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Delayed feedback induces motion of localized spots in reaction-diffusion systems

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We study the formation of localized structures, often called localized spots, in reaction-diffusion systems subject to time delayed feedback control. We focus on the regime close to a second-order critical point marking the onset of a hysteresis loop. We show that the space-time dynamics of the FitzHugh-Nagumo model in the vicinity of that critical point could be described by the delayed Swift-Hohenberg equation. We show that the delayed feedback induces a spontaneous motion of localized spots. We characterize this motion by computing analytically the velocity and the threshold above which localized structures start to move in an arbitrary direction. Numerical solutions of the governing equation are in close agreement with those obtained from the delayed Swift-Hohenberg equation.

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I. INTRODUCTION

Dissipative structures found far from equilibrium are a well-documented area of research since the seminal works of Turing [1] and Prigogine and Lefever [2]. They have been observed experimentally [3]. Dissipative structures could be either periodic or localized in space. Many driven systems exhibit localized structures (LSs), often called localized spots and localized patterns, which may be isolated, randomly distributed, or self-organized in clusters forming a well-defined spatial pattern. Generally speaking, the spatial confinement arises from the balance between a positive-feedback mechanism associated with chemical reactions tending to amplify spatial inhomogeneities and a diffusion process that, conversely, tends to restore spatial uniformity. In addition, these structures reflect the spontaneous appearance of symmetry breaking and self-organization phenomena through dissipation.

Localized structures are homoclinic solutions (solitary or stationary pulses) of partial differential equations such as reaction-diffusion models. The conditions under which LSs and periodic patterns appear are closely related. Typically, when the Turing instability becomes subcritical, there exists a pinning domain where localized structures are stable. This is a universal phenomenon and a well-documented issue (for recent overviews on this issue see [4,5]). The LSs occur in various fields of nonlinear science such as chemistry [6], plant ecology [7], and optics [8,9]. They may exhibits a curvature instability [10]. Experimental observation of LSs in driven nonlinear optical cavities has motivated further the interest in this field of research [9]. In particular, LSs could be used as bits for information storage and processing [9].

Localized structures are not necessarily a stationary solution of nonlinear equations; they can exhibit regular motion in space or oscillation in time. In particular, the origin of the motion can be induced by vorticity [11], finite relaxation rates [12–14], a phase gradient [15], an Ising-Bloch transition [16–18], a walkoff, a symmetry breaking due to off-axis feedback [19], a resonator detuning [20], or Hopf-Turing interaction bifurcations [21]. Self-induced motion and vibrations of a vortex ensemble were also reported by Staliunas and Weiss [22]. A drifting two-dimensional localized structure in the FitzHugh-Nagumo model has been reported in an excitable regime with long-range inhibition and global coupling [23]. In the framework of the complex Ginzburg-Landau equation, the Gray-Scott model, and a three-component reaction-diffusion type of model, a time-periodic solution called scattors have been reported in Ref. [24]. It has been shown that the inclusion of delayed feedback in the dynamics of spatially extended systems can lead to a drift instability of localized structures [25]. More recently, a bifurcation analysis showed that delayed feedback can lead to the formation of oscillons or soliton rings [26].

In this paper we investigate the effect of delayed feedback on the mobility two-dimensional localized structures and localized patterns. We consider a simple reaction-diffusion type of model: the FitzHugh-Nagumo system. The control and engineering of far-from-equilibrium systems by delayed feedback are important problems in nonlinear science [27]. We use the time delayed control, which is the simplest scheme that has been previously proposed [28]. Other studies of various spatially extended systems with time delay have motivated further investigations of this issue [29]. The analysis based on delayed partial differential equations shows that when the product of the delay and the strength of the feedback exceeds some threshold, LSs start to move in an arbitrary direction [25,26]. We focus our analysis on double limits of long-wavelength symmetry-breaking instability (often called Turing instability [1]) and close to a second-order critical point marking the onset of a hysteresis loop. We derive a delayed Swift-Hohenberg equation for the FitzHugh-Nagumo system with delayed feedback. Then we show the existence of an instability that leads to a spontaneous motion of localized structures. We show that when the product of the delayed time and the feedback amplitude exceeds a certain threshold, LSs and localized patterns exhibit motion in an arbitrary direction. We evaluate analytically and numerically the threshold and the velocity of moving LSs.

In the next section we briefly introduce the FitzHugh-Nagumo system with delayed feedback. The derivation of a delayed Swift-Hohenberg equation is presented in Sec. II as well. The spontaneous motion of localized structures and localized patterns is presented in Sec. III. We summarize in Sec. IV.

II. MODEL EQUATIONS

A. FitzHugh-Nagumo model

Our analysis is based on the well-known FitzHugh-Nagumo system, which constitutes a simplified version of the Hodgkin-Huxley model that has been derived to describe electric excitations in nervous membranes [30]. The excitation is mediated by an electrochemical reaction involving sodium and potassium ion flow. This model reads

$$\begin{aligned}\partial_t u &= u - u^3 - v + d_u \nabla^2 u,\\ \partial_t v &= \epsilon(\gamma u - v - a) + \xi[v(t-\tau) - v(t)] + d_v \nabla^2 v,\end{aligned} \tag{1}$$

where $\epsilon = T_u/T_v$ is the ratio of characteristic chemical relaxation times of the activator u and inhibitor v. The Laplace operator $\nabla^2 = \partial_{xx} + \partial_{yy}$ acts in the $\mathbf{r} = (x,y)$ plane and t is time. The parameters a and γ control the relative position and the number of nullcline intersections. The diffusion coefficients are d_u and d_v. The delayed feedback is characterized by the time delay τ and the feedback strength ξ. It is convenient to subtract the inhibitor $v(\mathbf{r},t)$ from its delayed value $v(\mathbf{r},t-\tau)$, so that when we set $\tau = 0$, we recover the homogeneous steady states of the system u_s and v_s and solutions of the equations $a = (-1 + \gamma + u_s^2)u_s$ and $v_s = \gamma u_s - a$. These spatially uniform states are monostable when $\gamma > 1$. They exhibit a bistable behavior when $\gamma < 1$. In the following we consider a regime where the homogeneous steady states are stable with respect to the steady bifurcation and focus on the regime of pattern-forming instabilities, namely, when $d = d_v/d_u > 1$.

B. Real order parameter description

In what follows we explore the space-time dynamics of the FitzHugh-Nagumo system with delayed feedback (1) in the neighborhood of the critical point associated with bistability and close to the long-wavelength pattern-forming regime. The coordinates of the critical point are determined by the two conditions $\partial a/\partial u_s = \partial^2 a/\partial u_s^2 = 0$. This yields the critical values $a_c = 0$, $\gamma_c = 1$, and $u_c = v_c = 0$. Let us now examine the linear space dynamics of the model (1) in the vicinity of the critical point. The linear stability analysis with respect to the finite-wavelength perturbation of the form $\exp(i\mathbf{q}\cdot\mathbf{r} + \lambda\mathbf{t})$ yields the transcendental characteristic equation

$$\begin{aligned}&\lambda^2 + \{(d+1)q^2 + \epsilon - 1 + 3u_s^2 + \xi[1 - \exp(-\lambda\tau)]\}\lambda\\ &\quad + (q^2 - 1 + 3u_s^2)\{dq^2 + \epsilon + \xi[1 - \exp(-\lambda\tau)]\} + \epsilon\gamma\\ &\quad = 0.\end{aligned} \tag{2}$$

A Turing instability occurs if $\lambda = 0$ with a finite q and $\partial_q\lambda = 0$. In a long-pattern-forming regime, i.e., $q \to 0$ and close to the critical point $u_s \to 0$, the conditions $\lambda = 0$ and $\partial_q \lambda = 0$ impose the constraint $\epsilon = \epsilon_c = d$.

We now explore the space-time dynamics in the vicinity of the critical point $(u_c,v_c,a_c,\gamma_c,\epsilon_c) = (0,0,0,1,d)$. To this end, we introduce a small parameter η that measures the distance from criticality as

$$(a,\gamma,\epsilon) = (0,1,d) + (a_1,\gamma_1,\epsilon_1)\eta + (a_2,\gamma_2,\epsilon_2)\eta^2 + \cdots . \tag{3}$$

In addition, we introduce the slow space and time variables $(x',y') = \eta^{1/2}(x,y)$ and $t' = \eta^2 t$. For the delay parameters we consider the scaling

$$\xi = \eta^2 K, \quad \tau' = \frac{\tau}{\eta^2}. \tag{4}$$

We seek corrections to the steady states at criticality of the form

$$(u,v) = (u_1,v_1)\eta + (u_2,v_2)\eta^2 + \cdots . \tag{5}$$

Our aim is to determine a slow time and slow space amplitude equation that depends on time and space through the slow variables. From the FitzHugh-Nagumo equations (1) we obtain a sequence of linear equations for the unknown variables (u_1,v_1) and (u_2,v_2). We then apply solvability conditions at each order. Substituting (3) and (5) in Eqs. (1), we find that at first order in η, $u_1 = v_1$ and $a_1 = \gamma_1 = 0$. At second order, we have $u_2 = v_2 - \nabla^2 v_1$ and $a_2 = 0$. At third order, the solvability condition reads

$$\begin{aligned}\frac{d-1}{d}\partial_{t'} v_1 &= a_3 - \gamma_2 v_1 - v_1^3 + \frac{\epsilon_1}{d}\nabla^2 v_1 - \nabla^4 v_1\\ &\quad + K[v_1(t'-\tau') - v_1(t')].\end{aligned} \tag{6}$$

Note that this equation is valid only if $d > 1$. In terms of the original variables and setting $v_1 = b/\eta$, Eq. (6) reads

$$\begin{aligned}\partial_t b(\mathbf{r},t) &= A - \Gamma b(\mathbf{r},t) - b(\mathbf{r},t)^3 + 2D\nabla^2 b(\mathbf{r},t)\\ &\quad - \nabla^4 b(\mathbf{r},t) + \sigma[b(\mathbf{r},t-\tau) - b(\mathbf{r},t)],\end{aligned} \tag{7}$$

where $A = a - a_c$, $\Gamma = (\gamma - \gamma_c)/3$, $D = (\epsilon - \epsilon_c)/2d$, $\sigma = \eta^2 K$, $t' = (d-1)\eta^2 t/d$, and $(x',y') = \eta^{1/2}(x,y)$. In the absence of delay $\sigma = 0$, Eq. (7) is the generalized Swift-Hohenberg equation [31]. The distributed real variable $b(x,y,t)$ is proportional to the deviation from the inhibitor value at criticality. In what follows we consider a regime where the homogeneous steady states are stable with respect to the steady bifurcation and focus on the regime of pattern-forming instabilities, namely, when $D < 0$.

C. Linear stability analysis

The homogeneous steady state solutions of Eq. (7) are $A = b_s(b_s^2 + \Gamma)$. The bistability between uniform solutions occurs when $\Gamma < 0$. The linear stability analysis of the homogeneous steady states with respect to perturbations of the form $\exp(i\mathbf{q}\cdot\mathbf{r} + \lambda\mathbf{t})$ leads to the transcendental characteristic equation

$$\lambda = -\Gamma - 3b_s^2 - 2Dq^2 - q^4 + \sigma[\exp(-\lambda\tau) - 1]. \tag{8}$$

The homogeneous steady states b_s undergo a Turing instability leading to the spontaneous formation of a stationary periodic pattern at $b_{\pm T} = \pm\sqrt{D^2 - \Gamma}/3$. These instabilities correspond to the occurrence of a zero eigenvalue $\lambda = 0$ at an intrinsic finite wave number $q = q_c = \sqrt{-D}$. It is the same at both bifurcation points $b_{\pm T}$. Thus spontaneous pattern-forming instability requires $D > 0$ and $D^2 > \Gamma$ in order to have $b_{\pm T}$ and q_c both real. Traveling wave instability occurs if a pair of complex conjugate eigenvalues has a vanishing real part and a nonzero imaginary part, i.e., $\lambda = \pm i\omega$. By replacing this expression in the dispersion reaction equation (8), we get

$$\begin{aligned}b_{\pm TW} &= \pm\sqrt{D^2 - \Gamma + \sigma[\cos(\omega\tau) - 1]}/3,\\ \sigma &= -\frac{\omega}{\sin(\omega\tau)}.\end{aligned} \tag{9}$$

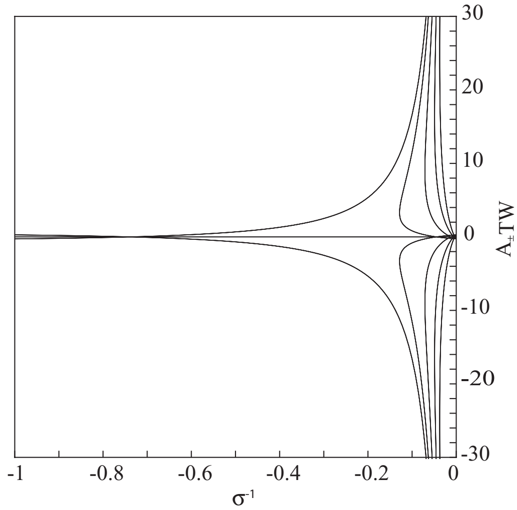

FIG. 1. Plot of the thresholds $A_{\pm TW}$ associated with traveling wave instabilities as a function of the delayed feedback strength σ in the monostable case $\Gamma > 0$. The parameters are $\Gamma = 1$, $D = -1$, and $\tau = 1$.

Note that the real part of the eigenvalue λ vanishes at $q = q_c = \sqrt{-\Gamma}$, which is the fastest growing wave number at the Turing instability even in the absence of the delayed feedback. The corresponding thresholds associated with the traveling wave instability are $A_{\pm} = (b_{\pm TW}^2 + \Gamma)b_{\pm TW}$. By taking into account Eq. (9), we get

$$A_{\pm TW} = \pm\frac{D^2 + 2\Gamma + F(\omega)}{3\sqrt{3}}\sqrt{D^2 - \Gamma + F(\omega)},\qquad \sigma = -\frac{\omega}{\sin(\omega\tau)}, \tag{10}$$

with $F(\omega) = \omega\tan(\omega\tau/2)$. We fix the parameters at $\tau = 1$ and $D = -1$ and let the feedback strength σ be the control parameter. The linear stability analysis is summarized in Figs. 1 and 2, where we plot the thresholds associated with traveling wave instability as a feedback amplitude. These figures are the parametric plot of Eqs. (10). The solid lines correspond to the traveling wave thresholds instabilities. When $\sigma < 0$, the instability domain associated with traveling wave instabilities increases with the increase of the feedback amplitude. For small σ, several pairs of traveling wave instabilities occur inside the instability domain. Each pair of traveling wave instabilities is connected to a different temporal frequency with finite wave number $q = q_c = \sqrt{-D}$. We have revisited the linear stability with respect to spatiotemporal perturbations in the presence of the delay. The study of the linearized problem shows that homogeneous steady state of Eq. (1) exhibits a pair of traveling wave instabilities.

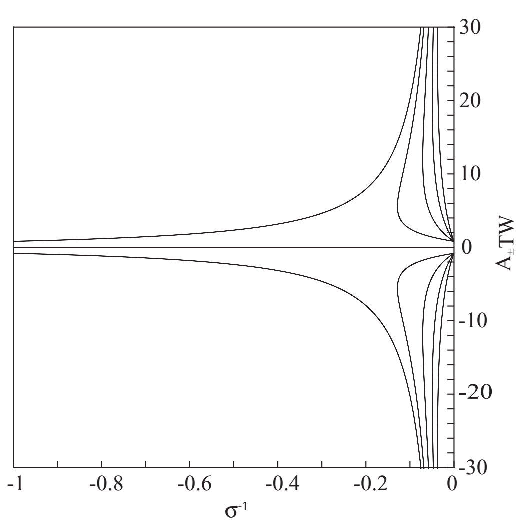

FIG. 2. Plot of the thresholds $A_{\pm TW}$ associated with traveling wave instabilities as a function of the delayed feedback strength σ in the bistable case $\Gamma < 0$. The parameters are $\Gamma = -1$, $D = -1$, and $\tau = 1$.

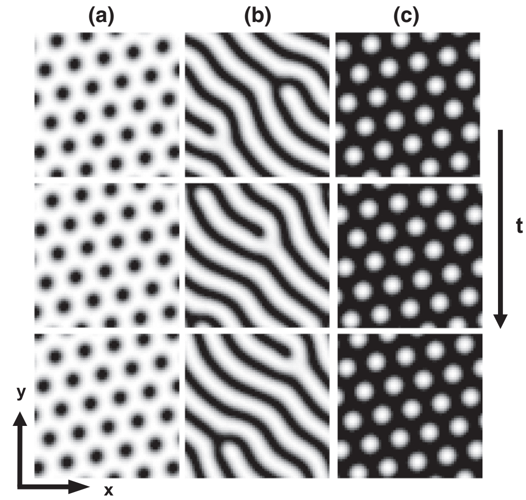

FIG. 3. Moving extended structures in time t under the effect of the delay: (a) hexagons $H0$, (b) stripes, and (c) hexagons $H\pi$. Maxima are plain white and mesh number integration is 60×60. The adimensional size of the system is 40×40. The parameters are $\Gamma = -1$, $D = -1$, $\tau = 1$, $\sigma = -1.02$, and (a) $A = 0.5$, (b) $A = -0.25$, and (c) $A = 0.5$.

We perform numerical simulations of Eq. (7) by using a finite-difference method with forward temporal Euler integration. The boundary conditions are periodic in both spatial directions. In the absence of the delayed feedback it is well known that the SH equation admits a sequence of stationary periodic hexagon- $(H0)$ stripe-hexagon $(H\pi)$ patterns. However, when varying the strength of the delayed feedback, these periodic structures start to move in an arbitrary direction with a constant speed. This behavior is shown in Fig. 3.

III. MOVING LOCALIZED STRUCTURES

The generalized Swift-Hohenberg equation (SHE) (7) without delayed feedback is one of the most studied models describing nonlinear dynamics in spatially extended systems. The SHE is a well-known paradigm in the study of spatial periodic or localized patterns. In this respect, it has been widely considered in hydrodynamics [31] and in other fields of natural science such as chemistry [32], plant ecology [33], and nonlinear optics [34]. An important property of the SHE is that it has a gradient structure, i.e., admits a potential

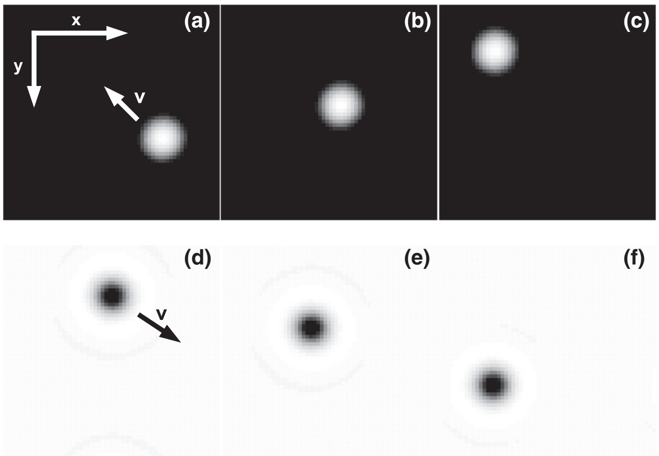

FIG. 4. Moving localized structures under the effect of delay in (x,y) space. Snapshots at different times showing the motion of LSs with a constant velocity: (a) and (b) bright and (c) and (d) dark localized structures obtained for the parameters $\Gamma = -1$, $D = -1$, $\tau = 1$, $\sigma = -1.05$, and (a)–(c) $A = -0.5$ and (d)–(f) $A = 0.5$. Maxima are plain white and mesh number integration is 60×60. The size of the system is 25×25.

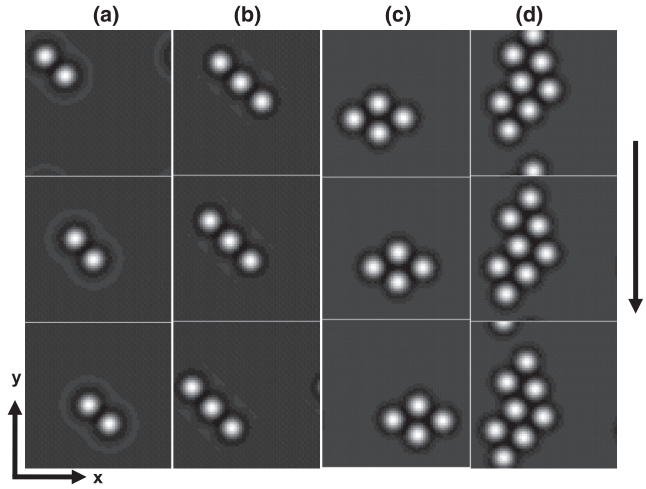

FIG. 5. Moving multipeak localized structures under the effect of the delay. Snapshots at different times showing the motion of (a) one peak, (b) two peaks, (c) three and four peaks, and (d) a cluster. Maxima are plain white and mesh number integration is 60×60. The size of the system is 40×40. The parameters are $\Gamma = -1$, $D = -1$, $A = 0.5$, $\tau = 1$, and $\sigma = -1.02$.

or a Lyapunov functional. Any perturbation evolves towards stationary homogeneous or inhomogeneous solutions. The existence of a Lyapunov functional pushes the time evolution towards the state for which the functional has the smallest possible value compatible with the boundary conditions. Stable stationary localized structures are homoclinic solutions of Eq. (7) with $\partial b/\partial t = 0$; they exist in the subcritical domain where a uniform solution and a branch of spatially periodic solutions are both linearly stable. There exist two types of stable LSs: dark and bright stationary pulses in two spatial dimensions. Their domain of stability for fixed values of the parameters $\Gamma = -1$ and $D = -1$ are for bright LSs $-0.52 < A < -0.11$ and for dark LSs $0.11 < A < 0.52$. However, in the presence of a delay term $b(x,y,t-\tau)$, the delayed SHE (7) loses the gradient structure and involves a nonvariational effect allowing LSs exhibiting uniform motion in an arbitrary direction. A direct manifestation of this nonvariational effect is the spontaneous motion of both bright and dark LSs as illustrated in the Fig. 4. The numerical simulations of Eq. (7) are performed on square-shaped domain with periodic boundary conditions. We use an initial condition consisting of a homogeneous steady state perturbed at ten grid points with an amplitude $\Delta b = 2.5$. This perturbation evolves rapidly towards the formation of moving single bright LSs [see Figs. 4(a)–4(c)]. For the dark localized spot, we use the same initial condition but with $\Delta b = -0.5$. By fixing the values of parameters $\Gamma = -1$ and $D = -1$, the stability domain for the bright LS occurs in the range $-0.60 < A < -0.23$ and the dark LS is stable in the range $0.60 < A < 0.23$. We clearly see a shift in the stability domain of the moving bright and dark LSs with respect to the one associated with stationary LSs. When LSs are sufficiently separated from each other, localized spots are independent and randomly distributed in space. However, when the distance between peaks decreases they start to interact via their oscillating, exponentially decaying, tails. This interaction leads then to the formation of clusters. When localized spots are moving, they exhibit a deformation along the direction of the motion. Examples of moving clusters formed by two more localized spots are shown in Fig. 5. These moving multispot solutions are obtained for the same parameters as in Figs. 4(a)–4(c), except that they differ only by the initial condition.

We calculate the threshold above which LSs exhibit motion with a constant velocity and derive an expression for the velocity. To do that, we assume that Eq. (7) without the delayed feedback, i.e., $\sigma = 0$, has a stable stationary radially symmetric localized structure $b = b_0(|\mathbf{r}|)$. The stability of this solution means that all the solutions Λ of the following eigenvalue problem $(\Gamma + 3b_0^2 + D\nabla^2 b_0 - \nabla^4)\phi = \Lambda\phi$ are real and negative except for a pair of zero eigenvalues corresponding to the translational invariance of Eq. (7), $\Lambda_{1,2} = 0$. Let us substitute a slightly perturbed single-spot solution $b(\mathbf{r},t) = b_0(|\mathbf{r}|) + \phi e^{\mu t}$ into Eq. (7). Then linearizing it with respect to a small perturbation ϕ, we obtain

$$\mu + (1 - e^{\mu\tau})\sigma = \Lambda. \tag{11}$$

In particular, for the twofold-degenerate eigenvalues $\Lambda_{1,2} = 0$, assuming that $|\mu| \ll 1$ and expanding Eq. (11) up to the second-order terms in μ, we get two real solutions

$$\mu_{1,2} = \frac{2(\sigma\tau + 1)}{\sigma\tau^2}, \quad \mu_{3,4} = 0, \tag{12}$$

where the zero solutions $\mu_{3,4}$ are associated with the translational symmetry of the model equations and $\mu_{1,2}$ change their sign at the drift instability point $\sigma\tau = -1$. At this point, Eq. (11) has fourfold-degenerate solutions $\mu_{1,2,3,4} = 0$. The stationary spot solution loses stability and exhibits a uniform motion that bifurcates from the stationary one. According to Eq. (12), the stationary spot is stable for $-1/\tau < \sigma < 0$ and becomes unstable for $\sigma\tau < -1$. The velocity of the moving single spot can be estimated by performing an expansion in terms of a small parameter ζ that measures the distance from the drift instability threshold $\eta\tau = -1 - \zeta^2$. Let us seek a solution of Eq. (7) in the form of a uniformly moving spot $b(\mathbf{r},t) = b_0(\mathbf{R}) + \zeta^3 \delta b(\mathbf{R}) + \cdots$, $\mathbf{R} = \mathbf{r} - \mathbf{v}t$, where b_0 is the stationary spot solution evaluated at the drift instability point, $\mathbf{v} = \zeta \mathbf{V}$ is the LS velocity, and δb is the correction to the LS shape due to its motion. We now plug this expression into Eq. (7) and use the expansion $b_0(\mathbf{R} - \zeta V\tau) = b_0(\mathbf{R}) - \zeta V\tau b_1(\mathbf{R}) + (\zeta V\tau)^2 b_2(\mathbf{R})/2 - (\zeta V\tau)^3 b_3(\mathbf{R})/6 + \cdots$, where $V = |\mathbf{V}|$ and $b_p = (\mathbf{V}\cdot\nabla b_{p-1})/V$ $(p = 1,2,3,4)$. By collecting the third-order terms in ζ, we obtain the inhomogeneous equation

$$\left(\Gamma + 3b_0^2 + D\nabla^2 b_0 - \nabla^4\right)\delta b = -Vb_1 + \frac{\sigma}{6}(V\tau)^3 b_3. \quad (13)$$

To satisfy the solvability condition, the right-hand side of this equation should to be orthogonal to the translational neutral modes $\phi_{x,y} = \partial_x b_0, \partial_y b_0$. By multiplying Eq. (13) with the linear combination of these modes $\mathbf{V}\cdot\nabla b_0/V \equiv b_1$ and integrating over two-dimensional space, we obtain the equation for the velocity

$$V(G_1 - \sigma V^2 \tau^3 G_2) = 0, \quad (14)$$

with $G_1 = \int_{-\infty}^{+\infty} b_1^2 dx\, dy$ and $G_2 = \int_{-\infty}^{+\infty} b_1 b_3 dx\, dy/6$. The nontrivial solution of Eq. (14) is given by

$$v = \zeta V = \frac{G}{\tau}\sqrt{-(1+\sigma\tau)}, \quad (15)$$

with $G = G_1/G_2$. This expression is valid not only for a single spot but also for any localized pattern. The spatial form of the pattern affects only the factor G in Eq. (15), which can be estimated numerically. The dependence of the spot velocity on the strength of the delay calculated by Eq. (15) is plotted for a fixed value of the feedback strength in Fig. 6.

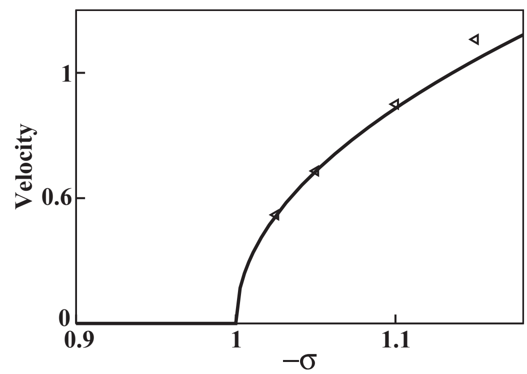

FIG. 6. Velocity of single-peak LSs as a function of the feedback strength. The parameters are $\gamma = -1$, $D = -1$, $G = 3.26$, and $\tau = 1$.

IV. CONCLUSION

We have studied the FitzHugh-Nagumo system with delayed feedback in two spatial dimensions. Our analysis focused on the neighborhood of the second-order critical point where a long-wavelength pattern-forming process takes place. We showed that in this regime the space-time dynamics of the FitzHugh-Nagumo model with delayed feedback is described by a generalized Swift-Hohenberg equation with time delay. We showed that there exists a threshold above which both dark and bright spots exhibit spontaneous motion in an arbitrary direction with a constant velocity. We provided an estimation of the velocity for single localized spots.

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