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Kardar-Parisi-Zhang universality class in (2 + 1) dimensions: Universal geometry-dependent distributions and finite-time corrections

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The dynamical regimes of models belonging to the Kardar-Parisi-Zhang (KPZ) universality class are investigated in $d = 2 + 1$ by extensive simulations considering flat and curved geometries. Geometry-dependent universal distributions, different from their Tracy-Widom counterpart in one dimension, were found. Distributions exhibit finite-time corrections hallmarked by a shift in the mean decaying as $t^{-\beta}$, where β is the growth exponent. Our results support a generalization of the ansatz $h = v_\infty t + (\Gamma t)^\beta \chi + \eta + \zeta t^{-\beta}$ to higher dimensions, where v_∞, Γ, ζ, and η are nonuniversal quantities whereas β and χ are universal and the last one depends on the surface geometry. Generalized Gumbel distributions provide very good fits of the distributions in at least four orders of magnitude around the peak, which can be used for comparisons with experiments. Our numerical results call for analytical approaches and experimental realizations of the KPZ class in two-dimensional systems.

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Almost three decades after Kardar, Parisi, and Zhang (KPZ) [1] proposed their celebrated equation to describe the coarse-grained regime of evolving surfaces, there has been renewed interest in the subject due the experimental realization of its universality class in a turbulent liquid crystal setup [2] and the achievement of invaluable analytical solutions for distinct dynamical regimes and geometries in $d = 1 + 1$ [3]. The KPZ equation reads as

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \xi, \quad (1)$$

where ξ is a white noise of mean zero and amplitude $\sqrt{D}$. Despite its original conception for evolving interfaces, the KPZ equation has also found its place in other important physical systems [4].

A great advance in the theoretical understanding of the KPZ universality class was begun in the early 2000's with the seminal works of Johansson [5] and Prähofer and Spohn [6] presenting analytical asymptotic solutions of some models in the KPZ class. These solutions link the height's stochastic fluctuations to universal distributions [7] of the random matrix theory. Inspired by these exact results, the ansatz

$$h = v_\infty t + s_\lambda (\Gamma t)^\beta \chi, \quad (2)$$

with the exactly known growth exponent $\beta = 1/3$, was conjectured as describing the asymptotic interface fluctuations of any model belonging to the KPZ class in $d = 1 + 1$ [4,8]. In this equation, $s_\lambda = \mathrm{sgn}(\lambda)$, while the asymptotic velocity v_∞ and Γ are model-dependent parameters and χ is a universal random variable with a time-independent distribution given by the Gaussian orthogonal ensemble (GOE) for flat geometries [5,6] and the Gaussian unitary ensemble (GUE) for the curved ones [6,8]. Notice that, in terms of the constants of the KPZ equation, the parameter Γ is given by $\Gamma = \frac{1}{2}A^2|\lambda|$, with $A = D/\nu$ [4]. These geometry-dependent universality was confirmed in turbulent crystal liquid experiments [2] and in stochastic simulations of several models without known analytical solutions [9–11].

Many fine-tuning results have been aggregated to the asymptotic height distributions (HDs) of one-dimensional KPZ systems. The limiting processes describing the surface fluctuations are known as Airy_1 and Airy_2 processes for flat [12,13] and curved geometries [14], respectively. Finite-time corrections to Eq. (2) were also analytically [3,15], experimentally [2], and numerically [16] observed, leading to the generalization

$$h = v_\infty t + s_\lambda (\Gamma t)^\beta \chi + \eta + \zeta t^{-\beta}, \quad (3)$$

where η and ζ are nonuniversal. The correction η introduces a shift in the distribution of the scaled height $q = \frac{h - v_\infty t}{s_\lambda (\Gamma t)^\beta}$ in relation to the asymptotic distributions. The hallmark of this correction, a shift in the mean vanishing as $\langle q \rangle - \langle \chi \rangle \sim t^{-1/3}$, has been verified in the crystal liquid experiments [2] and computer simulations of several models [9,10,16]. To our knowledge, only two exceptions have been reported. In the first one, Ferrari and Frings [15] analyzed the partially asymmetric simple exclusion process and found a specific value of the asymmetry parameter where there is no correction up order $O(t^{-2/3})$. Off-lattice simulations of an Eden model consistent with a decay $t^{-2/3}$ were reported [11], but a subsequent analysis showed that the unusual behavior is an artifact of low precision estimates of v_∞ and a long crossover to the scaling law $t^{-1/3}$ [16].

In contrast to the deep understanding of the KPZ class in $d = 1 + 1$, essentially no exact results are available in $d = 2 + 1$, the most important dimension for applications [17]. Indeed, available analytical approximations [18] fail in predicting the best numerical estimates of the scaling exponents [19]. The best we know about the KPZ class in $d = 2 + 1$ comes from simulations: The scaling exponents [19] and height distributions in the stationary regime [20] are accurately known and its universality has been verified. A few works, impaired by finite-size effects, had studied height distributions in the dynamical regime using flat geometry [21,22] when, very recently, Halpin-Healy [23] reported large-scale simulations of some KPZ models that convincingly suggest the universality of the height distributions. Halpin-Healy's analysis is in consonance with our results.

In the present Rapid Communication, a detailed study of the dynamical regime of several KPZ models in (2 + 1)

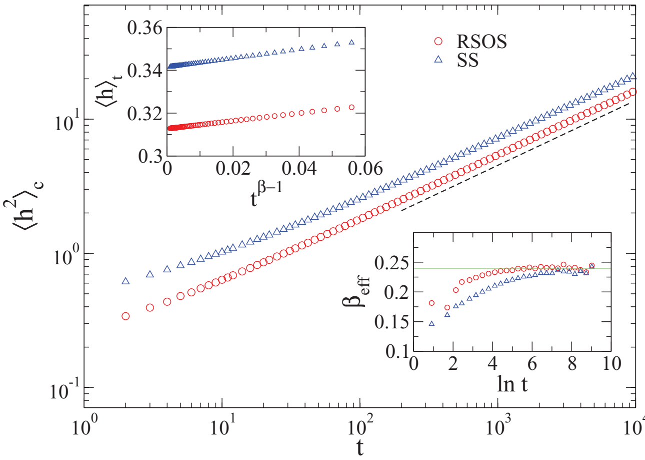

FIG. 1. (Color online) Main plot: Height variance against time for the RSOS and SS models. The dashed line represents the power law $t^{0.48}$. Bottom inset: Effective growth exponents against time. The horizontal line represents the accepted KPZ value in $d = 2 + 1$. Top inset: Interface velocity against $t^{\beta-1}$.

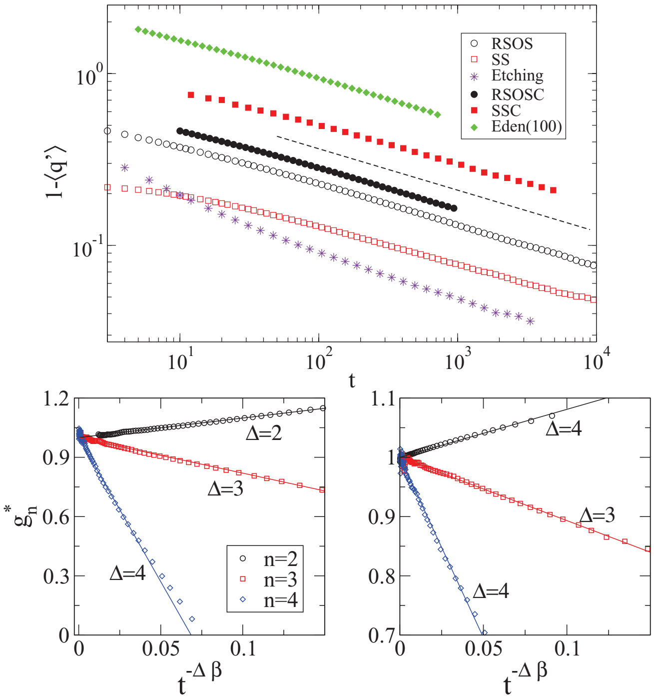

FIG. 2. (Color online) Top: Determination of the mean shift $\langle\eta\rangle$ for flat and curved geometries. The dashed line is the decay $t^{-0.24}$. Bottom: Normalized cumulants $g_n^* = g_n(t)/g_n(\infty)$ against scaled time for the SS (left) and RSOS (right) models in a flat geometry. Lines are (scaled) linear regressions used to determine $g_n(\infty)$.

dimensions is presented. Both flat and curved geometries are considered. We go beyond Halpin-Healy's results and show that the generalized KPZ ansatz given by Eq. (3) still holds in $d = 2 + 1$ with the proper growth exponent $\beta = 0.24$. The universality of χ, which differs from the counterparts in $(1 + 1)$ dimensions, is confirmed and its geometry dependence characterized. Also, we have verified that the corrections in the mean vanish as $t^{-\beta}$ and nonuniversal corrections were found for higher order cumulants. We compensate for the absence of an exact analytical expression for the HDs, showing that generalized Gumbel distributions [24] fit noticeably well the heights scaled accordingly Eq. (3).

Flat geometry. We performed extensive simulations of three models in the KPZ class, namely, the restricted solid-on-solid (RSOS) [25], single step (SS) [17], and etching [26] models. Square lattices with up to $2^{15} \times 2^{15}$ sites and periodic boundary conditions were used. Except for the SS model, for which a checkerboard initial condition was used, an initially smooth substrate was considered. Up to 10^3 runs were used in averages.

The growth exponent can be determined from $w^2 \equiv \langle h^2\rangle_c \sim t^{2\beta}$, where $\langle X^n\rangle_c$ represents the nth cumulant of X. Figure 1 shows the evolution of the variance for two models, while the corresponding effective growth exponents (the local slope in curves $\ln w$ vs $\ln t$) are shown in the bottom inset. The growth exponents obtained for all models are shown in Table I, in which an excellent agreement with the accepted KPZ exponent $\beta = 0.24$ is observed for all flat models. Differentiating $\langle h\rangle$ in Eq. (3), one finds $\langle h\rangle_t = v_\infty + s_\lambda \beta \Gamma^\beta \langle\chi\rangle t^{\beta-1} + \cdots$. A linear regression in $\langle h\rangle_t$ against $t^{\beta-1}$ for $t \to \infty$ yields v_∞. This procedure is illustrated in the top inset of Fig. 1 and the estimates for all investigated models are given in Table I. The quantity $\Gamma^\beta\langle\chi\rangle$ can be obtained from the asymptotic value of $g_1 = (\langle h\rangle_t - v_\infty)t^{1-\beta}/\beta$. It was shown that the value of Γ determined from g_1 is more reliable than using cumulants of order $n \geqslant 2$ [16].

The accuracy in determining universality in simulations may be very sensitive to the correction η depending on the model and the attainable simulation time. So, it is important to determine the strength of corrections before analyzing the height distributions. The mean $\langle\eta\rangle$ can be determined using the height scaled in terms of directly measurable parameters v_∞ and g_1 as $q' = (h - v_\infty t)/(s_\lambda g_1 t^\beta)$ [16]. Equation (3) implies $1 - \langle q'\rangle = -(s_\lambda\langle\eta\rangle/g_1)t^{-\beta} + \cdots$. Figure 2 shows that the power law $t^{-\beta}$ describes very precisely the shift,

TABLE I. Nonuniversal and universal quantities for the dynamical regime of KPZ models. Definitions are found in the text.

Model	v_∞	g_1	g_2	g_3	g_4	$\langle\eta\rangle$	β	R	S	K
RSOS	0.31270(1)	− 0.773(1)	0.1936(4)	0.0364(3)	0.0130(5)	− 0.5(1)	0.240(3)	0.324(3)	0.427(5)	0.347(8)
SS	0.341368(3)	− 0.881(1)	0.250(1)	0.0536(3)	0.0219(5)	− 0.4(1)	0.239(5)	0.322(2)	0.428(5)	0.35(1)
Etching	3.3340(1)	− 2.348(3)	1.715(3)	0.950(2)	1.00(1)	0.6(1)	0.235(5)	0.311(2)	0.423(2)	0.340(5)
RSOSC	0.3134(2)	− 2.116(2)	0.272(2)	0.0481(6)	0.0158(5)	−1.7(1)	0.24(1)	0.061(3)	0.339(8)	0.21(1)
SSC	0.12611(2)	− 0.797(1)	0.051(2)	0.0037(2)	0.00053(8)	−1.2(1)	0.23(2)	0.080(5)	0.32(4)	0.20(5)
Eden (001)	0.6495(3)	− 3.543(3)	0.785(8)	0.234(3)	0.13(1)	9.8(5)	0.243(7)	0.063(2)	0.336(9)	0.21(2)
Eden (111)	0.6242(2)	− 3.219(5)	0.610(8)	0.164(3)	0.083(5)	8.8(5)	0.239(6)	0.059(2)	0.34(1)	0.22(2)

analogously as observed in $d = 1 + 1$ [2,9,10,15,16]. So, using the prefactor of the power law $t^{-\beta}$, we determined $\langle\eta\rangle$ for all investigated models. The estimates are shown in Table I.

From Eq. (3), we have that scaled cumulants $g_n(t) = \langle h^n\rangle_c/(s_\lambda^n t^{n\beta})$, $n \geqslant 2$, converge to $\Gamma^{n\beta}\langle\chi^n\rangle_c$ for $t \to \infty$. Contrasting with the first cumulant, the corrections in g_n depend on the model. Figure 2 (bottom) shows the scaled cumulants against $t^{-\Delta\beta}$ where Δ was assumed to be an integer (the values used are indicated near each curve). For the sake of visibility, curves were normalized by the asymptotic value $\Gamma^{n\beta}\langle\chi^n\rangle_c$ obtained by extrapolation in plots of g_n vs $t^{-\Delta\beta}$. These estimates are shown in Table I. For the SS (bottom left in Fig. 2) and etching (data not shown) models, the corrections are quite consistent with $\langle q^n\rangle_c - \langle\chi^n\rangle_c \sim t^{-n\beta}$, in analogy to the exact solution of the KPZ equation with an edge initial condition and experimental results in $d = 1 + 1$ [2,3]. However, in RSOS the second cumulant presents a different behavior with the shift decaying approximately as $t^{-4\beta}$, demonstrating the nonuniversality of the corrections in cumulants of order $n \geqslant 2$.

The parameters g_i, $i = 1$–4, shown in Table I, depend on Γ, which cannot be determined directly from height distributions [16]. However, one can investigate dimensionless cumulant ratios that are independent of Γ and, therefore, are expected to be universal. In Table I, we show the ratios $R = g_2/g_1^2 = \langle\chi^2\rangle_c/\langle\chi\rangle^2$, $S = g_3/g_2^{3/2} = \langle\chi^3\rangle_c/\langle\chi^2\rangle_c^{3/2}$ (skewness), and $K = g_4/g_2^2 = \langle\chi^4\rangle_c/\langle\chi^2\rangle_c^2$ (kurtosis) for all investigated models. The ratios for different flat models are essentially the same, confirming the universality of χ conjectured initially. Notice that they are different from the ratios for GOE distributions expected for their one-dimensional counterparts [6]. Since an infinite hierarchy of cumulant ratios can be measured, in principle, we can determine all cumulants in terms of the first one. Our estimates for S and K are in good agreement with those found by Halpin-Healy in Ref. [23], but fluctuating estimates for $\langle\chi\rangle$ and $\langle\chi^2\rangle_c$ presented there do not allow a reliable estimate of R (values ranging from 0.33 to 0.51 are extracted from Ref. [23]). We believe that the corrections in distributions, mainly in the mean, are responsible by the apparent nonuniversality of R in Ref. [23]. Our estimates of S and K are also consistent with former, small-size simulations [21] and also with recent simulations of the Eden model on flat substrates [27], confirming the universality of the HDs.

Due to the lack of rigorous results in $(2 + 1)$ dimensions, we are currently not able to associate our numerical results with an analogous of the Tracy-Widom (TW) distributions. However, previous works dealing with linear systems have shown that the generalized Gumbel distribution with a noninteger parameter m fits the probability density functions of stationary quantities in several equilibrium and nonequilibrium systems [24,28,29]. We have obtained a very good agreement between our simulations and the so-called Gumbel's first asymptotic distribution of mean $\langle X\rangle$ and variance $\langle X^2\rangle_c$ [29],

$$G(X;m) = \frac{m^m b}{\Gamma(m)} \exp[-m(z_X + e^{-z_X})], \tag{4}$$

where $b = \sqrt{\psi_1(m)/\langle X^2\rangle_c}$, $z_\chi = b(\langle X\rangle - X + s)$, $s = [\ln m - \psi_0(m)]/b$, $\Gamma(X)$ is the gamma function, and $\psi_k(X)$ the polygamma function of order k [30]. The parameter m

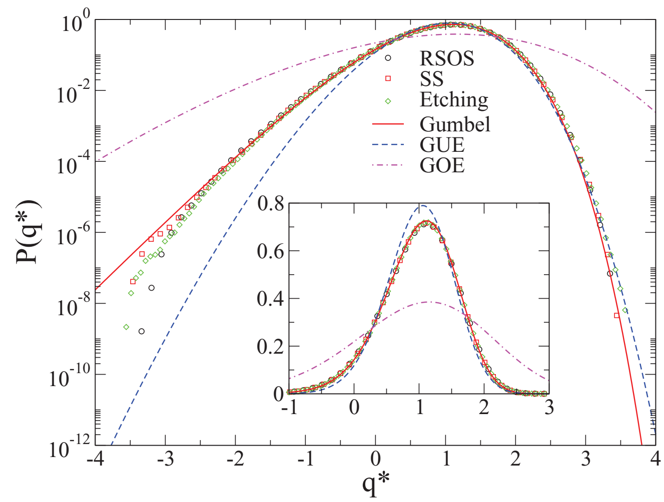

FIG. 3. (Color online) Height distributions for flat growth scaled to mean 1 compared with a Gumbel distribution for $m = 6$ and variance $R = 0.32$. The inset shows the same data in a linear scale. The growth times are $t = 10^4$ (RSOS), $t = 8000$ (SS), and $t = 2000$ (etching). Scaled TW distributions are included for the sake of comparison.

allows to change simultaneously, but not independently, the skewness and kurtosis of the distribution. For $m = 6$, one obtains a skewness $S_G = -0.4247$ and kurtosis $K_G = 0.3597$ that are very close to the universal values for flat models shown in Table I.

The height distribution scaled to a mean 1, accordingly the nonuniversal parameters, becomes $q^* = (h - v_\infty t - \langle\eta\rangle)/(s_\lambda g_1 t^\beta)$, leading to a variance $\langle q^{*2}\rangle_c \equiv R$. In the top panel of Fig. 3, the scaled heights for flat models are compared with a Gumbel distribution for $m = 6$, mean 1, and variance $R = 0.32$. A remarkable collapse is observed around the peak for at least four decades. From an experimental perspective, it is extremely hard to measure distribution extremes with an accuracy comparable to our simulations. Hence, the Gumbel approximation is a useful reference to check the KPZ universality class in $(2 + 1)$ dimensions. Notice that in a linear scale, simulations are indistinguishable from the Gumbel distribution, in contrast with the TW distributions that do not even barely fit the distribution's peak, as can be seen in inset of Fig. 3. Interestingly, the rightmost tail of the scaled distributions is well fitted by the scaled GUE distribution $\chi_{\mathrm{GUE}}/\langle\chi_{\mathrm{GUE}}\rangle$. It is worth mentioning that generalized Gumbel functions were compared with distributions of height extremes in the stationary regime of the KPZ and other nonlinear models in Ref. [31], where a good fit around the peak and large deviations in the tails were observed.

Curved geometry. We study radial geometry using the on-lattice Eden D model [16]. Due to the intrinsic anisotropy of on-lattice Eden clusters, we investigate surface fluctuations along axial (100) and diagonal (111) directions. We also considered curved surfaces using the RSOS and SS models growing in a corner (RSOSC and SSC), where fluctuations in the (111) direction are considered. Details of the models and simulation are presented in Ref. [16], where we carried out a

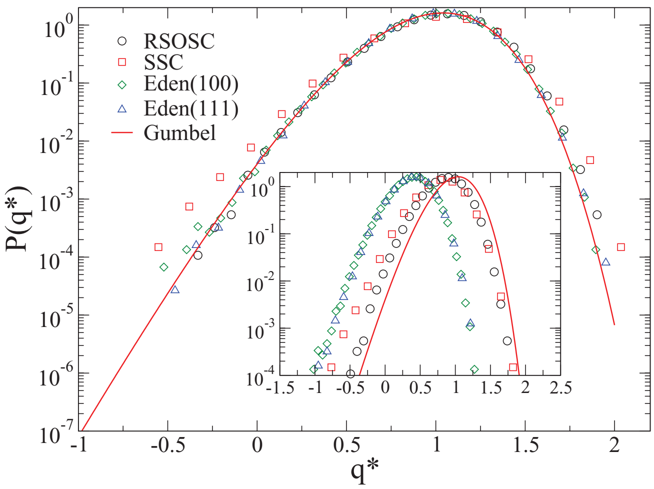

FIG. 4. (Color online) Height distributions for curved growth scaled to mean 1 compared with a Gumbel distribution for $m = 9.5$ and variance $R = 0.062$. The growth times are $t = 1012.4$ (RSOSC), $t = 4000$ (SSC), and $t = 549.7$ (Eden). Inset: Scaled height distributions disregarding the shift $\langle\eta\rangle$.

detailed study in $d = 1 + 1$ and obtained the expected KPZ scaling, GUE TW, for curved growth.

The growth exponents found for all models agree very well with the KPZ value $\beta = 0.24$, as shown in Table I. The nonuniversal parameters related to the curved growth models are shown in Table I. The asymptotic velocity of the SSC model has been under debate [32,33] and our estimate is in agreement with Ref. [33]. Again, the shift in the mean scales as $t^{-\beta}$ exactly as in the flat case (Fig. 2). However, the amplitude of the corrections is in general much larger than in the flat case, particularly for the Eden model, and plays a central role in the time scale simulated in the present work. Corrections in g_n are of order $t^{-2\beta}$ or faster, in analogy to the flat case.

Dimensionless cumulant ratios are also universal for curved geometries, as shown in Table I. These ratios differ from those of the flat case and are even further from the TW values known for $(1 + 1)$ dimensions. Our cumulant ratios are also in agreement with those reported by Halpin-Healy for a single model in the so-called point-point geometry [23]. Once again, the scaled height distributions are well fitted by a generalized Gumbel distribution with $m = 9.5$, which has $S_G = 0.335$ and $K_G = 0.224$. A very important remark is that curves do not collapse if the correction $\langle\eta\rangle$ is not explicitly included in the analysis, as shown in the inset of Fig. 4. Rescaling the distributions, accordingly to Eq. (3), to mean 1 and variance $R = 0.062$, we found a good data collapse, with the exception of the SS model (Fig. 4). This is due to its larger value of R (possibly produced by large fluctuations).

Assuming the last term in Eq. (3) has the form $\zeta t^{-\gamma}$, one has that

$$s_\lambda(\langle h\rangle_t - v_\infty)t^{1-\beta}/\beta = g_1 - \gamma s_\lambda \langle\zeta\rangle t^{-\gamma-\beta}/\beta. \quad (5)$$

Our simulations show that g_1 converges to its asymptotic value with a correction quite close to $t^{-2\beta}$ in all flat and curved growth models. So, the last term in Eq. (3) decays with an exponent $\gamma = \beta$. An equivalent result was obtained in the simulations of the KPZ models in $d = 1 + 1$, where a term $t^{-1/3}$ was identified in the KPZ ansatz [16]. So, we have additional evidence that the generalized KPZ ansatz in $d = 1 + 1$ has an equivalent counterpart in higher dimensions.

In conclusion, we have studied the height distributions in the dynamical regime of KPZ systems in $d = 2 + 1$ and confirmed the universality of geometry-dependent distributions found very recently by Halpin-Healy [23]. However, we have gone further and characterized also the finite-time behavior of the distributions. As in the $(1 + 1)$ case, the shift in the mean decays as $t^{-\beta}$ and the corrections in higher order cumulants are nonuniversal and decay faster than or are equal to $t^{-2\beta}$. We also show that generalized Gumbel distributions, commonly applied to fit distributions in linear systems [24,28,29], fit noticeably well the HDs of KPZ models that are nonlinear. Such distributions and the finite-time behaviors may play an import role in the experimental study of KPZ systems. Furthermore, they may motivate and guide analytical insights for the understanding of the KPZ universality class in $(2 + 1)$ dimensions.

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