Reexamination of explosive synchronization in scale-free networks: The effect of disassortativity

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Previous work [J. Gómez-Gardeñes, S. Gómez, A. Arenas, and Y. Moreno, Phys. Rev. Lett. **106**, 128701 (2011)] has reported that explosive synchronization can be achieved in heterogeneous networks with a microscopic correlation between the structural and dynamical properties of the networks. This phenomenon, however, cannot be observed in all heterogeneous networks even if this structure-dynamics correlation is preserved. It is therefore of particular interest to identify the general topological factors that can induce the first order synchronization transition and to understand the underlying mechanisms. Here we investigate this issue using the scenario of the smooth transformation from homogeneous Erdös-Rènyi networks to heterogeneous Barabàsi-Albert networks. Specifically, we scrutinize how local and global properties of the network change during this process, and how these properties are associated with the emergence of explosive synchronization. We find that the local degree-degree correlation in the network contributes primarily to explosive synchronization, other than the global topological property or starlike subgraphs. We furthermore demonstrate that the degree of disassortative mixing also has a great effect in the presence of explosive synchronization.

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I. INTRODUCTION

A network is an abstraction of a set of interconnected units which is extensively used to describe diverse systems in the context of sociology, biology, economy, physiology, engineering, and others. Certain structural features have been found to be ubiquitous for various kinds of networks in the real world. One of the most fundamental of such properties is captured by the power law degree distribution, which can be reproduced with the so-called scale-free network model proposed in the seminal work of Barabási and Albert [1]. A scale-free network with power law degree distribution necessarily implies that a few nodes possess a large number of connections while most of the nodes usually have a low degree. The presence of highly connected nodes (i.e., hubs) is the most prominent feature of scale-free networks, indicating that scale-free networks are heterogeneous. It has been shown that this heterogeneity has a significant impact on the dynamical processes taking place in the networked systems. For instance, heterogeneous networks typically exhibit the robust-yet-fragile property in the face of node failures [2,3]. Other networked processes like synchronization in networks have also been found to be significantly affected by the topological heterogeneity of the underlying network. In particular, it is much harder for heterogeneous networks to synchronize than for homogeneous ones, even though the average network distance of the former is smaller. [4].

More interestingly, it is known that a weakly coupled network with strongly heterogeneous natural frequency does not display any coherent behavior. However, recent work [5] has shown that an explosive synchronization transition (the first order transition to synchronization being quantified by an order parameter) occurs in the heterogeneous scale-free networks when the natural frequency is positively correlated with the degree of the nodes. Accordingly, the emergence of this first order synchronization transition is claimed to be the result of heterogeneity and the above mentioned correlation. To fully understand the relationship between the emergence of the explosive synchronization and the topological properties that lead to this phenomenon, we revisited explosive synchronization in the scenario where the network topology changes progressively from Erdös-Rènyi (ER) networks [6] to Barabàsi-Albert (BA) scale-free networks, and the network heterogeneity increases with the smooth transition from an ER to a BA network.

We find, however, that explosive synchronization is not present in all heterogeneous networks. More specifically, the degree of heterogeneity under which explosive synchronization can arise still remains unclear. In other words, is there a critical heterogeneity for the networks to bring about explosive synchronization? This leads to the relevant question of whether there are other topological properties responsible for this abrupt synchronization transition, apart from network heterogeneity. In this paper, we address these questions by exploring the interplay between synchronization transition and the characteristics of network topology from both a macroscopic and a microscopic level.

II. THE MODEL

A widely adopted model for studying synchronization of oscillating systems is the well-known Kuramoto model [7]. Considering a network consisting of N phase oscillators with reciprocal interactions, the motion of the ith oscillator can be

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formulated as follows:

$$\dot{\theta}_i(t) = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \sin(\theta_j(t) - \theta_i(t)) \tag{1}$$

for all $1 \leqslant i \leqslant N$, where ω_i represents the intrinsic frequency of the oscillator. The coupling matrix $(A_{ij})_{N\times N}$ describes the corresponding network connections; thus if there is a link between oscillator i and j then $A_{ij} = 1$, and otherwise, $A_{ij} = 0$. In addition, the parameter λ represents the coupling strength associated with all links. We follow the definition of the synchronized state given in Ref. [5]: A synchronization state is one for which all oscillators evolve with the identical frequency, i.e., $\dot{\theta}_i(t) = \dot{\theta}_j(t)$ for all $i \neq j$. In this paper, the dynamics of the N oscillators are studied by solving Eq. (1) using the Runge-Kutta method, where ω_i are set to be the degree of node i. Since discontinuous synchronization transitions have been observed when the natural frequency positively correlates with the node degree, we continue to use this correlation to study the effects of local connection patterns on explosive behavior. To quantify the degree of synchronization of the whole network, the following measure of synchronization has been proposed [8]:

$$r = \left\langle \frac{1}{N} \left| \sum_{j=1}^{N} e^{i\theta_j} \right| \right\rangle, \tag{2}$$

where $\langle\cdot\rangle$ denotes the average over time. The relationship between structural properties and synchronization transition then can be explored by investigating the behavior of the order parameter under different network topologies.

We employ the network model proposed in Ref. [9] to generate a series of networks, on top of which the oscillators interact with each other and evolve in time. The advantage of this network model is to allow us to construct networks ranging from homogeneous networks (ER networks) to heterogeneous networks(BA networks) while keeping the average degree $\langle k\rangle$ unchanged, by tuning a single parameter $\alpha(0 \leqslant \alpha \leqslant 1)$. Although, intuitively, the heterogeneity is increased during the transition from ER networks to BA networks, it is necessary to characterize this change in a quantitative manner. In this light, we use the heterogeneity index given by Ref. [10], which is measured according to the difference of every two degree values in a degree sequence, i.e.,

$$H = \frac{\sum_i \sum_j |d_i - d_j|}{2N^2 \bar{d}}. \tag{3}$$

III. THE ROLE OF DEGREE-DEGREE CORRELATION

First, we consider a set of oscillators organized with a BA scale-free topology corresponding to the network model for $\alpha = 0$. We randomly choose one realization of the network model for $\alpha = 0$ and shuffle the connections among the nodes. The order parameter r is then averaged on 100 time steps for each λ after the transient period of phase oscillation and λ is increased progressively with step $\delta\lambda = 0.02$ in the process of synchronization. For a desynchronizing process, the simulation is performed by decreasing λ from the maximal value with $\delta\lambda = 0.02$ to the starting point where the synchronization process begins. Unsurprisingly, the first

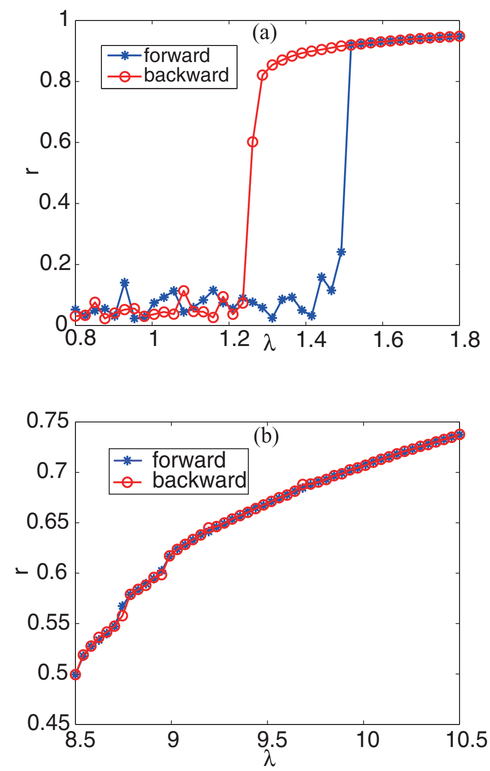

FIG. 1. (Color online) Synchronization diagrams for a BA network with $r_a = -0.078$ and its corresponding assortative mixing network with $r_a = 0.45$, respectively. Network size is $N = 1000$ and $\langle k\rangle = 6$ throughout this paper. (a) Hysteresis of the order parameter in the BA network and (b) the second order phase transition.

order phase synchronization transition is observed when the coupling strength λ is increased progressively, as is evidenced by the order parameter shown in Fig. 1(a). In contrast, by assortatively mixing the BA networks [11,12] (i.e., keeping the node degree unchanged while favoring nodes with similar degrees to be interconnected through random rewiring), we get a continuous synchronization transition; see Fig. 1(b). It should be pointed out that the reason λ used in Fig. 1(b) is larger than those in Fig. 1(a) is that the positive degree correlation has been proved to hinder the synchronization of a networked system [13], and thus a much stronger coupling strength is required to achieve an ordered state. The shuffling operations keep the degree sequence, and thus the heterogeneity of the network remains unchanged. Therefore this result indicates that not all heterogeneous BA networks are subject to explosive synchronization. Note that the assortative mixing here has changed the local patterns of connectivity. It is then natural to conjecture that the alteration in local topological properties may play an important role in the synchronization process. Following this line of thought, we focus on how the change in the local property of the network, i.e., degree-degree correlation, can affect explosive synchronization. The degree-degree correlation, also known as “assortativity in degree”, has been extensively studied in recent years [14–16]. The degree-degree correlation or assortativity is the tendency for nodes with similar degree to connect with each other. Formally, the assortativity of a network refers to the linear correlation coefficient defined in terms of the network degree distribution,

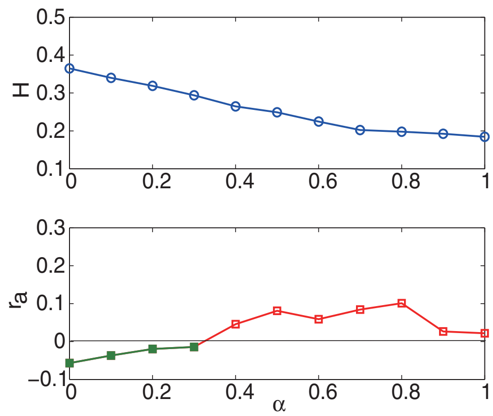

FIG. 2. (Color online) Heterogeneity and assortativity coefficients for different networks by tuning the model parameter α from 0 to 1. The solid squared line indicates parameter range in which explosive synchronization can occur. Note in all cases the same correlation between the natural frequency and the degree of the nodes is maintained.

its remaining degree distribution, and its link distribution. In practice, this quantity is evaluated by [14]

$$r_a = \frac{M^{-1}\sum_i j_i k_i - \left[M^{-1}\sum_i \frac{1}{2}(j_i + k_i)\right]^2}{M^{-1}\sum_i \frac{1}{2}(j_i^2 + k_i^2) - \left[M^{-1}\sum_i \frac{1}{2}(j_i + k_i)\right]^2}, \quad (4)$$

where j_i and k_i are the degrees of two end points of the ith link and M is the total number of links in the network. $r_a > 0$ implies that the network is assortative, while $r_a < 0$ means a disassortative mixing pattern of the network. Figure 2 is obtained by computing the heterogeneity index and assortativity simultaneously in the network series mentioned above, and the result is averaged over 20 realizations.

In Fig. 2 we can see that the heterogeneity decreases smoothly when the network topology transits from BA scale-free networks to ER random networks by tuning α from 0 to 1. It is therefore hard to associate the heterogeneity (or the change in it) of the network with the onset of explosive synchronization at this stage, or to conclude that the heterogeneity results in explosive synchronization, as the heterogeneous BA network is found to demonstrate a second order synchronization transition when assortatively mixed (illustrated in Fig. 1). However, when looking at the curve of assortativity versus α, one can easily spot that the assortativity value undergoes a transition from negative to positive, implying a topological transition from disassortative mixing to assortative mixing for the networks. What is of particular interest here is that the assortativity transition point coincides with the critical point corresponding to the onset of explosive synchronization, shown in Fig. 3, as well as the transition from solid-square to hollow-square lines in Fig. 2. For lack of a well-established quantity to indicate the first order transition, we define an error at this stage to measure the existence of the first order transition by comparing the difference between the maximal order parameter error and the average of all order parameter errors except the maximal one, in which the order parameter error δr is the variation of two successive order parameters, r, corresponding respectively to coupling strengths λ and $\lambda + \delta\lambda$. The reason is that we find that δr is very small for a continuous phase transition, while for a discontinuous phase transition there will be a big δr, and hence the difference between these two types of δr should be large in the case of explosive synchronization.

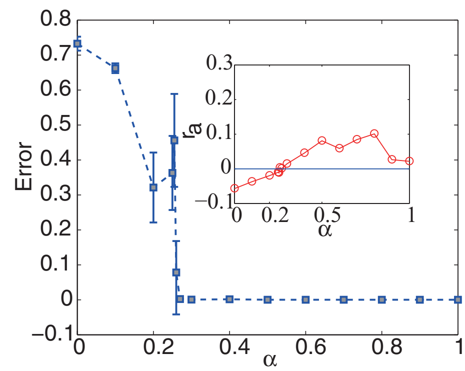

FIG. 3. (Color online) Synchronization errors under different heterogeneity parameters. The inset plot shows the transition of the assortativity property with heterogeneity parameter changing.

As shown in Fig. 3, the critical coupling strength is around 0.26. We have searched the critical point of explosive synchronization and an assortativity transition point using small steps (0.02) near $\alpha = 0.26$. Although the correspondence between these two critical points may not be exact, they appear to be close to each other, and most importantly we can find the strong association between the onset of explosive synchronization and the assortativity transition point. We also investigate the parameter space to manifest the role of assortativity in the emergence of explosive synchronization. High errors in Fig. 4 indicate the occurrence of the first order transition. It should be noted that due to the suppressing effect of positive degree correlation [13], the errors in homogeneous assortative networks are higher in a sense than disassortative networks for the same α. Additionally, in contrast to Ref. [12], the assortativity coefficient cannot approach two extrema, limited by the degree sequences. Further work will be required to systematically study this issue by fully exploring the parameter space. From these results, we can see that the explosive synchronization cannot be attributed only to heterogeneity, but is also intimately associated with the assortativity property of

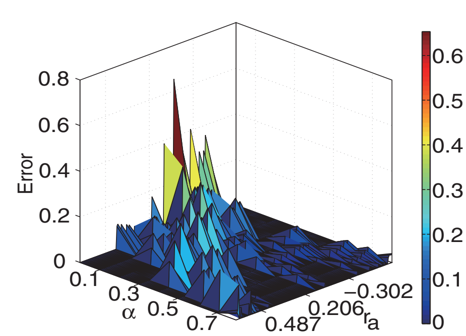

FIG. 4. (Color online) Synchronization errors under various topological parameters, namely, hetergeneity parameter α and assortativity coefficient r_a. As with the errors in Fig. 3, here the error is also measured by comparing the difference between the maximal synchronization error and the mean value of the rest of the synchronization errors.

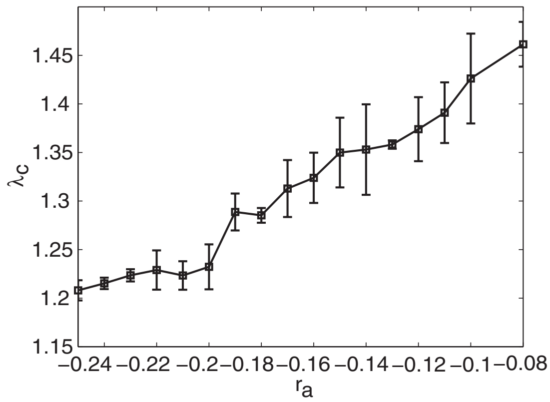

FIG. 5. Critical coupling strength λ_c changes under different assortativity coefficients. The error bars are computed by many realizations of explosive synchronization for each assortativity coefficient.

the network structures, on the condition that there is a positive correlation between the degree and the natural frequency of the oscillator. In other words, heterogeneity cannot solely determine the emergence of explosive synchronization.

Now we study how the extent of assortativity (measured by assortativity coefficient r_a) can affect the first order synchronization transition. We calculate r_a for each value of α and average it over many realizations of the network model. As is shown in Fig. 5, the critical coupling strength at which the synchronization phase curve has a takeoff (i.e., where the order parameter changes abruptly) decreases with r_a. That is, the more disassortative a network is, the more easily explosive synchronization can occur. However, our findings also indicate that too large of a disassortativity (for example, $r_a < -0.35$, as is shown in Fig. 6) will impede the occurrence of the first order synchronization transition. In this case, second order synchronization transitions are more likely occur. It should be pointed out that assortativity is also not uniquely sufficient to determine the emergence of explosive synchronization transition. Namely, one cannot expect to observe the explosive phase transition in a disassortative network with a low heterogeneity index (corresponding to homogeneous networks), as shown in Fig. 4.

The effects of disassortativity in a mixing pattern also manifest themselves in another aspect. Note that in the processes of synchronization (by a forward increase in the coupling strength) and desynchronization (by a backward decrease in the coupling strength), a hysteresis loop is formed. As the networks become more disassortative, the synchronization is enhanced, and therefore the critical point (r,λ) corresponding to λ_c on the curve which we call takeoff point in panel (b) is higher than that in panel (a), also showing a better synchrony of disassortative networks than the BA networks [13]. As a result, the size of the hysteresis loop is observed to be smaller than that of less disassortative networks (Fig. 7).

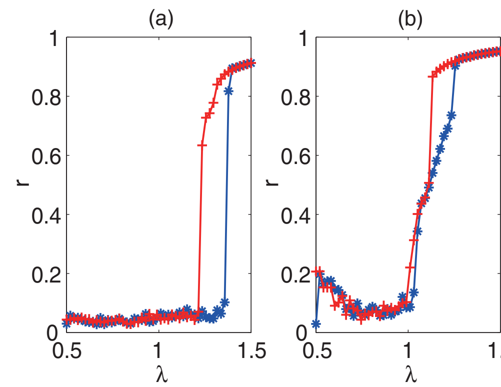

FIG. 7. (Color online) The synchronization transitions in two scale-free networks with different assortativity coefficients. (a) A BA network with $r_a = -0.0647$; (b) its corresponding reshuffled configuration, whose assortativity coefficient is -0.23. The blue line with star symbols corresponds to the synchronizing process, and the red line with plus signs corresponds to the desynchronizing process for both panels.

At the same time, degree assortativity has been revealed to be associated with other topological measures such as the clustering coefficient [17]. It has been reported in Ref. [18] that the clustering coefficient of a network affects the synchronization processes in small-world networks. Therefore, we also study the correlation between the emergence of explosive synchronization and the clustering coefficient. As shown in Fig. 8, the clustering coefficient remains almost unchanged for heterogeneous networks (i.e., $\alpha < 0.5$) and increases significantly as the networks become more homogeneous. Nevertheless, it is hard to associate this characteristic change with the occurrence of explosive synchronization, compared with the corresponding change of the assortativity coefficient.

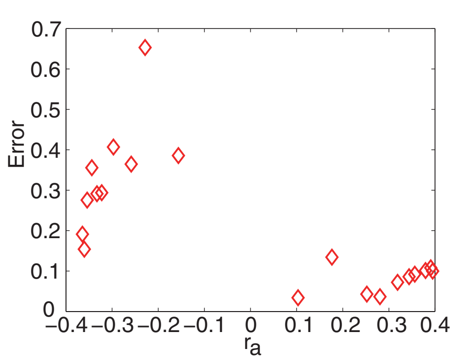

FIG. 6. (Color online) Synchronization errors under different assortativities with the same degree sequence for $\alpha = 0$.

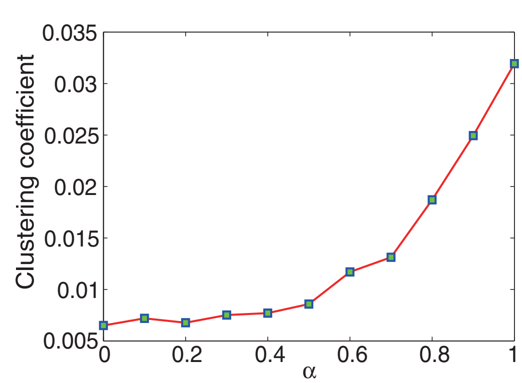

FIG. 8. (Color online) Clustering coefficient for the networks corresponding to variation of α.

In addition, the network model used in this work has a very low clustering coefficient for all values of α. Therefore, it is difficult to conclude with certainty the effect of this measure on explosive synchronization using this model. Future work will include more extensive investigation of the function of the transitivity property using more suitable models.

IV. SUMMARY

Heterogeneity is a global feature of network structure, reflecting the uniformity of the overall degree distribution, while the assortativity characterizes local connectivity patterns of network structure, i.e., how nodes are locally connected. In the presence of a positive correlation between the degree and the natural frequency of a node, the difference between a node and its neighbors in terms of their degrees can dramatically influence the corresponding synchronization process. A large difference in node degree implies a significant difference in intrinsic frequencies of the nodes, which in turn holds back the neighboring nodes from reaching a uniform frequency (and hence global synchronization) unless a high enough coupling strength is achieved. From this analysis we can see that assortativity plays a crucial role in the emergence of explosive synchronization on the premise of positive correlation between the structure and the dynamics of the system. Meanwhile, for explosive synchronization to take place, the level of heterogeneity of the network cannot be too low to ensure the existence of the difference in the intrinsic frequencies of the nodes.

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[1] A. Barabási and R. Albert, Science **286**, 509 (1999).
[2] R. Albert, H. Jeong, and A. Barabási, Nature (London) **406**, 378 (2000).
[3] J. Zhang, C. Zhou, X. Xu, and M. Small, Phys. Rev. E **82**, 026116 (2010).
[4] T. Nishikawa, A. E. Motter, Y. C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).
[5] J. Gómez-Gardeñes, S. Gómez, A. Arenas, and Y. Moreno, Phys. Rev. Lett. **106**, 128701 (2011).
[6] P. Erdös and A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
[7] Y. Kuramoto, Prog. Theor. Phys. **53**, 1302 (1975).
[8] S. Strogatz, Physica D **143**, 1 (2000).
[9] J. Gómez-Gardeñes and Y. Moreno, Phys. Rev. E **73**, 056124 (2006).
[10] H. Hu and X. Wang, Physica A **387**, 3769 (2008).
[11] R. Xulvi-Brunet and I. M. Sokolov, Phys. Rev. E **70**, 066102 (2004).
[12] R. Xulvi-Brunet and I. M. Sokolov, Acta Phys. Pol. B **36**, 1431 (2005).
[13] M. Di Bernardo, F. Garofalo, and F. Sorrentino, Int. J. Bifurcat. Chaos **17**, 3499 (2007).
[14] M. E. J. Newman, Phys. Rev. Lett. **89**, 208701 (2002).
[15] M. Small, X. Xu, J. Zhou, J. Zhang, J. Sun, and J. A. Lu, Phys. Rev. E **77**, 066112 (2008).
[16] X.-K. Xu, J. Zhang, J. Sun, and M. Small, Phys. Rev. E **80**, 056106 (2009).
[17] E. Estrada, Phys. Rev. E **84**, 047101 (2011).
[18] P.-J. Ma and B.-H. Wang, Chin. Phys. Lett. **25**, 3507 (2008).