Scattering by a boundary with complex structure

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The distribution of escape times is usually studied in open billiards theory. In this work, we will concentrate on another important question: The distribution of outgoing rays by exit directions, which we refer to as billiard's indicatrix. It can be obtained analytically and consists of two parts: the symmetric diffuse part and the asymmetric directed part. The criterion for the separation of the indicatrix into these two parts is established. The asymmetry of the directed part of the indicatrix and the influence of the billiard's borders on it is investigated. We also propose a method of the creation of a matte surface model using open billiards with a fully diffuse indicatrix.

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I. INTRODUCTION

Questions of the dispersion and reflection from boundaries with a complicated structure arise in many physically interesting environments. Examples are granulated substances [1], mediums with fractally arranged boundaries [2], and so on. Generally speaking, the boundary of any real substance, if it is not specially processed, as a rule is extremely difficult. Owing to this complexity, the reflection from it possesses many general properties caused by the boundary structure. In researching these properties we offer a model of a structurally difficult boundary made of open billiards. A billiard is called open when its boundary is not closed and has sites through which particles can both enter and leave the billiard (see, for example, [3,4]). Such billiards have been intensively studied recently [5–8]. Attention was mainly given to the distribution times of the particle residence inside an open billiard, the general case study of which is not yet complete. It has been shown that for the billiards whose closed forms demonstrate strongly chaotic dynamics, the exponential distribution law is typical, and in the case of regular dynamics, sedate (see, e.g., [9]). The properties of these distributions and open questions were discussed in detail in Ref. [10]. In this work, we will concentrate on another prominent aspect, namely on how the directions of the trajectories leaving the billiard are distributed. We will use for this distribution term indicatrix in the sense of a distribution of outgoing rays by their exit directions. In particular, we will investigate how this indicatrix is formed from some parallel beams falling on the hole in the billiard boundary. This question is important for a wide set of phenomena, in particular, the nonspecular type of reflection from the boundaries of real bodies. For example, the not specular reflection of light from the boundaries of scintillation crystals [11] is well known.

The classical description of the propagation of light in a scintillation crystal with a diffusively reflecting surface leads to the integral equation of light gathering with a singular kernel. This is a complex integral nonlinear equation, and the search for and analysis of its exact solutions meets with serious difficulties. One of the possible solutions in such situations is modeling the processes of light propagation in a crystal using billiards (see, for example, [12]).

To model the propagation of light in a scintillation crystal, it is quite natural to use a billiard of the same form as the crystal. However, in this billiard the reflection from the boundary follows the mirror law whereas in the crystal the indicatrix of the reflection can have a significantly diffuse component. We will show that it is possible to overcome this contradiction and model a diffuse type of reflection by means of a strictly mirror reflection. For this purpose we use a billiard with a boundary that does not have a trivial structure. Such a macroscopically flat boundary consists of "microscopic" open billiards, possibly of several different kinds. Possible variants of the construction of such a boundary are shown in Fig. 1. In the first case, the reflection from the boundary as a whole is mainly ideally specular, with the amendment caused by beams from microscopic billiards. In another formation of the boundary, microscopic billiards are imposed against each other so that the macroscopical global billiard boundary entirely consists of inputs in microscopic billiards (Fig. 1, right). In this case the ratio of the specular and diffuse components of the reflection depends only on the indicatrixes of the reflection of microscopic open billiards.

Apart from the reflection from a boundary with a complex structure, the indicatrixes of reflection generated by open billiards also concerns other important questions. For example, modern methods of lithography allow the creation of nanostructures, often heterostructures based in gallium arsenide. The transport properties of such structures have interesting "anomalies." The electrons at the Fermi level in these structures at low temperatures are in ballistic mode and form a two-dimensional electron gas. The size of the system is less than the length of the electron's free path, therefore its resistance is defined not by the usual diffuse dispersion of electrons, but, at least for short trajectories, by the reflection from structure boundaries [13]. The system border is defined by the electrostatic depletion of the chosen area of the two-dimensional electronic gas. The resistance of such structures appears [14,15] to not be additive and shows complex, nonmonotonic alteration depending on the weak magnetic field enclosed perpendicularly to the structure. It is possible [16] to successfully explain all these anomalies within the scope of the classical movement of a particle in an open billiard. There, the electron plays the role of a particle

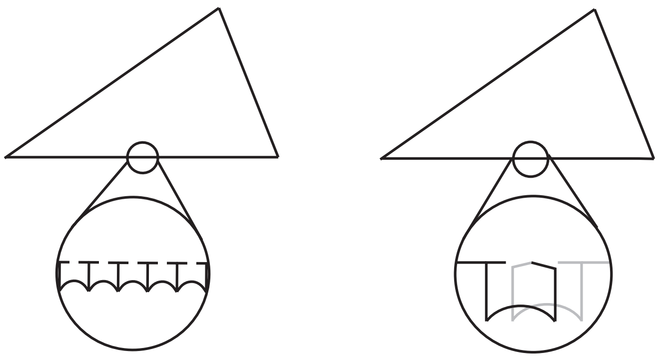

FIG. 1. Two ways of constructing global boundaries from microscopic billiards are shown. At left, a structurally complex border of a global billiard obtained by the simple joining of "microscopic" open disseminating billiards. The periodic top flat part of the boundary provides a considerable specular reflection component. On the right, the boundary of a global billiard obtained by overlaying of "microscopic" ones. In this case, the flat sites of the border are absent and the specular component is defined only by the properties of "microscopic" billiards.

moving in a billiard, and the border of the structure acts as the billiard boundary. The billiard's indicatrix is important for the resistance of these structures in a series. A comparison of the classical and quantum approaches shows [17–19] that the classical approach is in full agreement with the results of the quantum description of this system. Thus, questions of chaotic transport through open billiards are also important for the development of modern nanoelectronics.

II. MICROSCOPIC BOUNDARY STRUCTURE

The structure of a macroscopical boundary is defined by the periodic conjunction of microscopic billiards. Clearly, the choice of a microscopic billiard plays the key role in dispersion from such a boundary. In the case of a general position billiards possess chaotic dynamics. So as microscopic billiards we will choose those open billiards whose closed forms possess chaotic dynamics of beams with a positive Lyapunov's exponent, such as the disseminating billiard of Sinai [20]. The choice of the exact form of microscopic billiard is optional. As the simplest choice we will take the disseminating billiard shown in Fig. 2. In the closed form of such a billiard strong chaos is realized. The form of this billiard is defined by three dimensionless parameters, which may be easily constructed from the radius of the curvature of a concave site of border r, width of billiard l, heights h, and distance to the entrance window a. The presence of a disseminating site on a boundary with nonzero curvature guarantees chaos [21] in this billiard. A small number of the parameters characterizing the billiard form allows for the analysis of the influence of the billiard form on the reflection indicatrix.

III. DIFFUSE COMPONENT OF INDICATRIX

We will begin with general relationships. It should be noted that as soon as the "microscopic" billiard is open, the trajectories getting inside always leave the billiard after a certain number of collisions with its boundary. This fact

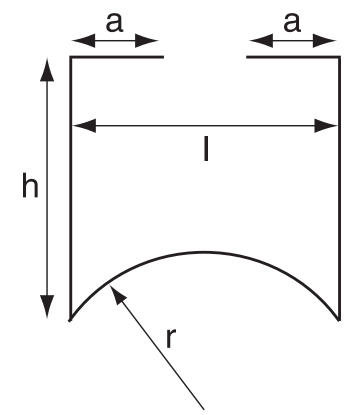

FIG. 2. Simple "microscopic" chaotic billiard. The form of this open billiard is defined by parameters r, h, a, and l.

naturally follows from Poincare's theorem of recurrence [22]. However, there is a difference between the trajectories leaving after several collisions and the trajectories that stayed in the billiard for quite a long time. Indeed, the loss of memory about the initial conditions occurs on times greater than the time of the uncoupling of correlations. For a strongly chaotic billiard this time $\tau_c \approx \frac{1}{\Lambda_1}$, where Λ_1 is a positive Lyapunov's exponent. In the case of $t < \tau_c$ memory may remain. In this case a narrow beam of falling trajectories will leave the billiard. This results in the reflections in certain directions depending on the angle of incidence and the form of the billiard.

The trajectories leaving the billiard after time $t > \tau_c$ will completely lose the memory of the angle of incidence and leave the microscopic billiard in directions independent of the initial data, having some universal distribution. This, in a sense, is a result of asymptotic chaos, while directed reflection can be thought of as the result of transient chaos [23]. Numerical modeling shows that this distribution completely coincides with the Lambert indicatrix [24] of diffuse reflection from an ideally matte surface. That is why it is possible to call this part of the indicatrix of the reflection, created by such trajectories, diffuse. Generally, an indicatrix generated by a "microscopic" billiard contains both diffuse and directed components. Some examples of the obtained indicatrixes are shown in Fig. 3.

For the billiard form described above (see Fig. 2) it is possible to derive in explicit form the criterion for the quantity

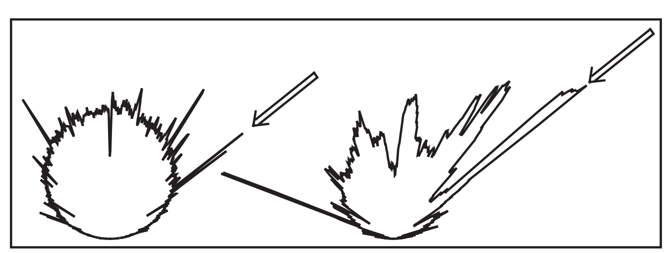

FIG. 3. On the left is a diffuse reflection indicatrix of a "microscopic" billiard with lateral walls and parameters $l = 100$, $h = 80$, $R = 55.1$, $a = 49.5$, and the angle of incidence $\varphi = 0.896$. On the right is the more general case of an indicatrix of reflection having diffuse and directed components. Billiard parameters are $l = 100$, $h = 80$, $R = 55.1$, $a = 45$, and $\varphi = 0.896$. The directions of falling beams are shown by arrows.

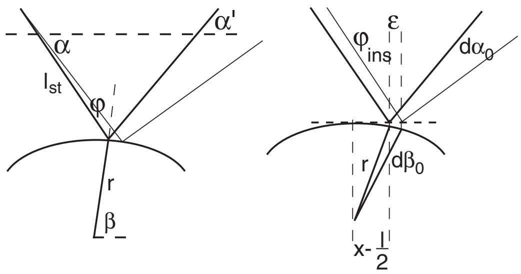

FIG. 4. Left: Reflection from the convex site of the border of the beam, falling from a point at distance l_{st} from the point of reflection, r is the radius of the curvature of the border. Right: Illustration explaining the choice of an initial angle of discrepancy of beams.

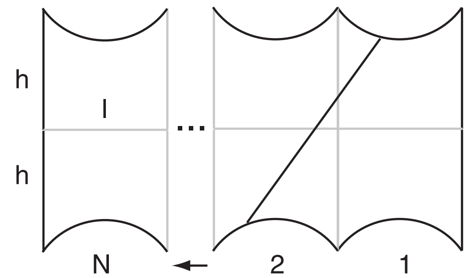

FIG. 5. Procedure of trajectory flattening on Nth construction step is shown here. This trajectory in the constructed billiard corresponds to collisions with concave sites of boundary.

of collisions with the convex part of a boundary, after which it is possible to consider the trajectory chaotic. Because of the presence of a disseminating boundary site, the discrepancy of trajectories, including the angle between directions of their movement, moves exponentially: $\Delta\alpha = \Delta\alpha_0 e^{\Lambda n}$. We will now estimate the Lyapunov exponent.

Let us consider two close beams, which fall on the convex part of a boundary having angles with horizontal axes α and $\alpha + d\alpha$ and the angle of incidence φ, as it is shown in Fig. 4. It is easy to see that after reflection the angle between these trajectories will increase on $2d\beta$

$$d\alpha' = d\alpha + 2d\beta. \tag{1}$$

Since $r d\beta \cos\varphi = l_{st} d\alpha$, we will receive

$$d\alpha' = \left(1 + 2\frac{l_{st}}{r\cos\varphi}\right) d\alpha. \tag{2}$$

Comparing with $d\alpha' = e^{\Lambda} d\alpha$, for the Lyapunov exponent we have

$$\Lambda = \ln\left(1 + \frac{2\bar{l}}{r}\left\langle\frac{1}{\cos\varphi}\right\rangle\right), \tag{3}$$

where $\bar{l}$ is the mean path in the billiard between collisions with the concave part. Since we cannot perform exact averaging along the trajectory, as an estimate here the average product is replaced by the product of average values. The average value of the cosine of the angle of incidence is not equal to zero because it can accept values only in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Further, for estimation we will use $\langle\frac{1}{\cos\varphi}\rangle \approx 1$.

To estimate $\bar{l}$ we will use the well-known Sabine's law [25]

$$\bar{l} = \frac{\pi S}{P}. \tag{4}$$

Here S is the billiard area and P its perimeter. However, its direct application to the closed billiard form will establish only the mean free path between collisions with all walls, not only with the convex site. To overcome this difficulty we will use the principle of trajectory flattening in collisions with rectilinear walls. As the result of such a procedure on the Nth construction step we will receive the billiard shown in Fig. 5. In this billiard, collisions occur only with convex sites ($N \gg 1$). Therefore the average free path calculated for it will give the length of the path between collisions with the scattering boundary part.

The average trajectory length in this billiard is easily calculated as

$$\bar{l} = \frac{\pi N 2S}{4h + N2l_r}. \tag{5}$$

Here $S = hl + \frac{l}{4}\sqrt{4r^2 - l^2} - r^2 \arcsin(\frac{l}{2r})$ is the the area of initial billiard, and $l_r = 2r\arcsin(\frac{l}{2r})$ is the length of the concave part. According to the obtained relations, at $N \gg 1$ the average length of the trajectory between concave parts is

$$\bar{l}(h,l,r) = \frac{\pi S}{l_r} = \frac{\pi\left[hl + \frac{l}{4}\sqrt{4r^2 - l^2} - r^2 \arcsin\left(\frac{l}{2r}\right)\right]}{2r\arcsin\left(\frac{l}{2r}\right)}. \tag{6}$$

This is indeed the mean free path between concave sites because the contribution of collisions with flat lateral walls in average length at N going to infinity goes to zero.

The same result can be obtained by noting that, owing to strong chaos in the considered billiard, collisions are distributed uniformly along its boundary, so its concave part has $\frac{l_r}{P}$ of the total number of collisions. The average mean path between them is a corresponding number of times greater $\bar{l}(h,l,r) = \frac{\pi S}{P}\frac{P}{l_r} = \frac{\pi S}{l_r}$.

Thus in order of magnitude we obtain

$$\Lambda = \ln\left(1 + \frac{2}{r}\left\langle\frac{\bar{l}}{\cos\varphi}\right\rangle\right) \approx \ln\left(1 + \frac{S}{l_r r}\right). \tag{7}$$

The value of the Lyapunov exponent depends on all three parameters that define the form of the microscopic billiard. We will now estimate the number of collisions after which the memory of the initial conditions is lost. The initial angle of the discrepancy of trajectories after the first collision with the scattering part (see Fig. 2) we estimate as

$$d\alpha_0 = 2d\beta_0 = 2\frac{l - 2a}{\sqrt{r^2 - \left(x - \frac{l}{2}\right)^2}}, \tag{8}$$

where $x = h\tan\varphi_{\text{ins}}\ mod\ l$. Considering that memory is lost as angle α reaches 2π we obtain

$$n_{cr} \approx \frac{\ln\frac{\pi\sqrt{r^2 - (x - \frac{l}{2})^2}}{l - 2a}}{\ln\left(1 + \frac{S}{l_r r}\right)}. \tag{9}$$

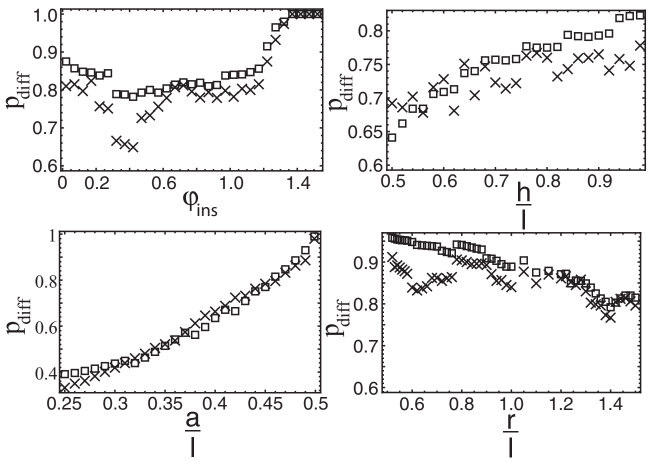

FIG. 6. Top left: Dependence of diffuse component part in indicatrixes of reflection on the angle of incidence φ, at $l = 100$, $h = 85$, $r = 150$, $a = 46$. Top right: Dependence of diffuse component part in indicatrix on height h, at $l = 100$, $a = 46$, $r = 150$, $\varphi = 0.57$. Bottom left: Dependence of same part on parameter a, at $l = 100$, $h = 85$, $r = 150$, $\varphi = 0.77$. Bottom right: Dependence of same part on curvature radius r, at $l = 100$, $a = 46$, $h = 85$, $\varphi = 0.77$. Small squares correspond to the diffuse part, calculated using criterion (9), crosses are obtained from the analysis of shape of the indicatrix of reflection.

The criterion thus obtained has a qualitative character defining a characteristic order of values. Thus, the trajectories leaving the billiard after a number of collisions greater than n_{cr} lose the memory of their entry conditions. Such trajectories leave the billiard in directions which do not depend on the initial data and follow the general distribution for such trajectories. These trajectories form the diffuse component of the indicatrix of reflection.

The next important question is the dependence of characteristics of the full reflection indicatrix on the form of the "microscopic" billiard. First, it is important to find out what portion of the falling beams goes in the diffuse reflection component and what in the directed one. Then it is necessary to consider the dependence of these on the parameters of the microscopic billiard and the angle of incidence of the beams. The influence of certain parameters at a qualitative level can be understood with simple considerations. For example, the increase in the curvature of the scattering part leads to an increase of the Lyapunov exponent and therefore the portion of the diffuse component. It is possible to come to the same conclusion analyzing the dependence n_{cr} on the curvature of the scattering part. The critical value decreases with an increase in the curvature $1/r$ and therefore the beams will fill up the diffuse component after a smaller number of collisions.

The diffuse component portion in the reflection indicatrix was calculated numerically and shown in Fig. 6. It was defined in two ways: the first, by the division of trajectories according to the criterion received above $n \geqslant n_{cr}$ (shown by thesquares) and the second by the direct analysis of the shape of the calculated indicatrix of reflection (shown by the crosses). This analysis is based on the choice of a parameter of Lambert distribution that best fits the constructed indicatrix. The results of these two methods correlate well. Some divergences arise if the falling beams hit a corner of the billiard or in a case of of a large curvature in the scattering part. In these cases it is necessary to consider the initial angle of divergence more precisely. Thus, generally criterion $n \geqslant n_{cr}$ defines the diffuse component portion well.

The diffuse component portion of the indicatrix depends mostly on the size of the entrance window $\varepsilon = l - 2a$. It is clear that with its reduction the trajectories will have more collisions with the scattering part before leaving the billiard. Hence the portion of the diffuse component increases with increase of a that corresponds to decrease of the entrance window. Shown on the bottom left of Fig. 6 is the corresponding dependence derived as a result of numerical calculation. For fairly large entrance windows the portion of the diffuse component is too small to be obtained from the indicatrix, therefore only values for $a > 0.25$ are shown.

Now we will discuss the influence of parameter h on the diffuse component. It is easy to see that with an increase of h the mean free path also increases, and hence Lyapunov's factor also increases. It in turn leads to an increase in the diffuse component part in the indicatrix of reflection according to criterion 1. At the top right of Fig. 6 this dependence is shown, obtained numerically. It is possible to note the increase of the diffuse component as parameter h increases.

It remains to discuss the dependence of the indicatrix on the angle of incidence. At small angles of incidence a small reduction of the diffuse component part is observed. A possible explanation is that with an increase in the angle of incidence, the bunch of trajectories getting into the billiard becomes narrower. A narrower bunch requires more time for the divergence and formation of diffuse trajectories. At angles of incidence close to horizontal the mean free path increases considerably, as does the angle of initial divergence, which results in an increase of diffuse component part to unity.

IV. SPECULAR REFLECTION COMPONENT

We will now discuss the directed components observed in the indicatrix of reflection. If the entrance window is big enough, of the order of characteristic billiard size, an appreciable portion of the falling beams will leave the billiard after only several reflections from its border. In this case the number of reflections will be insufficient for the mechanism of memory loss to work. Therefore the dispersion indicatrix will contain several directions on which the local maxima of the indicatrix (see Fig. 3) will be observed. It is convenient to begin with the simple case of $r \to \infty$. In this case the beams move in a rectangular billiard (Fig. 7) $n_{cr} = \infty$ and hence the diffuse component is absent. It is easy to see that in a rectangular billiard, irrespective of the number of collisions, the beams extend only in four directions. These are the direction of the initial input into the open billiard, the direction equivalent to the specular reflection of the falling beam, and its opposite. Therefore there are only two directions of exit from the open billiard: the direction of a specular reflection and the direction directly opposite to the falling direction. In other words, part of the beams after a number of reflections leave this open billiard in a direction strictly opposite to their falling direction.

If we choose a microscopic billiard that is a polygon with rational angles, the number of all possible directions of

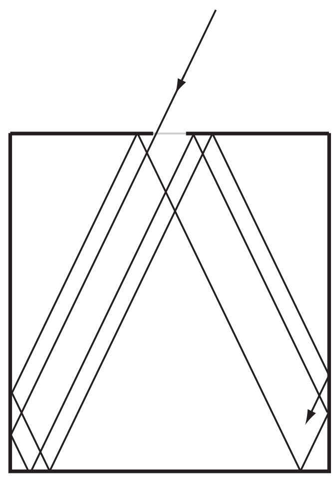

FIG. 7. A beam path in an open rectangular billiard is shown here. All possible directions of the movement of the beam and therefore the possible directions of its exit are visible.

movement of the trajectories increases but remains finite. The number of possible exit directions from the billiard increases. It is interesting to note that the specular component among them will be absent as a rule, while the opposite exit direction will be observed in half of such microscopic billiards.

With a finite radius of curvature the situation becomes more complicated. The number of possible exit directions becomes more than two and the number of observed prominent directions in the indicatrix depends on the angle of incidence and the exact form of the billiard. The formation mechanism of these directions is as follows. The falling stream of beams is divided into bunches, each of which will leave the billiard after a certain number of reflections from the billiards border. Bunches with a small number of collisions form corresponding splashes in the indicatrix of dispersion. With an increase in the number of collisions, as discussed above, the beams lose the memory of the initial conditions and form a diffuse component.

Following these concepts, we consider trajectories leaving the billiard after one collision (Fig. 8). It is easy to understand

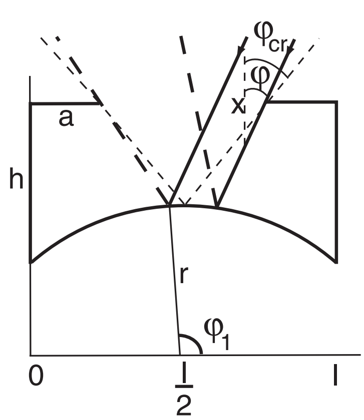

FIG. 8. A beam of trajectories leaving the billiard after one collision with its boundary is shown. This beam always maintains the direction of a specular reflection, corresponding to the reflection from the central point of the concave segment. The trajectory defining the critical angle is indicated by a dotted line.

that there is a critical angle φ_{cr}, such that if the angle of incidence $\varphi > \varphi_{cr}$ trajectories leaving after one collision do not exist. However, if $0 < \varphi < \varphi_{cr}$ they do exist. To find φ_{cr} we note, that if such trajectories exist, they necessarily contain the trajectory reflected from the center of the convex part of the boundary, i.e., a point with the coordinates $(\frac{l}{2},r)$. Therefore the critical angle is defined by the condition of the existence of this trajectory. At the moment of its disappearance the trajectory touches the right edge of the entrance window and passes through the central point of the scattering segment (see Fig. 8). This allows the calculation of the critical angle $\varphi_{cr} = \arctan \frac{\frac{l}{2}-a}{h+\sqrt{r^2-\frac{l^2}{4}}-r}$ from these simple geometrical considerations. Thus, the considered beams, if any, are reflected in the range of angles containing the angle of specular reflection. Hence part of the beams leaving the billiard after only one collision with its boundary contributes to the specular reflection component

We will now evaluate what part of the stream falling into the entire entrance window goes to this beam. As its measure we will choose the relation of the width of the beam reflected after one collision to the width of the entrance window $\frac{x}{l-2a}$. It is easy to see that x is the distance from the right edge of the entrance window to the beam which, after reflection, touches the left edge of the entrance window (see Fig. 8). This leads to a simple geometrical problem, after calculations we will see that under conditions $\arctan \frac{\frac{l}{2}-a}{h+\sqrt{r^2-\frac{l^2}{4}}} < \varphi < \arctan \frac{\frac{l}{2}-a}{h+\sqrt{r^2-\frac{l^2}{4}}-r}$

$$\frac{x}{l-2a} = 1 - \frac{\sin 2\left[\varphi_1 - \left(\frac{\pi}{2}-\varphi\right)\right]}{(l-2a)\sin\left[2\varphi_1 - \left(\frac{\pi}{2}-\varphi\right)\right]\sin\left(\frac{\pi}{2}-\varphi\right)} \times \left(h + \sqrt{r^2-\frac{l^2}{4}} - r\sin\varphi_1\right), \tag{10}$$

where φ_1 is defined implicitly by the equation

$$\left(\frac{l}{2}-a\right)\sin\left[2\varphi_1 - \left(\frac{\pi}{2}-\varphi\right)\right] + r\sin\left[\varphi_1 - \left(\frac{\pi}{2}-\varphi\right)\right] + \left(h+\sqrt{r^2-\frac{l^2}{4}}\right)\cos\left[2\varphi_1 - \left(\frac{\pi}{2}-\varphi\right)\right] = 0. \tag{11}$$

If the angle of incidence of this beam is $\varphi < \arctan \frac{\frac{l}{2}-a}{h+\sqrt{r^2-\frac{l^2}{4}}}$, then the extreme right of the beams passing through the entrance window does not leave the billiard after one collision with its border. Therefore the maximum of the considered portion of the beams does not lie at the normal angle of incidence of the bunch. The occurrence of such beams leads to a change of parities describing the part of beams leaving billiard after one collision. It is possible to prove that the maximum dependence of energy of the considered reflected beam is reached at $\varphi = \arctan \frac{\frac{l}{2}-a}{h+\sqrt{r^2-\frac{l^2}{4}}}$. The obtained analytical dependence (10) is quite complex, therefore the plot of this function is shown in Fig. 9.

Similarly, it is possible to consider the beams leaving after two, three, or more collisions with the boundary, till n_{cr} number of collisions with the scattering part. With a growth in the number of collisions the contribution of the considered beam

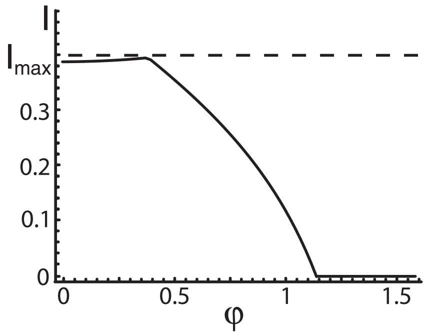

FIG. 9. Dependence of the portion of falling beams, corresponding with the one collision beam, on the angle of incidence of the stream $0 \leqslant \varphi \leqslant \pi/2$, for a billiard with parameters $l = 100$, $h = 40$, $r = 63$, $a = 17$.

to the indicatrix will decrease and the relations defining it will become more complicated. Thus, in principle, the directed part of the indicatrix of reflection can be constructed analytically.

The beam leaving after only one reflection is one of the most essential for the formation of the directed part of the indicatrix and, within a certain range of angles of incidence, exists at any admissible choice of billiard parameters. It can be used to establish a qualitative criterion with which it is possible to divide all billiards of a considered form into two classes: billiards with a purely diffuse indicatrix of reflection and billiards with a mixed indicatrix of reflection, containing a more or less clearly expressed directed reflection. It is possible to consider billiards with a purely diffuse reflection indicatrix those billiards with the energy density of the beam leaving after one reflection $\frac{p_{\max}}{2(\varphi_1+\varphi-\frac{\pi}{2})}$ at an angle of incidence when the energy of this beam is at its maximum, much less than the density of the energy of purely diffuse reflection $\frac{l-2a}{2}\cos\varphi$. It is similarly possible to receive values for the portion of energy in beams leaving the billiard after two and more collisions. These beams also form corresponding splashes in the reflection indicatrix.

Thus, the specular part of the reflection from the boundary as a whole in the first means of the formation (see Fig. 1) is made up of two parts: the reflection from the top flat sites of the boundary and the specular reflection generated by the interior of the microscopic billiard. The part of the energy of the falling beam, reflected in the specular direction, in this case weakly depends on the falling direction. The presence of flat sites between the entrance windows of microscopic billiards is not necessary. By superimposing microscopic billiards it is possible to make the whole macroscopic border only out of windows. Basically this makes such a border even more realistic than a boundary with a periodic structure of flat sites between entrance windows. In this case the influence of beams from the billiard onto the specular component will be more significant.

It should be noted that the construction of a boundary with microscopic billiards of several types and the modification of their form allow to obtain the indicatrix of reflection with different properties. For example, in real crystals, reflection occurs not only strictly mirror like, but in a certain narrow range of angles including the direction of the specular reflection. This imperfection can be modeled with a special billiard. Choosing the lateral sides for this billiard in the form of parabola sites we will receive the billiard shown in Fig. 10.

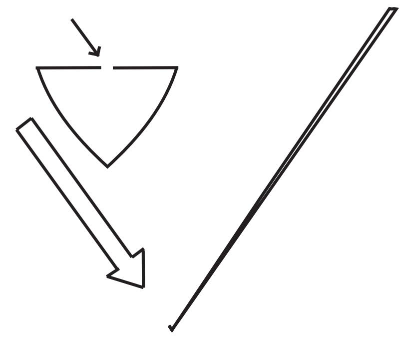

FIG. 10. The indicatrix of a nonideal specular reflection. Billiard parameters correspond to completely regular dynamics.

This billiard type can be referred to as a Bunimovich focusing billiard [26]. Certain parameters of this billiard correspond to completely regular dynamics, and the indicatrix of reflection for all angles of incidence looks like those shown in Fig. 10. The width of the specular reflected bunch varies depending on the size of the entrance window. Reflection in the direction opposite to the direction of the beams falling is practically absent for all angles of incidence.

As noted earlier, generally the reflection indicatrix has a significant contribution from the return reflection. The presence of this type of dispersion is characteristic for superconducting systems with Andreev type reflection [27], but is not observed in crystals, for example. The reflection indicatrix in scintillating crystals contains only specular and diffuse components [11], without reflection in opposite direction. We will consider the possibility of the suppression of such reflection by directing the form of a microscopic billiard. First of all we will note that reflection in the opposite direction will be obviously observed after transition to a rectangular billiard $r \to \infty$. Reflections from the lateral walls of the microscopic billiard are important for this effect. Therefore it is necessary to considerably modify the microscopic billiard, having removed the lateral walls. From the physical point of view it means a change in the billiard's topology and the transition to a billiard on a cylinder. Such a billiard can be received gluing the lateral walls to one another. The typical numerically obtained indicatrix of reflection of the modified billiard is shown in Fig. 11. It can be seen that this indicatrix has a corresponding maximum in the direction of the specular reflection.

Thus, by changing the billiard form, it is possible to receive various indicatrixes of dispersion containing diffuse

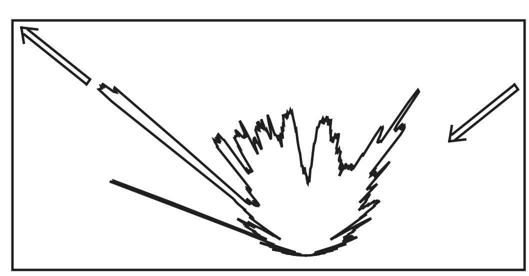

FIG. 11. Diffuse reflection indicatrix of a "microscopic" billiard without lateral walls with parameters $l = 100$, $h = 80$, $R = 55.1$, and $a = 45$ for an angle of incidence $\varphi = 0.675$. The direction of falling and specular reflection of beams are indicated by arrows.

components, strictly specular reflections, or reflections in some range of angles close to the specular direction.

V. INDICATRIX OF REFLECTION ASYMMETRY

We will now discuss in detail one more general property of indicatrixes, connected with return reflection. The indicatrix of reflection generated by open billiards is not symmetric, which is a consequence of the asymmetry of the directed part of reflection. Note that all trajectories leaving an open billiard can generally be divided into trajectories leaving to the left and to the right. It is roughly possible to name trajectories of "mirror" reflection as those that leave the billiard in the direction of movement of a falling beam. The second type of trajectories which leave the billiard roughly in the opposite to falling direction we will refer to as trajectories of the return "reflection." In this case we will consider only the movement of the beam projection on an axis parallel to the entrance window. To estimate the value of the indicatrix asymmetry it is possible to ask the following question: What part of the exiting trajectories changed their movement direction to the opposite, and, accordingly, what part of the beams kept their initial direction of movement? Let the falling beams have a positive x velocity component, i.e., it falls on the entrance window from the left, then we will denote as p the part of the beams which after exit have a negative x velocity component. In a sense, p is the part of the beams which are reflected in the opposite direction and form an echo effect. This part depends on all parameters changing the indicatrix of reflection and also on whether a case of a billiard with or without lateral walls is being considered.

As shown above, the part of the directed component is mostly influenced by the size of the entrance window. Therefore the dependencies of part p from the size of the billiard entrance window were obtained numerically. To track the influence of lateral walls, they were calculated for pairs of billiards with identical parameters and angles of incidence where the only distinction was the presence or absence of lateral walls. The typical dependencies obtained are shown in Figs. 12 and 13. It is shown that for a small size of the entrance window, reflection occurs symmetrically, which corresponds to purely diffuse reflection indicatrixes. However, starting from a certain size of entrance window asymmetry arises, and its character remains unchanged with a further increase of the

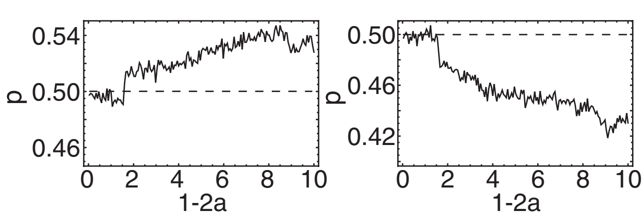

FIG. 12. Dependence on the size of the entrance window of the portion of trajectories that changed the sign of x, the velocity component after exit from the billiard, calculated for a billiard with parameters $h = 80$, $l = 100$ and $r = 55.11$ and an angle of incidence $\varphi = 1.15$. On the left for a billiard with lateral walls, on the right, without. The level of symmetric reflection is shown with a dotted line.

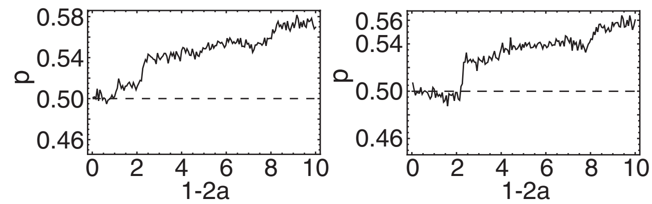

FIG. 13. Dependence of the part of trajectories that changes sign of the x velocity component after exit from the billiard on the size of entrance window calculated for a billiard with parameters $h = 100$, $l = 100$, and $r = 53.11$ for an angle of incidence $\varphi = 1.15$. On the left for a billiard with lateral walls, on the right, without. The level of symmetric reflection is shown with a dotted line.

entrance window. For the parameters of the billiard in Fig. 12, the dominance of the return reflection for a billiard with lateral walls is visible, and of a "mirror" reflection in the case of their absence. With different parameters in Fig. 13 the dominance of reflection in the opposite to falling direction and hence the occurrence of an echo effect for both forms of billiards is very visible. Therefore lateral walls can lead to the occurrence of an echo effect, however, their absence does not necessarily mean its disappearance. In changing the form of the billiard and the angle of incidence we also found that the critical value of window size, after which asymmetry appears, is not a constant and widely varies. A reasonably typical case is when even for $\frac{a}{l} = 0.4$ ($a = 40$ at $l = 100$) asymmetry does not yet appear.

Thus the echo effect depends on the parameters defining the form of the billiard. After the occurrence of asymmetry in the reflection indicatrix, the preservation of asymmetrical character is typical with a further increase in the entrance window size. This allows us to attribute a certain kind of asymmetry to defining the billiard form from the parameters $(h/l, r/l)$ and the angle of incidence φ, and also to investigate its dependence on the billiard parameters. These dependencies have been calculated for two types of microscopic billiards and certain angles of incidence and shown in Fig. 14. The plane of the billiard parameters has appeared divided into zones with reasonably smooth boundaries. In a sense these are bifurcation diagrams of asymmetry indicatrixes of the dispersion of open billiards.

The removal of the lateral walls in a billiard leads to an essential change in reflection asymmetry. As shown in gray, the "mirror" reflection type becomes prevalent. In both cases it is possible to notice a semblance of periodic behavior with a change in the height of the billiard h. In the case of a billiard without lateral walls in Fig. 14 it is clearly visible that the quasiperiod is equal to $2l$, and significantly less for a billiard with walls in the range of small r. The presence of such quasiperiodic behavior can be understood, having recalled that asymmetry is created by the beams of trajectories leaving the billiard after a small number of collisions with its boundary. Clearly, if one such beam leaves a billiard of height h under a certain angle to the horizontal α, then with an increase of the billiard height to $h + 2l\tan\alpha$ it will again get to the entrance window and leave on the same angle. Certainly, if the angle of incidence of the falling beams is not equal to $\frac{\pi}{2} - \alpha$, the initial falling bunch will not get on the same disseminating part of the boundary. However, despite

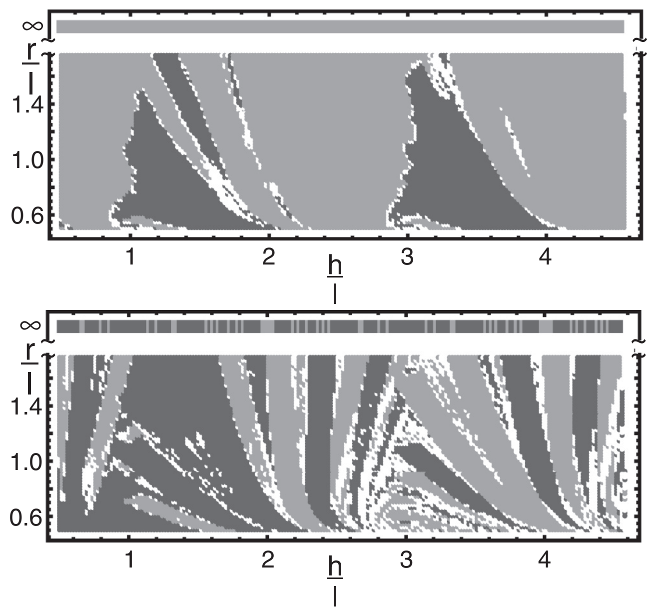

FIG. 14. The plane of billiard parameters (h,r) is shown. Above: A billiard without lateral walls. Below: With walls. Considering that the initial beams falls from the right, parameters corresponding to mainly leftward exit from billiard are shown in gray. For parameters in black: To the right (echo). In white: Reflection occurs symmetrically, and there is no dominating exit direction. Angle of incidence $\varphi = 1.15$, $l = 100$, and $a = 45$.

this, as we see in Fig. 14, some periodicity of asymmetry still remains. Another simple explanation of the quasiperiod structure of the zones of reflection can be obtained considering open rectangular billiards (corresponding $r \to \infty$). For this billiard the periodic structure of the zones of "mirror" and return reflection asymmetry can be obtained analytically. With the advent of curvature these zones "are washed away," become wider, but do not totally disappear.

VI. CONCLUSION

In this paper, the indicatrix of reflection of open billiards was studied, i.e., the distribution of exit directions of trajectories. It is shown that this indicatrix consists of two parts: the directed reflection and the diffuse part. Short, quickly exiting trajectories form the directed part of indicatrix, and reasonably long trajectories form the diffuse part. The criterion for the number of collisions with the scattering part of the billiards boundary, after which it is possible to consider the trajectory belonging to the diffuse part, was obtained. This part of the indicatrix corresponds to a universal distribution which does not depend on the parameters of an open billiard or the angle of incidence of beams and completely coincides with the Lambert indicatrix of reflection from an ideally matte surface. The dependence of the part of the diffuse component in reflection from the parameters of an open billiard and the angle of incidence was studied. The most significant influence on it is the size of billiard entrance window. For small entrance windows, the reflection indicatrix appears completely diffuse, which allows to model a macroscopic ideally matte boundary by superimposing many microscopic billiards.

The directed part of the reflection can, in principle, be constructed analytically. The simple part of the directed indicatrix component, retaining the direction of the specular reflection, was constructed. It is similarly possible to consider beams of trajectories leaving after a greater number of collisions with the scattering part of the boundary, until an $n_{\rm cr}$ number of collisions. All other trajectories belong to the diffuse part and have a corresponding distribution. Thus, we show that an indicatrix can be constructed completely analytically.

The asymmetry of the reflection indicatrix was considered. The space of the billiard parameters appears to be divided into smooth, quasiperiodic zones with different prevailing reflection types. The transition to a billiard on a cylinder without lateral walls significantly changes these zones, and the portion of the "mirror" reflection areas increases.

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