Velocity distribution for quasistable acceleration in the presence of multiplicative noise

A. V. Kargovsky,* E. I. Anashkina, and O. A. Chichigina
Faculty of Physics, Lomonosov Moscow State University, Leninskie Gory, 119992 Moscow, Russia

A. K. Krasnova
Faculty of Physics, Lomonosov Moscow State University, Leninskie Gory, 119992 Moscow, Russia and Institute of Physics of the Earth, Russian Academy of Sciences, B. Gruzinskaya 10, 123995 Moscow, Russia
(Received 14 February 2013; revised manuscript received 16 April 2013; published 29 April 2013)

Processes that are far both from equilibrium and from phase transition are studied. It is shown that a process with mean velocity that exhibits power-law growth in time can be analyzed using the Langevin equation with multiplicative noise. The solution to the corresponding Fokker-Planck equation is derived. Results of the numerical solution of the Langevin equation and simulation of the motion of particles in a billiard system with a time-dependent boundary are presented.

DOI: 10.1103/PhysRevE.87.042133 PACS number(s): 05.40.−a, 05.10.Gg

I. INTRODUCTION

The problem of processes that are far from equilibrium is not completely solved nowadays. The theory of phase transitions is one of the greatest achievements in this field. The interpretation of nonequilibrium steady states that are far from phase transition is the most challenging and current tasks for researchers. A wide variety of systems in physics, technology, and biology can be modeled as a set of chaotically moving noninteracting accelerated particles whose mean velocity is always proportional to t^α. The balance of random and deterministic effects remains constant, and therefore thermodynamic parameters of the system can be defined. It is obvious that such a process is irreversible, and we call it a *quasistable* process. Experimental and computational studies presented in the literature have explored the similarities in the behavior of these systems.

We will demonstrate some examples of acceleration processes in different physical models corresponding to different values of α. Two types of models are considered. In the billiard-like systems (particles in nonstationary random force fields, Galton board, and billiards with moving walls) particles gain energy for acceleration from collisions with scatterers and external fields. Such an interaction of a system with a thermostat depends on velocities of particles. On the contrary, active Brownian particles have an internal source of energy. They can be used for description of living or technical objects. We assume that probability distribution is known and can be factored into the distributions of velocity and coordinates. The probability density function (PDF) of velocities differs from the Maxwell distribution function and is time-dependent. This circumstance makes it possible to investigate the transport properties of this system and to define effective temperature, entropy, and other thermodynamic parameters.

When $\alpha < 1/2$, fast particles move at relatively large times in the presence of nonstationary random force fields with correlations that rapidly decrease in space, but not necessarily in time. The general feature of the time dependence of the averaged kinetic energy lies in the fact that it depends only on whether the force is conservative or not. When it is conservative, the velocity asymptote is $t^{1/5}$ regardless of the details of the potential and of the dimension of space. For a nonconservative force, the power law is different: $t^{1/3}$ in all dimensions [1]. The acceleration law is also $t^{1/3}$ if the oscillation axis of scatterers is fixed and uniform throughout the Lorentz gas [2].

If $\alpha = 1/2$, the friction for the Brownian particle is negligibly small and the kinetic energy of surrounding particles is greater than the energy of Brownian particle. The noise intensity does not depend on velocity.

The Galton board is a vertical (or inclined) board with interleaved rows of pegs. A ball thrown into the Galton board moves under gravity and bounces off the pegs on its way down [3]. The dynamics of ball velocity corresponds to $\alpha = 2/3$. It is one of the simplest mechanical devices where nonstationary transport occurs and a good example of a billiard system.

A billiard is a system where noninteracting point particles (or one particle) in free motion undergo elastic collisions with a set of fixed scatterers. A billiard with oscillating scatterers is the model of a thermodynamic process in an open system. The noise intensity linearly depends on velocity, since the rate of collisions with scatterers is proportional to the velocity of a particle. Consequently $\alpha = 1$, and a linear increase in mean velocity is known as Fermi acceleration [4–6]. Such an exactly linear increase corresponds to negligibly small displacement of scatterers. Thus, the scenario $1/2 < \alpha \leqslant 1$ is possible for the system with noticeable displacement of scatterers when the rate of collision depends on the velocities of particle(s) and scatterers.

The aforementioned billiards are important for the analysis of fundamental problems. It is shown that macroscopic behavior of several far-from-equilibrium systems can be interpreted in terms of equilibrium statistical mechanics [7]. Quantum manifestation of classical chaos in a microwave resonator is also described using a billiard system [8].

The billiard model explains anomalous transfer properties, such as superdiffusion [9] and gives an opportunity to improve the efficiency of thermoelectric materials and to create systems in which transport coefficients are independent [10]. For instance, an extremely high value of the diffusion coefficient for gold nanoclusters on a graphite substrate [11,12] is a result of the Fermi acceleration.

*kargovsky@yumr.phys.msu.ru

Another interesting example is the phenomenon of thermal rectification. A thermal rectifier is a device in which the magnitude of the heat flow depends on the sign of the imposed temperature gradient. It can also be modeled as a billiard system [13].

From a physical point of view, the above acceleration processes result from the deposition of external energy.

The scenario $\alpha \to \infty$ corresponds to the exponential growth of mean velocity [14]. This is also realized in the theory of active Brownian particles, such as a conglomerate of almost noninteracting cells and organisms. The energy supply is provided by velocity-dependent friction that may become negative in the range of small velocities [15–17].

The basic method of investigation of the above acceleration processes is numerical simulation. The purpose of this work is the analytical description of statistical characteristics of such processes. The acceleration with the power-law time dependence of the mean velocity can be due to deterministic and noise sources that are taken into account using the Langevin equation. This equation is specifically constructed to balance the intensities of both acceleration sources. We have found the solution to the corresponding Fokker-Planck equation (FPE) [18] for an arbitrary initial distribution. The results are supported by numerical simulations of linear and nonlinear growth of mean velocity.

II. LANGEVIN AND FOKKER-PLANCK EQUATIONS FOR ACCELERATION PROCESS

The the above quasistable process can be described using the Langevin stochastic differential equation (SDE):

$$\dot{v}(t) = av^{\mu}(t) + v^{\gamma}(t)\xi(t), \tag{1}$$

where $\xi(t)$ is the Gaussian white noise with $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t+\tau)\rangle = 2D\delta(\tau)$, $\gamma < 1$, and $a \geqslant 0$.

When the noise term $v^{\gamma}\xi(t)$ is absent, we have a simple differential equation, and its solution is proportional to $t^{1/(1-\mu)}$. On the other hand, if the deterministic term av^{μ} is omitted, we obtain the Wiener process for $v^{1-\gamma}$ and the mean velocity is proportional to $t^{1/(2-2\gamma)}$. Thus, the noise or deterministic term dominates on a large time scale if $\mu \neq 2\gamma - 1$. Below we assume that $\mu = 2\gamma - 1$. Then Eq. (1) describes a Bessel process [19] for $\gamma = 0$ and squared Bessel process for $\gamma = 1/2$. Note, however, that there are different naming conventions, and sometimes a squared Bessel process is referred to as Bessel process [20].

If this equation is interpreted in the Stratonovich sense, the corresponding Fokker-Planck equation for PDF $w(v,t)$ can be written as [18,21]

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial v}[(a + D\gamma)v^{2\gamma-1}w] + \frac{\partial^2}{\partial v^2}[Dv^{2\gamma}w]. \tag{2}$$

The function $w(v,t)$ satisfies the initial and boundary conditions

$$\begin{gathered} w(v,0) = f(v), \quad v^{a/D+\gamma}\frac{\partial}{\partial v}v^{\gamma-a/D}w(v,t)\big|_{v=0} = 0, \\ \lim_{v\to\infty} v^{a/D+\gamma}\tfrac{\partial}{\partial v}v^{\gamma-a/D}w(v,t) = 0. \end{gathered} \tag{3}$$

Here $f(v)$ is a nonnegative function that satisfies normalization condition. The boundary conditions correspond to the fact that the probability current vanishes at the boundaries $v = 0$ and $v = \infty$.

A. Uniqueness of the solution

Introducing the operator

$$\hat{\mathcal{L}} = D\frac{\partial}{\partial v}v^{a/D+\gamma}\frac{\partial}{\partial v} - v^{a/D-\gamma}\frac{\partial}{\partial t}, \tag{4}$$

we rewrite Eq. (2) as

$$\hat{\mathcal{L}}\, v^{\gamma-a/D}w(v,t) = 0. \tag{2$'$}$$

The corresponding conjugate operator is

$$\hat{\mathcal{L}}^* = D\frac{\partial}{\partial v}v^{a/D+\gamma}\frac{\partial}{\partial v} + v^{a/D-\gamma}\frac{\partial}{\partial t}. \tag{5}$$

For differentiable functions u_1 and u_2 we have

$$\begin{aligned} u_1\hat{\mathcal{L}}u_2 - u_2\hat{\mathcal{L}}^*u_1 &= \frac{\partial}{\partial t}[-v^{a/D-\gamma}u_1u_2] \\ &\quad - \frac{\partial}{\partial v}\left[Dv^{a/D+\gamma}\left(u_2\frac{\partial u_1}{\partial v} - u_1\frac{\partial u_2}{\partial v}\right)\right] \\ &\equiv \frac{\partial Q}{\partial t} - \frac{\partial P}{\partial v}. \end{aligned} \tag{6}$$

Let G be simple region in the plane (t,v) with boundary ∂G. Let u_1 and u_2 be such that $Q(v,t)$ and $P(v,t)$ are defined and continuous and have continuous partial derivatives $\partial Q/\partial t$ and $\partial P/\partial v$ in closed region $\bar{G}$. Integrating with respect to v and t, we obtain [22]

$$\begin{aligned} &\int_G (u_1\hat{\mathcal{L}}u_2 - u_2\hat{\mathcal{L}}^*u_1)\,dv\,dt \\ &\quad = -\oint_{\partial G} v^{a/D-\gamma}u_1u_2\,dv + Dv^{a/D+\gamma} \\ &\qquad \times \left(u_1\frac{\partial u_2}{\partial v} - u_2\frac{\partial u_1}{\partial v}\right)dt, \end{aligned} \tag{7}$$

where the contour integral is calculated in the positive direction. Expression (7) is Green's formula for operator $\hat{\mathcal{L}}$.

Now assume that the solution to Eq. (2) exists and satisfies the initial and boundary conditions (3). Also assume that

$$\lim_{v\to\infty} v^{2\gamma}w(v,t)\frac{\partial}{\partial v}v^{\gamma-a/D}w(v,t) = 0. \tag{8}$$

Let us show that the solution is unique under the above assumptions.

We will proceed by contradiction. Assume that two different solutions to Eq. (2) $w_1(v,t)$ and $w_2(v,t)$ exist and satisfy identical initial and boundary conditions. Then, function $\tilde{w}(v,t) = w_1(v,t) - w_2(v,t)$ is the solution to the following system of equations:

$$\begin{gathered} \hat{\mathcal{L}}\, v^{\gamma-a/D}\tilde{w}(v,t) = 0, \quad \tilde{w}(v,0) = 0, \\ v^{a/D+\gamma}\tfrac{\partial}{\partial v}v^{\gamma-a/D}\tilde{w}(v,t)\big|_{v=0} = 0. \end{gathered} \tag{9}$$

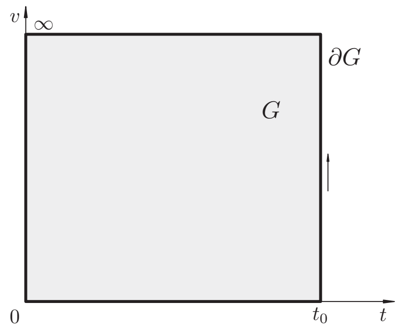

FIG. 1. Integration domain.

Let us use Green's formula (7). Figure 1 demonstrates region G, which is bounded on the right-hand side by arbitrary t_0. Choosing $u_1 = 1$ and $u_2 = [v^{\gamma-a/D}\tilde{w}(v,t)]^2$, we obtain

$$\int_G \hat{\mathcal{L}}[v^{\gamma-a/D}\tilde{w}(v,t)]^2 dv\,dt = -\oint_{\partial G} v^{\gamma-a/D}\tilde{w}^2(v,t)dv + Dv^{a/D+\gamma}\frac{\partial}{\partial v}[v^{\gamma-a/D}\tilde{w}(v,t)]^2 dt. \tag{10}$$

Using condition (9), we derive

$$\begin{aligned}\hat{\mathcal{L}}(v^{\gamma-a/D}\tilde{w})^2 &= 2D\frac{\partial}{\partial v}v^{a/D+\gamma}v^{\gamma-a/D}\tilde{w}\frac{\partial}{\partial v}v^{\gamma-a/D}\tilde{w} - 2v^{\gamma-a/D}\tilde{w}\frac{\partial\tilde{w}}{\partial t}\\ &= 2v^{\gamma-a/D}\tilde{w}\,\hat{\mathcal{L}}\,v^{\gamma-a/D}\tilde{w} + 2Dv^{a/D+\gamma}\left(\frac{\partial}{\partial v}v^{\gamma-a/D}\tilde{w}\right)^2\\ &= 2Dv^{a/D+\gamma}\left(\frac{\partial}{\partial v}v^{\gamma-a/D}\tilde{w}\right)^2.\end{aligned} \tag{11}$$

Integrating along the boundary ∂G with allowance for Eqs. (8) and (9), we obtain

$$2D\int_G v^{a/D+\gamma}\left(\frac{\partial}{\partial v}v^{\gamma-a/D}\tilde{w}\right)^2 dv\,dt + \int_{t_0} v^{\gamma-a/D}\tilde{w}^2\,dv = 0. \tag{12}$$

The integrands are essentially nonnegative, hence $\partial(v^{\gamma-a/D}\tilde{w})/\partial v = 0$ inside G and $\tilde{w}(v,t_0) = 0$. We have $\tilde{w}(v,t) = 0$ for any v and t, since $t_0 > 0$ is arbitrary. Two different solutions that satisfy the initial and boundary conditions (3) and Eq. (8) cannot exist. Thus, the solution is unique.

Since the probability current is defined as

$$\Pi(v,t) = -Dv^{a/D+\gamma}\frac{\partial}{\partial v}v^{\gamma-a/D}w(v,t), \tag{13}$$

condition (8) is equivalent to

$$\lim_{v\to\infty} v^{\gamma-a/D}w(v,t)\Pi(v,t) = 0. \tag{8$'$}$$

The limiting values are $w(\infty,t) = 0$ and $\Pi(\infty,t) = 0$, since the infinite velocity is impossible.

B. Existence of the solution

Let us find a particular solution to Eq. (2) represented as

$$w(v,t) = \phi(v)\psi(t). \tag{14}$$

Substituting expression (14) into Eq. (2) and separating variables, we obtain

$$\frac{\psi'(t)}{\psi(t)} = \frac{-[(a+D\gamma)v^{2\gamma-1}\phi(v)]' + [Dv^{2\gamma}\phi(v)]''}{\phi(v)} = -\lambda. \tag{15}$$

Then,

$$\psi(t) = Ae^{-\lambda t}, \tag{16}$$

where A is constant and $\mathrm{Re}\lambda \geqslant 0$ because we consider only bounded solutions. For $\phi(v)$, we obtain equation

$$\frac{d^2}{dv^2}[Dv^{2\gamma}\phi(v)] - \frac{d}{dv}[(a+D\gamma)v^{2\gamma-1}\phi(v)] + \lambda\phi(v) = 0. \tag{17}$$

Changing the variable $z = \frac{v^{1-\gamma}}{1-\gamma}\sqrt{\frac{\lambda}{D}}$ and introducing a new function $\phi[v(z)] = z^{\frac{1+a/D-3\gamma}{2(1-\gamma)}}\chi(z)$, we can transform this equation to standard form of Bessel equation:

$$z^2\frac{d^2\chi}{dz^2} + z\frac{d\chi}{dz} + \left[z^2 - \left(\frac{a/D+\gamma-1}{2(1-\gamma)}\right)^2\right]\chi = 0. \tag{17$'$}$$

The general solution is

$$\phi(v) = v^{\frac{1+a/D-3\gamma}{2}}\left[c_1(\lambda)J_\nu\left(\frac{v^{1-\gamma}}{1-\gamma}\sqrt{\frac{\lambda}{D}}\right) + c_2(\lambda)Y_\nu\left(\frac{v^{1-\gamma}}{1-\gamma}\sqrt{\frac{\lambda}{D}}\right)\right], \tag{18}$$

where $\nu = \frac{a}{2D(1-\gamma)} - \frac{1}{2} \geqslant -\frac{1}{2}$.

The solution to Eqs. (2) and (3) can be presented as a superposition of particular solutions such as Eq. (14). Due to continuity of parameter λ, the solution can be written as

$$w(v,t) = v^{\frac{1+a/D-3\gamma}{2}}\int_0^\infty e^{-\lambda t}\left[c_1(\lambda)J_\nu\left(\frac{v^{1-\gamma}}{1-\gamma}\sqrt{\frac{\lambda}{D}}\right) + c_2(\lambda)Y_\nu\left(\frac{v^{1-\gamma}}{1-\gamma}\sqrt{\frac{\lambda}{D}}\right)\right]d\lambda. \tag{19}$$

Let $c_2(\lambda) = 0$, and we find $c_1(\lambda)$ using the initial condition. Substituting $t = 0$, we obtain from Eq. (19):

$$f(v)v^{-\frac{1+a/D-3\gamma}{2}} = \int_0^\infty c_1(\lambda)J_\nu\left(\frac{v^{1-\gamma}}{1-\gamma}\sqrt{\frac{\lambda}{D}}\right)d\lambda. \tag{20}$$

The right-hand side of this equation is the Hankel transform of the function $c_1(\lambda)$. Assuming that $f(v)v^{-\frac{a/D+\gamma-1}{2}} \in \mathrm{L}^1(\mathbb{R}^+)$, we can find it by means of inverse transform ($\nu \geqslant -1/2$) [23]:

$$c_1(\lambda) = \int_0^\infty f(\tilde{v})\tilde{v}^{-\frac{1+a/D-3\gamma}{2}}J_\nu\left(\frac{\tilde{v}^{1-\gamma}}{1-\gamma}\sqrt{\frac{\lambda}{D}}\right)\frac{\tilde{v}^{1-2\gamma}}{2(1-\gamma)D}d\tilde{v}. \tag{21}$$

Substituting this expression into Eq. (19), changing the order of integration, and using [24], we obtain

$$w(v,t) = \int_0^\infty e^{-\lambda t} \int_0^\infty f(\tilde{v}) \left(\frac{v}{\tilde{v}}\right)^{\frac{1+a/D-3\gamma}{2}} J_\nu\left(\frac{\tilde{v}^{1-\gamma}}{1-\gamma}\sqrt{\frac{\lambda}{D}}\right) \times \frac{\tilde{v}^{1-2\gamma}}{2(1-\gamma)D} d\tilde{v} J_\nu\left(\frac{v^{1-\gamma}}{1-\gamma}\sqrt{\frac{\lambda}{D}}\right) d\lambda$$
$$= \frac{v^{1-2\gamma}}{2(1-\gamma)Dt}\int_0^\infty f(\tilde{v})\left(\frac{v}{\tilde{v}}\right)^{\frac{a/D+\gamma-1}{2}} \times e^{-\frac{v^{2-2\gamma}+\tilde{v}^{2-2\gamma}}{4(1-\gamma)^2Dt}} I_{\frac{a/D+\gamma-1}{2(1-\gamma)}}\left[\frac{(v\tilde{v})^{1-\gamma}}{2(1-\gamma)^2Dt}\right]d\tilde{v}, \quad (22)$$

where $I_\nu(z)$ is the Infeld function. Introducing a new dimensionless parameter

$$\rho = \frac{v^{2-2\gamma}}{4(1-\gamma)^2Dt}, \quad (23)$$

we can rewrite probability distribution in a short form:

$$w(v,t) = \frac{(2-2\gamma)\rho}{v}\int_0^\infty f(\tilde{v})\left(\frac{\rho}{\tilde{\rho}}\right)^{\frac{a/D+\gamma-1}{4(1-\gamma)}} \times e^{-\rho-\tilde{\rho}} I_{\frac{a/D+\gamma-1}{2(1-\gamma)}}(2\sqrt{\rho\tilde{\rho}})d\tilde{v}. \quad (22')$$

Let v tend to zero. To check that the solution satisfies the boundary condition we represent the Infeld function as a power series [25]:

$$I_\nu(2\sqrt{\rho\tilde{\rho}}) = (\rho\tilde{\rho})^{\nu/2}\sum_{k=0}^{\infty}\frac{(\rho\tilde{\rho})^k}{k!\Gamma(\nu+k+1)}. \quad (24)$$

Substituting this expression into Eq. (22′), we obtain

$$w(v,t) \sim v^{a/D-\gamma}e^{-\rho}\sum_{k=0}^{\infty}\frac{\rho^k}{k!\Gamma(\nu+k+1)}\int_0^\infty f(\tilde{v})\tilde{\rho}^k e^{-\tilde{\rho}}d\tilde{v}$$
$$\leqslant v^{a/D-\gamma}e^{-\rho}\sum_{k=0}^{\infty}\frac{\rho^k k^k e^{-k}}{k!\Gamma(\nu+k+1)}. \quad (25)$$

By the ratio test [26] the series converges absolutely if ρ is bounded. Substituting this expression into Eq. (3), we see that the resulting solution satisfies the boundary condition:

$$v^{a/D+\gamma}\frac{\partial}{\partial v}[v^{\gamma-a/D}w(v,t)]\Big|_{v=0} \sim v^{a/D+1-\gamma}|_{v=0} = 0. \quad (26)$$

The proof that the PDF (22) satisfies the second boundary condition (3), and Eq. (8) is trivial. Thus, the derived solution to the Fokker-Planck equation is unique.

For $f(v)v^{-a/2D} \in L^2(\mathbb{R}^+)$ the conditions imposed upon the parameter a can be relaxed [27]:

$$\frac{a}{D} > \gamma - 1. \quad (27)$$

III. RESULTS AND DISCUSSION

A. The moments of velocity

Let us derive an analytical expression for the moments of velocity. Using Ref. [24], we obtain the nth moment from Eq. (22):

$$M_n(t) = \frac{\Gamma\left[\frac{n+1+a/D-\gamma}{2(1-\gamma)}\right]}{\Gamma\left[\frac{1+a/D-\gamma}{2(1-\gamma)}\right]}[4(1-\gamma)^2Dt]^{\frac{n}{2(1-\gamma)}}\int_0^\infty f(\tilde{v}) \times {}_1F_1\left[\frac{n}{2(\gamma-1)},\frac{1+a/D-\gamma}{2(1-\gamma)},-\frac{\tilde{v}^{2-2\gamma}}{4(1-\gamma)^2Dt}\right]d\tilde{v}. \quad (28)$$

If the initial condition is a δ function

$$w(v,0) = \delta(v-v_0), \quad (29)$$

where v_0 is the initial particle velocity, the solution to FPE is

$$w(v,t) = \frac{v^{1-2\gamma}}{2(1-\gamma)Dt}\left(\frac{v}{v_0}\right)^{\frac{a/D+\gamma-1}{2}} e^{-\frac{v^{2-2\gamma}+v_0^{2-2\gamma}}{4(1-\gamma)^2Dt}} \times I_{\frac{a/D+\gamma-1}{2(1-\gamma)}}\left[\frac{(vv_0)^{1-\gamma}}{2(1-\gamma)^2Dt}\right]. \quad (30)$$

Then the mean velocity is

$$\bar{v}(t) = \frac{\Gamma\left[\frac{2+a/D-\gamma}{2(1-\gamma)}\right]}{\Gamma\left[\frac{1+a/D-\gamma}{2(1-\gamma)}\right]}[4(1-\gamma)^2Dt]^{\frac{1}{2(1-\gamma)}} \times {}_1F_1\left[\frac{1}{2(\gamma-1)},\frac{1+a/D-\gamma}{2(1-\gamma)},-\frac{v_0^{2-2\gamma}}{4(1-\gamma)^2Dt}\right]. \quad (31)$$

For large time $t \gg \frac{v_0^{2-2\gamma}}{4(1-\gamma)^2D}$, the information on the initial distribution is lost, and the PDF tends to

$$w(v,t) = \frac{2(1-\gamma)}{\Gamma\left[\frac{1+a/D-\gamma}{2(1-\gamma)}\right]}[4(1-\gamma)^2Dt]^{-\frac{1+a/D-\gamma}{2(1-\gamma)}} v^{a/D-\gamma}e^{-\frac{v^{2-2\gamma}}{4(1-\gamma)^2Dt}}, \quad (32)$$

while the mean velocity tends to

$$\bar{v}(t) = \frac{\Gamma\left[\frac{2+a/D-\gamma}{2(1-\gamma)}\right]}{\Gamma\left[\frac{1+a/D-\gamma}{2(1-\gamma)}\right]}[4(1-\gamma)^2Dt]^{\frac{1}{2(1-\gamma)}}. \quad (33)$$

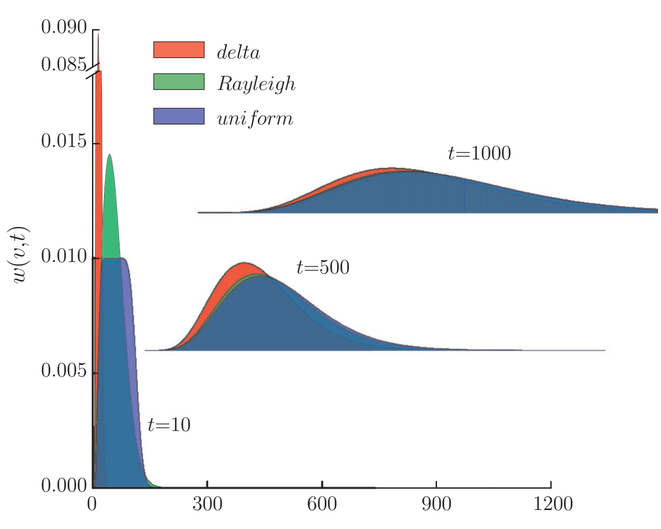

FIG. 2. (Color online) Time evolution of various initial distributions ($\gamma = 1/2$, $a = 0.54$, $D = 0.08$).

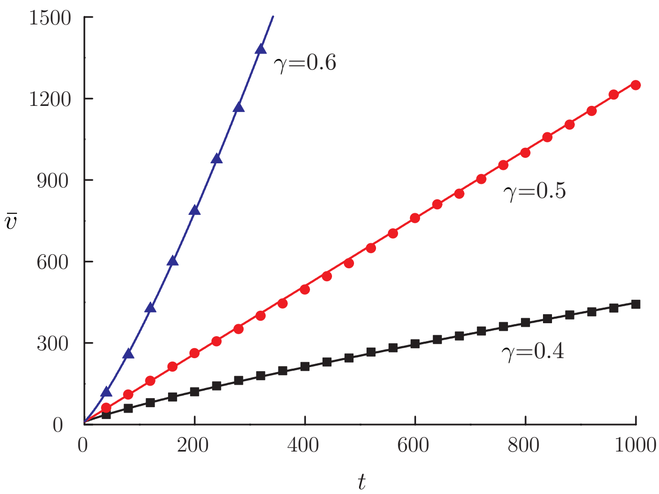

FIG. 3. (Color online) Plot of the mean velocity vs time at various values of γ ($a = 1$, $D = 0.5$): results of (solid lines) analytical and (dots) numerical calculations.

Note that distribution (32) for $\gamma = 1/2$ and $a = 0$ coincides with distribution from Ref. [4].

If the initial distribution is localized at small or medium velocities, expression (22) is transformed into formula (32) for sufficiently large times. Figure 2 illustrates this fact for the δ function and uniform and Rayleigh initial distributions:

$$w_1(v,0) = \delta(v - v_0), \quad \delta \quad \text{function}$$
$$w_2(v,0) = \frac{1}{\omega}, \quad v \in [v_0,\omega + v_0], \quad \text{uniform}$$
$$w_3(v,0) = \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}}, \quad \text{Rayleigh}$$

where $v_0 = 10$, $\omega = 100$, and $\sigma = 40$.

B. Numerical simulation

Here we present results of the numerical simulation of Eq. (1), which was interpreted in the Stratonovich sense:

$$dv = av^{2\gamma-1}dt + \sqrt{2D}\, v^{\gamma}\, dW_t, \tag{34}$$

where W_t is the Wiener process. The calculations were performed using SDE Toolbox [28] adopted for our purposes. We used the Mersenne twister as a pseudo-random number generator and a Milstein integration scheme with a time step of 1. Figure 3 compares analytical and numerical results for the mean velocity for various values of γ under the δ function initial condition. Note that 1000 realizations were averaged.

Figure 4 shows analytical and numerical results for the PDF at $t = 200$. We use the parameters $\gamma = 0.7$, $a = 0.7$, and $D = 0.05$, and Rayleigh initial distribution with $\sigma = 20$. The averaging was performed over 100 000 realizations of random process.

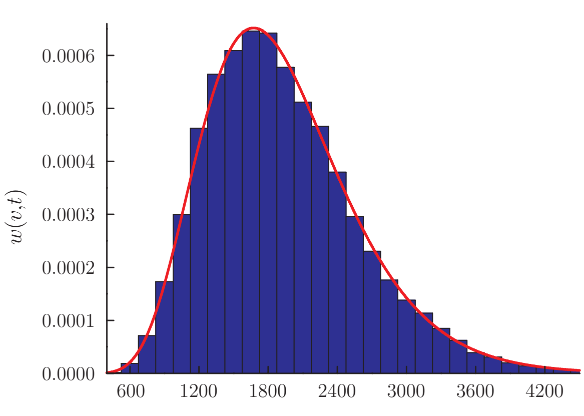

FIG. 4. (Color online) Probability density function at $t = 200$ ($\gamma = 0.7$, $a = 0.7$, $D = 0.05$): (solid line) analytical solution and (histogram) numerical results.

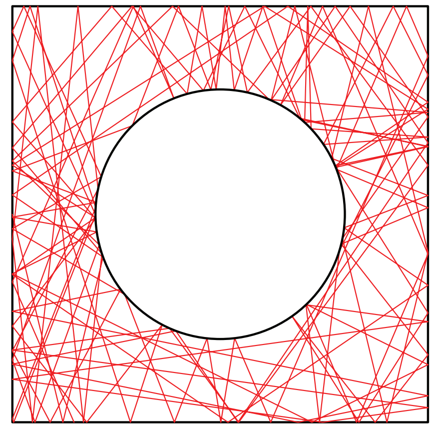

FIG. 5. (Color online) Trajectory of particle in the billiard.

C. Application to billiards with oscillating scatterers

The billiard theory is well developed and provides the methods to find Fermi acceleration for different types of billiard geometry and boundary motion. Here we compare the results of a direct simulation of the motion of a particle in the Sinai billiards with stochastically oscillating scatterer boundary and the results obtained using the Fokker-Planck equation.

We simulate the billiard system as a 20×20 square with a circular scatterer with a radius of 6 that is located at the center of the square. The scatterer with fixed center exhibits stochastic oscillations. Let the scatterer boundary oscillate with velocity $u(t) = u_0 \cos\phi(t)$, where u_0 is constant amplitude and $\phi(t)$ is

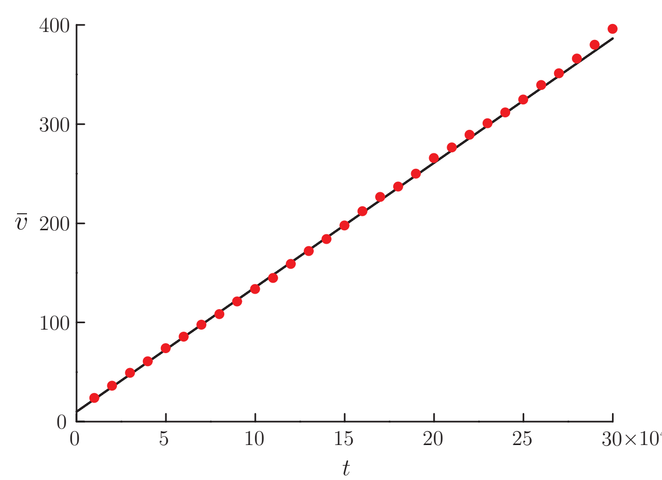

FIG. 6. (Color online) Plot of the mean velocity of particle vs time: (solid line) analytical solution and (circles) results of simulation.

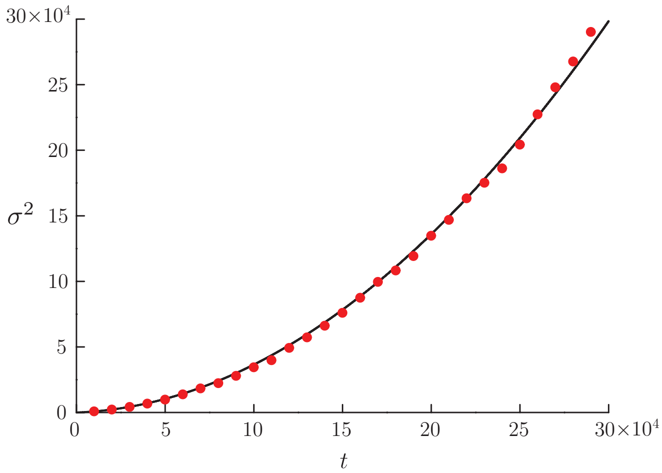

FIG. 7. (Color online) Plot of the dispersion of velocity vs time: (solid line) analytical solution and (circles) results of simulation.

a δ-correlated random phase with uniform distribution on the interval $[0, 2\pi]$. We use the static wall approximation (SWA) [29], in which the displacement of the scatterer boundary is disregarded and the momentum exchange between particle and scatterer upon impact is taken into account. Figure 5 demonstrates an example of the trajectory in such a billiard. The Fermi acceleration can be written as [5]

$$a_F = \frac{2\langle u^2(t)\rangle}{3\lambda}, \tag{35}$$

where $\lambda = \pi\Omega/P$ is the mean free path, Ω is the area of the accessible billiard regionm and P is the scatterer perimeter. Drift and diffusion coefficients of the Fokker-Planck equation are $D^{(1)}(v) = a_F$ and $D^{(2)}(v) = 2a_F v$, respectively [5]. Therefore, we have $\gamma = 1/2$, $a = 0$, and $D = a_F/2$. We average 5000 realizations with random directions of initial velocity of particle $v_0 = 10$. The amplitude of boundary velocity oscillations is $u_0 = 0.3$.

Figures 6–10 show the results of the numerical simulation and analytical solution. For coefficient of skewness γ_1 and kurtosis excess γ_2 limiting values are $2\sqrt{2}$ and 12, respectively.

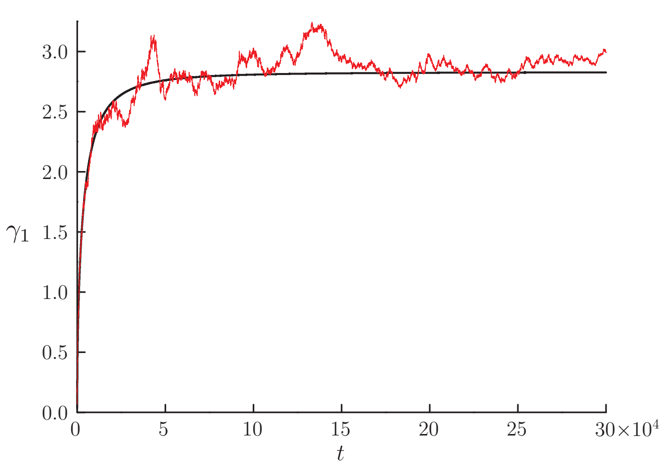

FIG. 8. (Color online) Plot of the coefficient of skewness of velocity vs time: (smooth line) analytical solution and (jagged line) results of simulation.

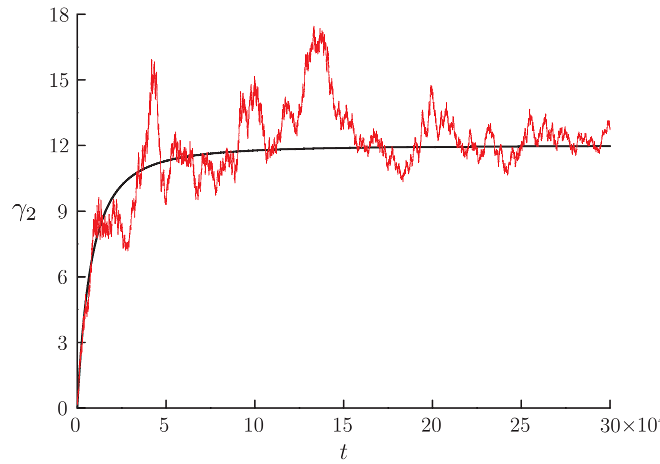

FIG. 9. (Color online) Plot of the coefficient of kurtosis excess of velocity vs time: (smooth line) analytical solution and (jagged line) results of simulation.

Clearly, the analytical and numerical results for a billiard system are in agreement.

Note that the Fokker-Planck equation technique correctly describes behavior of particle in billiard only after sufficiently large number of collisions. An alternative approach based on the use of the Chapman-Kolmogorov forward equation and lacking this disadvantage was introduced in Ref. [30].

IV. SUMMARY

There are many physical processes of different nature with a mean velocity that exhibits power-law growth in time. This acceleration can have regular and random sources. In this paper we describe them using the Langevin equation with multiplicative noise. This equation is specifically constructed in such a way that none of acceleration sources dominates. A wide variety of different physical systems can be described in a similar manner. Therefore, it is important to develop the fundamental thermodynamic approach to these processes [6,31]. We propose a semiphenomenological approach that

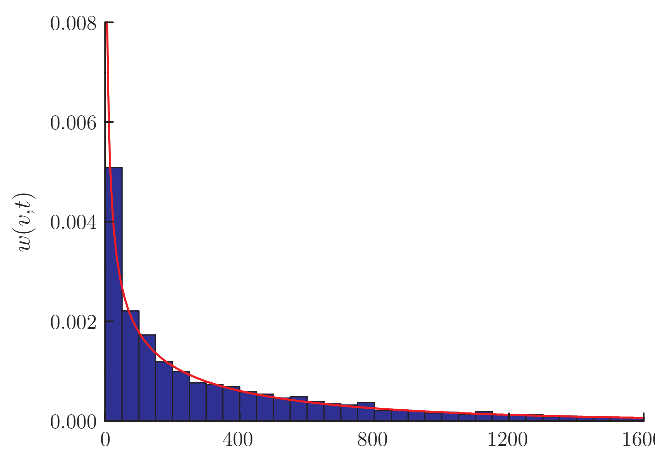

FIG. 10. (Color online) Probability density function at $t = 300\,000$: (solid line) analytical solution and (histogram) results of simulation.

makes it possible to describe the general features of time dynamics.

In this work the solution to the corresponding Fokker-Planck equation with an arbitrary initial distribution is derived. The existence and uniqueness of the solution are proven. It is demonstrated that the results obtained using the solution to FPE are similar to the results of the billiard simulation. It is also shown that the asymptotic behavior is independent of the initial conditions.

The results can be useful not only in physical applications. A particular case of the problem under study is the generalized Bessel process that is widely used in, e.g., population dynamics and financial markets.

ACKNOWLEDGMENT

We are very grateful to Dr. Andrey Chikishev for fruitful discussions.

APPENDIX: VELOCITY DISTRIBUTION FOR QUASISTABLE DECELERATION

Here we briefly present the results for a particle whose mean velocity decreases with time. A particular example of this model is one-dimensional active transport with particles with hard-core interactions (single-file system). In such models, the biased particle decelerates, and its mean velocity decreases as $t^{-1/2}$ ($\gamma = 2$) [32].

Let $\gamma > 1$. We assume that $f(v)v^{-\frac{a/D+\gamma-1}{2}} \in \mathrm{L}^1(\mathbb{R}^+)$ and $a \leqslant 0$, or $f(v)v^{-a/2D} \in \mathrm{L}^2(\mathbb{R}^+)$ and

$$\frac{a}{D} < \gamma - 1. \tag{A1}$$

Using the approach from Sec. II B, we obtain the solution to the Fokker-Planck equation (2):

$$w(v,t) = \frac{v^{1-2\gamma}}{2(\gamma-1)Dt} \int_0^\infty f(\tilde{v}) \left(\frac{v}{\tilde{v}}\right)^{\frac{a/D+\gamma-1}{2}} \times e^{-\frac{v^{2-2\gamma}+\tilde{v}^{2-2\gamma}}{4(1-\gamma)^2 Dt}} I_{\frac{a/D+\gamma-1}{2(1-\gamma)}} \left[\frac{(v\tilde{v})^{1-\gamma}}{2(1-\gamma)^2 Dt}\right] d\tilde{v}. \tag{A2}$$

It is easy to demonstrate that the conditions of uniqueness of the solution are satisfied. Here, as distinct from Eq. (22), we deal with a heavy-tailed distribution. Thus, the nth moment of velocity converges only for

$$\frac{a}{D} < \gamma - 1 - n. \tag{A3}$$

In this case Eq. (28) is still valid.

[1] B. Aguer, S. De Bièvre, P. Lafitte, and P. E. Parris, J. Stat. Phys. **138**, 780 (2010).

[2] A. K. Karlis, P. K. Papachristou, F. K. Diakonos, V. Constantoudis, and P. Schmelcher, Phys. Rev. Lett. **97**, 194102 (2006).

[3] N. Chernov and D. Dolgopyat, Phys. Rev. Lett. **99**, 030601 (2007).

[4] A. Loskutov, A. B. Ryabov, and L. G. Akinshin, J. Exp. Theor. Phys. **89**, 966 (1999).

[5] A. Loskutov, A. B. Ryabov, and L. G. Akinshin, J. Phys. A **33**, 7973 (2000).

[6] A. Loskutov, O. Chichigina, and A. Ryabov, Int. J. Bifur. Chaos **18**, 2863 (2008).

[7] D. A. Egolf, Science **287**, 101 (2000).

[8] H. D. Gräf, H. L. Harney, H. Lengeler, C. H. Lewenkopf, C. Rangacharyulu, A. Richter, P. Schardt, and H. A. Weidenmüller, Phys. Rev. Lett. **69**, 1296 (1992).

[9] A. Loskutov, O. Chichigina, A. Krasnova, and I. Sokolov, Eur. Phys. Lett. **98**, 10006 (2012).

[10] G. Casati, C. Mejía-Monasterio, and T. Prosen, Phys. Rev. Lett. **101**, 016601 (2008).

[11] L. J. Lewis, P. Jensen, N. Combe, and J.-L. Barrat, Phys. Rev. B **61**, 16084 (2000).

[12] A. K. Krasnova and O. A. Chichigina, Mosc. Univ. Phys. Bull. **67**, 48 (2012).

[13] J.-P. Eckmann and C. Mejía-Monasterio, Phys. Rev. Lett. **97**, 094301 (2006).

[14] O. A. Chichigina, A. A. Dubkov, D. Valenti, and B. Spagnolo, Phys. Rev. E **84**, 021134 (2011).

[15] F. Schweitzer, W. Ebeling, and B. Tilch, Phys. Rev. Lett. **80**, 5044 (1998).

[16] W. Ebeling, F. Schweitzer, and B. Tilch, BioSystems **49**, 17 (1999).

[17] B. Lindner and L. Schimansky-Geier, Phys. Rev. Lett. **89**, 230602 (2002).

[18] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer, Berlin, 1996).

[19] K. Itô and H. P. McKean, Jr., *Diffusion Processes and Their Sample Paths* (Springer, Berlin, 1996).

[20] E. Martin, U. Behn, and G. Germano, Phys. Rev. E **83**, 051115 (2011).

[21] W. T. Coffey, Y. P. Kalmykov, and J. T. Waldron, *The Langevin Equation: With Applications to Stochastic Problems in Physics, Chemistry, and Electrical Engineering* (World Scientific, Singapore, 2004).

[22] W. Kaplan, *Advanced Calculus* (Addison-Wesley, Reading, 2002).

[23] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1945).

[24] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Elsevier/Academic Press, Amsterdam, 2007).

[25] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953).

[26] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1963).

[27] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Chelsea Publishing Company, New York, 1986).
[28] U. Picchini, SDE Toolbox: Simulation and Estimation of Stochastic Differential Equations with MATLAB, http://sdetoolbox.sourceforge.net (2009).
[29] M. A. Lieberman and A. J. Lichtenberg, Phys. Rev. A **5**, 1852 (1972).
[30] A. K. Karlis, F. K. Diakonos, and V. Constantoudis, Chaos **22**, 026120 (2012).
[31] P. Collet, J.-P. Eckmann, and C. Mejía-Monasterio, J. Stat. Phys. **136**, 331 (2009).
[32] O. Bénichou, C. Mejía-Monasterio, and G. Oshanin, Phys. Rev. E **87**, 020103 (2013).