classical Quantum Gravity **27**, 135006 (2010).
[15] O. Kopáček, J. Kovář, V. Karas, and Z. Stuchlík, AIP Conf. Proceedings **1283**, 278 (2010).
[16] O. Kopáček, V. Karas, J. Kovář, and Z. Stuchlík, Astrophys. J. **722**, 1240 (2010).
[17] J. Kovář, O. Kopáček, V. Karas, and Y. Kojima, Classical Quantum Gravity **30**, 025010 (2013).
[18] J. Kovář, P. Slaný, Z. Stuchlík, V. Karas, C. Cremaschini, and J. C. Miller, Phys. Rev. D **84**, 084002 (2011).
[19] A. N. Aliev and D. V. Galtsov, Gen. Relativity Gravitation **13**, 899 (1981).
[20] M. A. Abramowicz and W. Kluźniak, Astrophys. Space Sci. **300**, 127 (2005).
[21] G. Török and Z. Stuchlík, Astron. Astrophys. **437**, 775 (2005).
[22] P. Bakala, E. Šrámková, Z. Stuchlík, and G. T. Örök, Classical Quantum Gravity **27**, 045001 (2010).
[23] P. Bakala, M. Urbanec, E. Šrámková, Z. Stuchlík and G. Török, Classical Quantum Gravity **29**, 065012 (2012).
[24] C. Cremaschini, J. C. Miller, and M. Tessarotto, Phys. Plasmas **17**, 072902 (2010).
[25] C. Cremaschini, J. C. Miller, and M. Tessarotto, Phys. Plasmas **18**, 062901 (2011).
[26] C. Cremaschini and M. Tessarotto, Phys. Plasmas **19**, 082905 (2012).
[27] C. Cremaschini, M. Tessarotto, and J. C. Miller, J. Magnetohydrodynamics **48**, 3 (2012).
[28] C. Cremaschini, J. C. Miller, and M. Tessarotto, Symposium–Int. Astron. Union **274**, 228 (2011).
[29] C. Cremaschini, J. C. Miller, and M. Tessarotto, Symposium–Int. Astron. Union **274**, 236 (2011).
[30] C. Cremaschini, M. Tessarotto, and J. C. Miller, Phys. Rev. Lett. **108**, 101101 (2012).
[31] C. Cremaschini and M. Tessarotto, Phys. Plasmas **18**, 112502 (2011).
[32] C. Cremaschini and M. Tessarotto, Phys. Plasmas **20**, 012901 (2013).
[33] R. Narayan, R. Mahadevan, and E. Quataert, in *Theory of Black Hole Accretion Discs*, edited by M. Abramowicz, G. Bjornsson, and J. Pringle (Cambridge University Press, Cambridge, UK, 1998).
[34] D. A. Uzdensky and J. Goodman, Astrophys. J. **682**, 608 (2008).
[35] J. Goodman and D. Uzdensky, Astrophys. J. **688**, 555 (2008).
[36] C. J. Saxton, K. Wu, M. Cropper, and G. Ramsay, Mon. Not. R. Astron. Soc. **360**, 1091 (2005).
[37] C. J. Saxton, K. Wu, J. B. G. Canalle, M. Cropper, and G. Ramsay, Mon. Not. R. Astron. Soc. **379**, 779 (2007).
[38] C. Cremaschini and M. Tessarotto, Eur. Phys. J. Plus **126**, 42 (2011).
[39] C. Cremaschini and M. Tessarotto, Eur. Phys. J. Plus **126**, 63 (2011).

[40] C. Cremaschini and M. Tessarotto, Eur. Phys. J. Plus **127**, 4 (2012).
[41] C. Cremaschini and M. Tessarotto, Eur. Phys. J. Plus **127**, 103 (2012).
[42] C. Cremaschini and M. Tessarotto, Phys. Rev. E **87**, 032107 (2013).
[43] B. Paczyński and P. J. Wiita, Astron. Astrophys. **88**, 23 (1980).
[44] Z. Stuchlík and J. Kovář, Int. J. Mod. Phys. D **17**, 2089 (2008).
[45] S. M. Mahajan, Phys. Fluids B **1**, 43 (1989).
[46] S. M. Mahajan, Phys. Fluids B **1**, 2345 (1989).
[47] V. Krishan, T. D. Sreedharan, and S. M. Mahajan, Mon. Not. R. Astron. Soc. **249**, 596 (1991).
[48] P. Bhaskaran and V. Krishan, Astrophys. Space Sci. **232**, 65 (1995).
[49] M. S. Janaki and Brahmananda Dasgupta, Phys. Plasmas **19**, 032113 (2012).
[50] A. Beklemishev and M. Tessarotto, Astron. Astrophys. **428**, 1 (2004).
[51] L. D. Landau and E. M. Lifschitz, *Mechanics*, Vol. 1 (Addison-Wesley, New York, 1957).
[52] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Boston, 1980).
[53] C. G. Wells and S. T. C. Siklos, Eur. J. Phys. **28**, 105 (2007).