Scaling behavior of an airplane-boarding model

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An airplane-boarding model, introduced earlier by Frette and Hemmer [Phys. Rev. E **85**, 011130 (2012)], is studied with the aim of determining precisely its asymptotic power-law scaling behavior for a large number of passengers N. Based on Monte Carlo simulation data for very large system sizes up to $N = 2^{16} = 65\,536$, we have analyzed numerically the scaling behavior of the mean boarding time $\langle t_b \rangle$ and other related quantities. In analogy with critical phenomena, we have used appropriate scaling *Ansätz*e, which include the leading term as some power of N (e.g., $\propto N^{\alpha}$ for $\langle t_b \rangle$), as well as power-law corrections to scaling. Our results clearly show that $\alpha = 1/2$ holds with a very high numerical accuracy ($\alpha = 0.5001 \pm 0.0001$). This value deviates essentially from $\alpha \simeq 0.69$, obtained earlier by Frette and Hemmer from data within the range $2 \leqslant N \leqslant 16$. Our results confirm the convergence of the effective exponent $\alpha_{\text{eff}}(N)$ to $1/2$ at large N as observed by Bernstein. Our analysis explains this effect. Namely, the effective exponent $\alpha_{\text{eff}}(N)$ varies from values about 0.7 for small system sizes to the true asymptotic value $1/2$ at $N \to \infty$ almost linearly in $N^{-1/3}$ for large N. This means that the variation is caused by corrections to scaling, the leading correction-to-scaling exponent being $\theta \approx 1/3$. We have estimated also other exponents: $\nu = 1/2$ for the mean number of passengers taking seats simultaneously in one time step, $\beta = 1$ for the second moment of t_b, and $\gamma \approx 1/3$ for its variance.

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I. INTRODUCTION

A wide variety of systems in nature can be regarded as many-particle ensembles with extremely intricate dynamics of their elements. Despite simple rules for the behavior of individual elements, such systems often exhibit a rather nontrivial collective dynamics. Numerous examples can be mentioned here, ranging from many-particle systems in physics to traffic flow and biological systems. Stochastic approaches to describe such systems are well known [1–5]. Random processes of simple particle hopping models, like the asymmetric simple exclusion process with open boundaries [6], are often used to describe the basic properties of driven nonequilibrium systems. Here we focus on the complex dynamics of an airplane-boarding process. It has been investigated in several works—see, e.g., [7–9] and references therein.

Recently, a simple model of boarding an airplane has been proposed by Frette and Hemmer [8]. In this model, N passengers have reserved seats, but enter the airplane in arbitrary order ($N!$ possibilities). A simplified situation is considered with a single aisle of rows and only one seat in each row. It is assumed that a passenger occupies a place equal to the distance between rows. In addition, he or she requires one time step to place carry-on luggage and get seated, the time for walking along the aisle being neglected. However, a passenger must wait for a possibility to move forwards to his or her seat if the motion is blocked by other passengers stopping or taking seats in front of him or her (see [8] for more details and examples).

This model shows a power-law scaling behavior proportional to N^{α} of the mean boarding time depending on the number of passengers N. The exponent $\alpha = 0.69 \pm 0.01$ was found in [8], considering system sizes $2 \leqslant N \leqslant 16$. Subsequently, an interesting discussion arose in [9] about the scaling exponent α; there it was found that the scaling behavior is different for large N, where the effective exponent $\alpha_{\text{eff}}(N)$ tends to $1/2$. However, no explanation of this phenomenon is given in [9]. In fact, the value $\alpha = 1/2$ is consistent with the Lorenz estimate obtained in [7]. We have looked at this problem from the point of view of the widely known scaling behavior observed in critical phenomena (see, e.g., [10–12]), where the power-law singularity is usually described by the leading scaling exponent and also by confluent corrections to scaling, including powers with different exponents. Our analysis of the Monte Carlo (MC) simulation data for N up to $N = 2^{16} = 65\,536$ has confirmed the convergence of $\alpha_{\text{eff}}(N)$ to $1/2$ at $N \to \infty$ and allowed us to explain it by corrections to scaling with a nontrivial correction-to-scaling exponent $\theta \approx 1/3$.

II. SCALING EXPONENTS

Considering an ensemble of all possible boarding scenarios ($N!$ permutations), each specific boarding scenario can be considered as a stochastic realization with a certain probability ($1/N!$). In each such realization, the number of waiting passengers $n(t)$ is changed from $n(t=0) = N$ to $n(t = t_b) = 0$, where $t = 0,1,2,\ldots$ is the discrete time and t_b is the final boarding time. Let us denote by $\Delta n(t) = n(t-1) - n(t)$ the number of passengers taking seats simultaneously at a certain time step and define the mean Δn over a given boarding scenario as

$$\overline{\Delta n} = \frac{1}{t_b} \sum_{t=1}^{t_b} \Delta n(t) = \frac{N}{t_b}. \tag{1}$$

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Consider now the average boarding time $\langle t_b \rangle$. According to (1), we have the following relation:

$$\langle t_b \rangle = N \left\langle \frac{1}{\Delta n} \right\rangle. \quad (2)$$

Since the distribution of t_b becomes very narrow at $N \to \infty$, we have

$$\langle t_b \rangle \simeq \frac{N}{\langle \overline{\Delta n} \rangle} \quad \text{at} \quad N \to \infty. \quad (3)$$

Consequently, the power-law *Ansätze* for $\langle t_b \rangle$ and $\langle \overline{\Delta n} \rangle$ read

$$\langle t_b \rangle \propto N^{\alpha} \quad \text{at} \quad N \to \infty, \quad (4)$$

$$\langle \overline{\Delta n} \rangle \propto N^{\nu} \quad \text{at} \quad N \to \infty, \quad (5)$$

where

$$\nu = 1 - \alpha. \quad (6)$$

Here the exponent for $\langle \overline{\Delta n} \rangle$ is denoted by ν because of some analogy between $\langle \overline{\Delta n} \rangle$ and the correlation length in theories of critical phenomena, where the classical value of this exponent is $\nu_{\text{class}} = 1/2$. This value is classical also from the point of view that it coincides with the analytical estimate $\alpha = 1/2$ of [7] via (6).

One can consider also moments of the boarding time $\langle t_b^m \rangle$, where $m = 1,2,3$, and so on. Since the distribution of t_b becomes very narrow at $N \to \infty$, the expected scaling is

$$\langle t_b^m \rangle \propto N^{m\alpha} \quad \text{at} \quad N \to \infty. \quad (7)$$

Apart from the leading asymptotic terms, $\langle t_b \rangle(N)$ and $\langle \overline{\Delta n} \rangle(N)$ can have correction terms. Here we allow the possibility that one or both of these quantities has a constant contribution like an analytical background term in critical phenomena. There are also multiplicative corrections to scaling in the form of powers with different exponents. Thus, we have the expansions

$$\langle t_b \rangle(N) = C_t + AN^{\alpha}(1 + a_1 N^{-\theta_1} + a_2 N^{-\theta_2} + \cdots), \quad (8)$$

$$\langle \overline{\Delta n} \rangle(N) = C_n + BN^{\nu}(1 + b_1 N^{-\theta_1} + b_2 N^{-\theta_2} + \cdots) \quad (9)$$

at large N, where $\theta_1 > 0$, $\theta_2 > \theta_1$, etc., are the correction-to-scaling exponents. Because $\langle t_b \rangle(N)$ and $\langle \overline{\Delta n} \rangle(N)$ are related to each other, it is normally expected that these quantities have a common set of correction-to-scaling exponents, as written in (8) and (9). It is possible, however, that some of the coefficients are zero. The constant contributions C_t and C_n can be represented as correction terms $(C_t/A)N^{-\alpha}$ and $(C_n/B)N^{-\nu}$ in the parentheses of (8) and (9), respectively. However, these are not numerically detectable from our simulation data, and therefore we set $C_t = C_n = 0$. If there is a correction $\sim N^{-\theta_k}$, then higher-order corrections $\sim N^{-2\theta_k}$, $\sim N^{-3\theta_k}$, etc. are also expected. Thus, we can have $\theta_k = k\theta$ with $k = 1,2,\ldots$ as a particular case. The exponent $\theta = 1$ corresponds to simple analytical corrections in critical phenomena. From this point of view, 1 is the classical value of θ, i.e., $\theta_{\text{class}} = 1$.

According to (8) and (9), where we set $C_t = C_n = 0$ (or represent these terms as corrections in the parentheses), the average slope of the log-log plot within $N \in [N'/c, cN']$ with constant c (it corresponds approximately to the local slope at $N = N'$) gives us effective exponents, which behave as

$$\alpha_{\text{eff}}(N) = \alpha + a' N^{-\theta} + o(N^{-\theta}), \quad (10)$$

$$\nu_{\text{eff}}(N) = \nu + b' N^{-\tilde{\theta}} + o(N^{-\tilde{\theta}}), \quad (11)$$

where $-\theta$ is the largest (smallest in magnitude) exponent of a nonvanishing term (a term with nonzero coefficient) in the parentheses of (8). Similarly, $-\tilde{\theta}$ is the largest exponent of a non-vanishing term in the parentheses of (9). Accordingly, the true asymptotic exponents α (or ν) can be accurately estimated by making a linear fit of the effective exponent plotted versus $N^{-\theta}$ (or $N^{-\tilde{\theta}}$) at large enough N.

Our numerical analysis, described in detail in Sec. III, shows that $\alpha_{\text{eff}}(N)$ converges to $1/2$ almost linearly in $N^{-1/3}$ at very large N. According to (6), this means that $\nu = \alpha = 1/2$. Our correction-to-scaling analysis then implies that $\theta = \tilde{\theta} = \theta_1 \approx 1/3$ holds. In this way, it turns out that ν has the classical value $1/2$, whereas $\theta \neq \theta_{\text{class}} = 1$. Not only the mean boarding time $\langle t_b \rangle(N)$ but also its second moment $\langle t_b^2 \rangle(N)$ and the variance $\text{Var}(t_b) = \langle t_b^2 \rangle - \langle t_b \rangle^2$ are important in our analysis. In analogy with (8), we have

$$\langle t_b^2 \rangle(N) = A^* N^{\beta}(1 + a_1^* N^{-\theta_1} + a_2^* N^{-\theta_2} + \cdots), \quad (12)$$

where we find that $\beta = 1$ holds with a high numerical accuracy. The estimated values $\alpha = 1/2$ and $\beta = 1$ obey the scaling relation

$$\beta = 2\alpha \quad (13)$$

following from (7).

Our numerical results are perfectly consistent with

$$\langle t_b \rangle(N) \propto N^{\alpha}[1 + bN^{-\theta} + cN^{-2\theta} + o(N^{-2\theta})], \quad (14)$$

$$\langle \overline{\Delta n} \rangle(N) \propto N^{\nu}[1 + dN^{-\theta} + eN^{-2\theta} + o(N^{-2\theta})], \quad (15)$$

corresponding to the case where $\theta_k = k\theta$ holds at least for $k = 1,2$. Here $o(N^{-2\theta})$ denotes a remainder term of a higher order than $N^{-2\theta}$, containing terms proportional to $N^{-3\theta}$, proportional to $N^{-4\theta}$, and so on, as well as possible corrections of other kinds. The validity of (14) and (15) follows from the fact that the remainder terms in Eqs. (10) and (11) appear to be almost quadratic in $N^{-\theta}$ at large N. This is consistent with the fact that $\text{Var}(t_b) = \langle t_b^2 \rangle - \langle t_b \rangle^2$ scales as t^{γ} with $\gamma \approx 1/3$ at large N. According to (12) and (8), where $\alpha = 1/2$ and $\beta = 1$, it is possible at $a_1^* = 2a_1$ and $C_t = 0$ provided that $\theta_k = k\theta$ holds for $k = 1,2$ with $\theta \approx 1/3$.

According to (14) and (15), it is meaningful to include corrections quadratic in $N^{-\theta}$ by fitting the plots of the effective exponents to

$$\alpha_{\text{eff}}(N) = \alpha + b' N^{-\theta} + c' N^{-2\theta} + o(N^{-2\theta}), \quad (16)$$

$$\nu_{\text{eff}}(N) = \nu + d' N^{-\theta} + e' N^{-2\theta} + o(N^{-2\theta}) \quad (17)$$

in order to obtain more precise asymptotic values.

III. MC SIMULATION

According to Sec. II, we have to calculate results for large system sizes N if we want to check whether the boarding time is really a power law. Clearly we have to consider $N \gg 16$ (the maximal value of N used to determine the power-law exponent

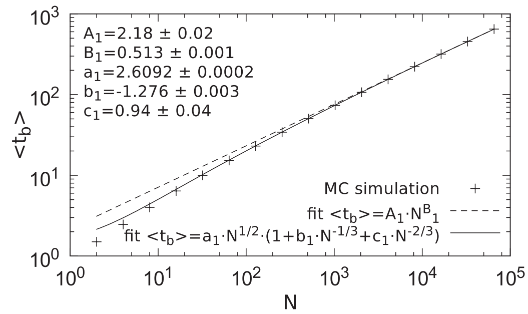

FIG. 1. The mean boarding time depending on the number of passengers N. The last five points (with the largest values of N) are used for the fitting.

in [8]). However, as the possible number of permutations is $N!$, we cannot numerically calculate boarding times by taking into account all possible permutations for large values of N.

The problem can be solved by Monte Carlo simulations. From all possible $N!$ permutations we can randomly choose M of them and calculate the average boarding time and other quantities we are interested in. Clearly, the larger is M the smaller is the statistical error, which scales with $\sqrt{N/M}$. For all simulations here the value $M = 5 \times 10^7$ was used.

The MC simulation results for average boarding times with values of N up to $N = 2^{16}$ can be seen in Fig. 1. For a realistic number of passengers $N = 128$, the simulated mean boarding time is about 7 min, if one time step of the model is 20 s. Much larger N values are considered here to estimate precisely the scaling exponents. We can confirm by our MC simulations that the average boarding time shows a power-law behavior, as already stated in [8],

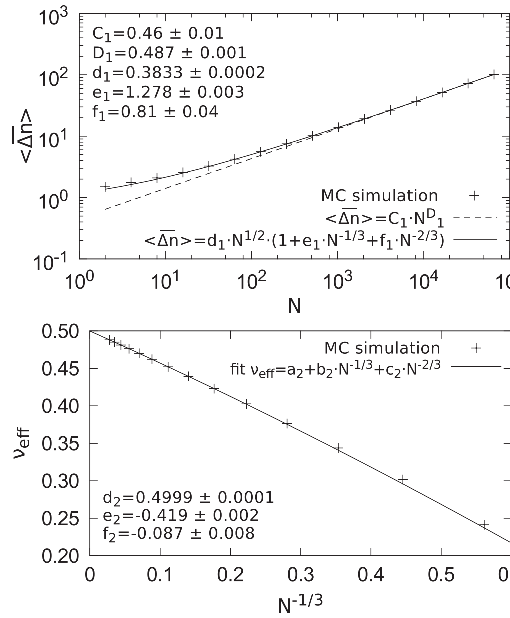

FIG. 3. Determination of the exponent ν. In the upper figure, the mean number of passengers that take seats simultaneously, averaged over the ensemble, is shown depending on the total number of passengers N. The lower figure shows the effective exponent ν_{eff}. It is used to determine the exponent ν. From the quadratic fit we conclude that $\nu = 0.4999 \pm 0.0001 \approx 1/2$.

but with a smaller value of the exponent α, which seems to be $1/2$. To check this more precisely, using the analysis of Sec. II, first we need to find the correction-to-scaling exponent θ and then plot the effective exponent versus $N^{-\theta}$. At large enough N, the relation should be linear. Here we have

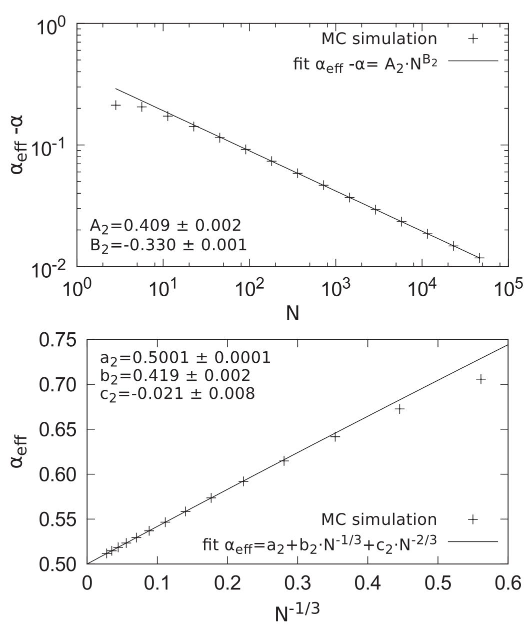

FIG. 2. Determination of the exponent α and the correction-to-scaling exponent θ. The upper figure is used to determine the correction-to-scaling exponent θ. From the linear fit in the log-log plot we see that $\theta \approx 1/3$. The lower figure is used to determine the exponent α. From the quadratic fit we see that $\alpha = 0.5001 \pm 0.0001 \approx 1/2$.

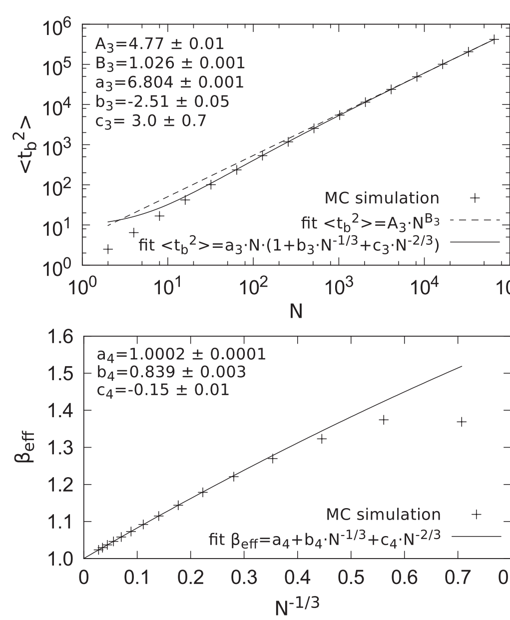

FIG. 4. The second moment of the boarding time. The upper figure shows how the second moment of the boarding time changes with the total number of passengers N. In the lower figure, the effective exponent β_{eff} vs $N^{-1/3}$ is plotted for the second moment of the boarding time. From the quadratic fit we see that $\beta = 1.0002 \pm 0.0001 \approx 1$.

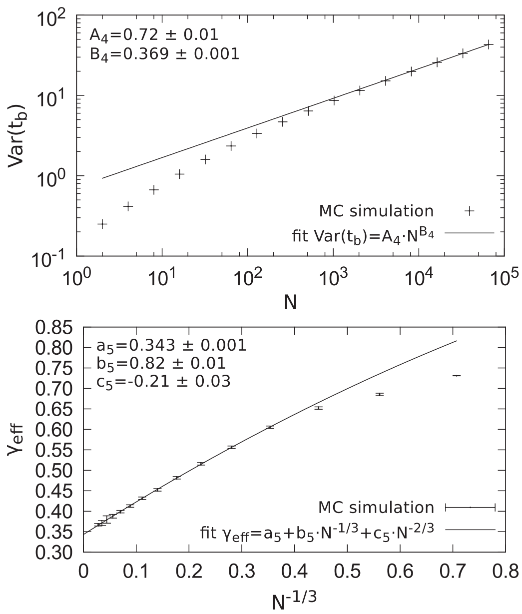

FIG. 5. Variance of the boarding time. The upper figure shows the variance of the boarding time depending on the total number of passengers N. The effective exponent γ_{eff} for the variance of the boarding time is plotted vs $N^{-1/3}$ in the lower figure. From the quadratic fit we estimate the exponent $\beta = 0.343 \pm 0.001 \approx 1/3$.

defined the effective exponent α_{eff} as

$$\begin{aligned}\alpha_{\text{eff}}(N) &= \frac{\ln[\langle t_b\rangle(\sqrt{2}N)] - \ln\left[\langle t_b\rangle\left(\frac{1}{\sqrt{2}}N\right)\right]}{\ln(\sqrt{2}N) - \ln\left(\frac{1}{\sqrt{2}}N\right)} \\ &\approx \alpha - b\frac{2^{\theta/2} - 2^{-\theta/2}}{\ln(2)}N^{-\theta} \\ &\quad + \left(\frac{b^2}{2} - c\right)\frac{2^{\theta} - 2^{-\theta}}{\ln(2)}N^{-2\theta} \qquad (18)\end{aligned}$$

according to (14). The results can be seen in Fig. 2. From these fits we find that the effective exponent α_{eff} is an almost linear function of $N^{-1/3}$ at large N, implying that $\theta \approx 1/3$. These plots confirm the convergence of the effective exponent $\alpha_{\text{eff}}(N)$ to 1/2 at large N as observed by Bernstein [9]. Our fit gives $\alpha = 0.5001 \pm 0.0001$ in agreement with the $\sqrt{N}$ law proposed in [7]. This means also that $\nu \approx 1/2$ holds, according to the scaling relation (6). Thus we have obtained the classical value of the exponent ν, but the scaling exponent θ is nontrivial. From Figs. 1 and 2 it is clear why our results for the exponent α differ from those obtained in [8] (there $\alpha \approx 0.69$).

We can also try to determine the exponent ν directly from the $\langle\overline{\Delta n}\rangle$ data using Eq. (15). This is done in Fig. 3, from which we obtain $\nu \approx 1/2$, as expected. Similarly we can find scaling exponents also for other moments of t_b and $\overline{\Delta n}$. So we have found $\langle t_b^2\rangle \propto N^{\beta}$ for the second moment of the boarding time, where $\beta \approx 1$ (see Fig. 4), as expected from (7) with $\alpha \approx 1/2$. However, we get a more unexpected result for the variance of the boarding time, i.e., $\text{Var}(t_b) = \langle t_b^2\rangle - \langle t_b\rangle^2 \propto N^{\gamma}$ with $\gamma = \theta \approx 1/3$ (see Fig. 5).

For a more complete picture, we have calculated also various probability distributions. In particular, we have determined the probability distributions $p(t_b)$ for the boarding time t_b, $p(\Delta n)$ for the number of passengers Δn taking seats simultaneously, as well as $p(\overline{\Delta n})$ for Δn averaged over a boarding scenario, i.e., for $\overline{\Delta n}$. In addition, we have also determined from our simulations the distribution $p(s)$, where s is the number of sequences of increasing values for numbers (seat numbers) assigned to passengers, as introduced in [8]. These probability distributions are plotted in Fig. 6, where we see that all distributions are narrow already for $N = 2^{15}$, especially $p(\overline{\Delta n})$. This confirms the assumption used in Sec. II.

We have compared our numerical values of the moments and variance of s with the known exact results [8] to conclude that the relative errors in our data for $\langle s\rangle$ are as small as 7.80×10^{-5}, 5.97×10^{-6}, 2.27×10^{-6}, and 2.09×10^{-6} at $N = 2$,

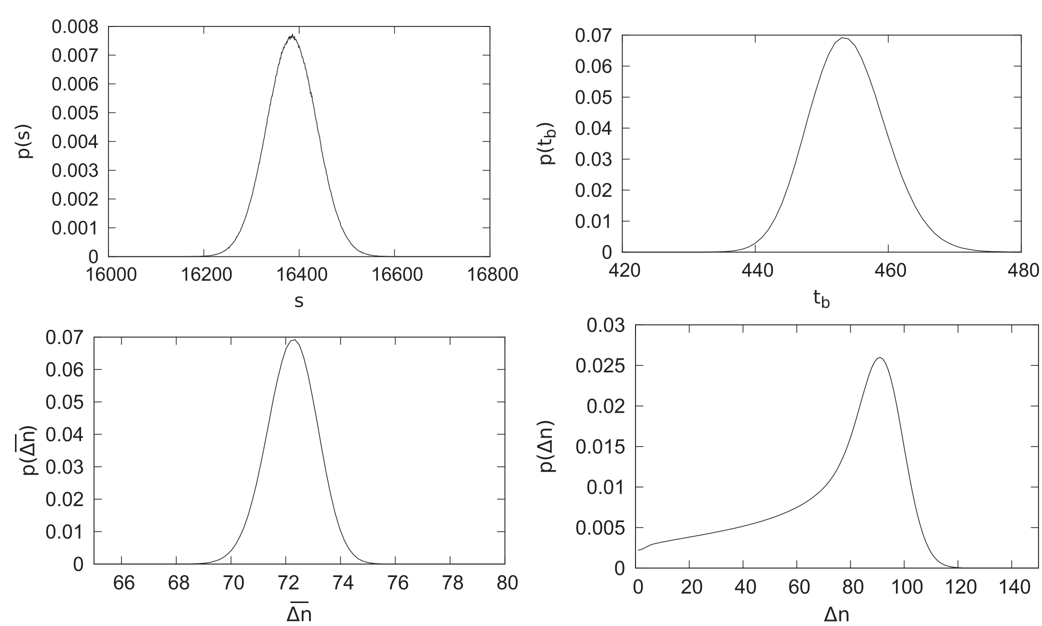

FIG. 6. Probability distributions for s, t_b, $\overline{\Delta n}$, and Δn in the case of $N = 2^{15} = 32\,768$.

$N = 64$, $N = 2048$, and $N = 65\,536$, respectively. The corresponding errors in $\langle s^2 \rangle$ are 1.40×10^{-4}, 1.46×10^{-5}, 4.48×10^{-6}, and 4.19×10^{-6}, and those in $\mathrm{Var}(s)$ are 5.60×10^{-8}, 5.25×10^{-4}, 4.09×10^{-4}, and 2.51×10^{-4}.

IV. CONCLUSIONS

In the current work, Monte Carlo simulations of the airplane-boarding model, introduced earlier by Frette and Hemmer [8], have been performed for very large numbers of passengers N up to $N = 2^{16} = 65\,536$ in order to determine precisely the asymptotic power-law scaling behavior. Based on an analogy with critical phenomena, our scaling analysis of the mean boarding time $\langle t_b \rangle$ and other related quantities takes into account the leading power-law behavior [e.g., $\langle t_b \rangle(N) \propto N^\alpha$ for $\langle t_b \rangle$], as well as corrections to scaling in the form of powers with different exponents. Our results confirm the observed convergence of the effective exponent $\alpha_{\mathrm{eff}}(N)$ to $1/2$ at large N obtained by Bernstein [9] and we explain this effect by corrections to scaling with the correction-to-scaling exponent $\theta \approx 1/3$. Our numerical value of the exponent α, obtained from very large system sizes, is $\alpha = 0.5001 \pm 0.0001$, in agreement with the Lorenz estimate of [7]. It differs very essentially from the values $\alpha = 0.69 \pm 0.01$ [8] and $\alpha = 0.692 \pm 0.004$ [9], provided by log-log fits over the range $2 \leqslant N \leqslant 16$, despite the fact that these fits look good. This observation is interesting also as a toy example for discussions of effective and true (asymptotic) critical exponents in critical phenomena [13], showing that the true critical exponents can be remarkably different from numerical estimates extracted from relatively small system sizes.

Apart from α, we have also estimated other exponents. In particular, we have found that the mean number of passengers taking seats simultaneously in one time step scales with the exponent $\nu = 1/2$ ($\nu = 0.4999 \pm 0.0001$), and $\langle t_b^2 \rangle$ scales with the exponent $\beta = 1$ ($\beta = 1.0002 \pm 0.0001$) as $N \to \infty$. We note that $1/2$ can be regarded as a classical value of the exponent ν and these exponents obey the scaling relations (6) and (13). The leading correction-to-scaling exponent $\theta \approx 1/3$ is nontrivial, i.e., $\theta \neq 1$. It turns out that the variance of $\langle t_b \rangle$ scales with a similar exponent, i.e., $\mathrm{Var}(t_b) \propto N^\gamma$ with $\gamma \approx 1/3$. Our current numerical estimate, however, slightly deviates from $1/3$, i.e., $\gamma = 0.343 \pm 0.001$.

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