Weak turbulence in two-dimensional magnetohydrodynamics

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A weak wave turbulence theory is developed for two-dimensional (2D) magnetohydrodynamics. We derive and analyze the kinetic equation describing the three-wave interactions of pseudo-Alfvén waves. Our analysis is greatly helped by the fortunate fact that in 2D the wave kinetic equation is integrable. In contrast with the three-dimensional case, in 2D the wave interactions are nonlocal. Another distinct feature is that strong derivatives of spectra tend to appear in the region of small parallel (i.e., along the uniform magnetic field direction) wave numbers leading to a breakdown of the weak-turbulence description in this region. We develop a qualitative theory beyond weak turbulence describing subsequent evolution and formation of a steady state.

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I. INTRODUCTION

Magnetohydrodynamics (MHD) is of great interest for modeling turbulence in magnetically confined and unconfined plasmas. In astrophysics its applications range from solar wind [1], to the Sun [2], the interstellar medium [3], and beyond [4]. Additionally, MHD is also relevant for describing the large-scale stability of fusion plasmas such as tokamaks [5]. One of the pioneering results of incompressible MHD turbulence has been obtained by Iroshnikov [6] and Kraichnan [7] (hereafter IK) who proposed an extension of the Kolmogorov phenomenology [8], originally derived for hydrodynamics (HD). For simplicity the assumptions of homogeneity and isotropy were made by Kolmogorov and the energy cascade was supposed to be dominated by local (in scale) interactions between eddies of similar size. Then, the Kolmogorov phenomenology leads to the well-known one-dimensional kinetic-energy spectrum, $E(k) \sim k^{-5/3}$, where k is the wave number. The associated cascade properties for its inviscid invariants differ for three-dimensional (3D) and 2D turbulences: while in 3D the energy and the kinetic helicity exhibit direct cascades, in 2D the energy cascades inversely—still with a $-5/3$ scaling—whereas a direct cascade is found for the enstrophy which leads to the spectrum, $E(k) \sim k^{-3}$, at small scales.

IK modified the Kolmogorov phenomenology by taking into account the magnetic field. They also assumed homogeneity, isotropy, and locality of interactions. However, there exist fundamental differences between the Kolmogorov and the IK theories. First of all, in MHD the energy cascade is supposed to be dominated not by the interactions between eddies, but between Alfvén wave packets propagating in opposite directions; this modification leads to the energy spectrum $E(k) \sim k^{-3/2}$. Furthermore, unlike hydrodynamics the cascades of the ideal MHD invariants exhibit some similarities in their behavior in 2D and 3D [9]: in both cases, the cascades have the same direction with a direct cascade for the energy and cross helicity, and an inverse cascade for the magnetic helicity in 3D, or the mean-squared magnetic potential—called anastrophy—in 2D. The differences between MHD and HD turbulences go beyond these classical properties. In the IK theory the large-scale magnetic field is supposed to play the role of external field which is necessary for the existence of Alfvén waves but its main effect, i.e., anisotropy, is not taken into account. The importance of an external magnetic field has been discussed many times during the last two decades [10–18] and the anisotropic behavior has been shown in direct numerical simulations for both 2D [11] and 3D [12].

Despite some similarities—like the cascade directions—the question about the identification of differences between 2D and 3D MHD turbulences still represents an important issue. In early numerical studies [11], mainly 2D simulations were performed because of the limited numerical resources available and the nonaccessibility of 3D calculations at high Reynolds numbers. Nowadays, 3D MHD numerical simulations are commonly achieved [19–25] but 2D simulations are still used for the illustration of new numerical techniques [26]. There is also an interest in the understanding of freely decaying MHD turbulence because the Reynolds numbers can be higher in 2D than in 3D [27–29]. In the context of solar flares triggered by a magnetic reconnection mechanism, 2D MHD is often used to estimate the reconnection rate. The challenge is to find a fast reconnection rate in order to explain the explosive flares. With a laminar configuration [30,31] only slow rates are found whereas the introduction of turbulence may lead to fast rates [32,33]. When the magnetic field perturbations are small compare to a uniform background magnetic field, the 2D MHD equations are sometimes used to model turbulence. Such a situation is particularly relevant for solar coronal

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loops [34]. Strong statements were made by some authors that 2D simulations can be safely used to model 3D situations because the properties of the 2D and the 3D MHD turbulences are essentially the same [27,35]. One of the motivations of the present paper is to test the validity of this claim in a special case when the external magnetic field is strong.

In this paper, we consider 2D MHD in the presence of a strong background magnetic field which implies the realization of the weak-turbulence regime. One of the main advantages of this regime is the fact that it allows one to derive accurate analytical results for the spectrum. An explicit comparison will be made between the weak turbulence regimes in 2D and in 3D; the latter was analyzed rigorously in Ref. [36]. The weak wave turbulence approach is widely familiar to the plasma physics community [37–50]. It is a statistical description of a large ensemble of weakly interacting dispersive waves. The formalism leads to wave kinetic equations from which exact power-law solutions can be found for the energy spectra. There were several reasons which postponed development of weak turbulence for Alfvén waves. The first one is their semidispersive nature. Typically, the wave kinetic approach cannot be used for nondispersive waves since such wave packets propagate with the same group velocity even if their wave numbers are different; the energy exchange between such waves may not be considered small and may lead to possible energy accumulation over a long time of interaction. Alfvén waves represent a unique exception to this rule because co-propagating wave packets do not interact and the nonlinear interaction is present only for counter-propagating wave packets: the latter pass through each other in some finite time and no long-time cumulative effect occurs. That is why the Alfvén waves represent a unique example of semidispersive waves for which the wave turbulence theory applies. The second reason which renders the weak turbulence theory for Alfvén waves very subtle is the fact that the domination of three-wave interactions—as assumed by IK—might be questionable. While in Ref. [51] the three-wave interactions were declared absent, the IK argument has been reestablished in Refs. [13,36,52]. The weak-turbulence theory for 3D incompressible MHD was developed in Ref. [36] (see also Refs. [53,54]) where the three-wave kinetic equations were derived with their exact solutions via a systematic asymptotic expansion in powers of small nonlinearities.

The main goal of the present paper is to derive the weak turbulence equation in 2D MHD, analyze it and make a comparison with the 3D case. The crucial technical step which allows a comprehensive theoretical analysis of the solutions consists of transforming the wave kinetic equation into an integrable form by Fourier transforming it and separating the transverse and the parallel dynamics by using a self-similar "effective time" variable.

This article is organized as follows: In Sec. II, we derive the weak-turbulence kinetic equation through a general perturbative procedure. In Secs. III and IV, we proceed with a detailed investigation of its properties in the anisotropic limit. The main goal of such a study is to verify whether or not turbulence is local. First of all, we will consider the steady-state behavior by looking for Kolmogorov-Zakharov type solutions and check their locality. Next, we will proceed with investigation of an unsteady spectrum evolution by considering two different cases with a Gaussian-shaped source and different kinds of dissipation: a uniform friction and a viscosity. Due to the integrability in the first case of the weak wave kinetic equation it is possible to find an exact solution. In the second case, a qualitative analysis for the steady state is complemented by a numerical simulation of the spectrum evolution. The goal of Sec. V is to develop some qualitative reasoning about the turbulent behavior of our system near the applicability margin of the wave kinetic formalism and beyond. Formation of steady state is also discussed. Finally, we present a summary of our results in Sec. VI.

II. WAVE KINETIC DESCRIPTION

A. Alfvén waves

In 3D incompressible MHD there exists two different kinds of Alfvén waves [15]: the first kind—called shear-Alfvén waves (SAWs)—have fluctuations of velocity and magnetic field transverse to the background magnetic field $\mathbf{B}_0$, whilst the other kind—called pseudo-Alfvén waves (PAWs)—have fluctuations along $\mathbf{B}_0$. Both waves propagate along $\mathbf{B}_0$ at the same group velocity (the Alfvén velocity).

The weak-wave-turbulence formalism for incompressible MHD applies for a small nonlinearity, $\epsilon \sim b_\perp k_\perp/(B_0 k_{||}) \ll 1$, where $b_\perp$ is the perpendicular magnetic field perturbation and $k_{||}$ and $k_\perp$ are, respectively, the wave numbers in the parallel and perpendicular directions to $\mathbf{B}_0$. Additionally, a strong anisotropy condition is often used, $\sigma = k_\perp/k_{||} \gg 1$. In the 3D case, it was shown that at the leading order of the weak nonlinearity ($\epsilon \ll 1$) and strong anisotropy ($\sigma \gg 1$) the SAWs interact only among themselves and evolve independently from the PAWs. At the same time, the PAWs scatter from the SAWs without amplification or damping, and they do not interact with each other. Such behavior does not rule out a possibility for the PAW to interact among themselves in the next order of expansion in $1/\sigma$. However, in the 3D case such a process is subdominant to a stronger interaction with the SAW and has not been considered yet.

In the 2D case, due to the geometrical restrictions, it is only possible to have PAWs. In this paper, we will see that three-wave interactions of PAWs do occur in 2D in the next order of expansion in $1/\sigma$ and represent the dominant process in the nonlinear evolution.

B. Interaction representation

The ideal incompressible MHD system in Elsässer variables $\mathbf{z} = \mathbf{v} + s\mathbf{b}$, with $s = \pm 1$, is given by [35]

$$(\partial_t - s\mathbf{B}_0 \cdot \nabla + \mathbf{z}^{-s} \cdot \nabla)\mathbf{z}^s = -\nabla P_*, \tag{1}$$

$$\nabla \cdot \mathbf{z}^s = 0, \tag{2}$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{b}$ is the magnetic-field fluctuation (in velocity units), $\mathbf{B}_0$ is a uniform background magnetic field (also in velocity units, i.e., the Alfvén speed), and P_* is the total (thermal plus magnetic) pressure. In what follows we suppose that the background magnetic field is directed along the $\widehat{\mathbf{x}}$ axis, $\mathbf{B}_0 = B_0\widehat{\mathbf{x}}$. In coordinate notation we have

$$(\partial_t - sB_0\partial_x)\, z_j^s = -\, z_n^{-s}\partial_n z_j^s - \partial_j P_*, \tag{3}$$

$$\partial_i z_i^s = 0. \tag{4}$$

The nonlinear terms in Eq. (3) include only the Elsässer variables of the opposite signs. Therefore, the nonlinear interactions take place only between counter-propagating wave packets.

The first step in the general procedure of the wave kinetic formalism is to identify the linear modes. Neglecting the nonlinear terms in the right-hand side (r.h.s.) of Eq. (3) (which includes the pressure term) and looking for solutions in the form of a wave

$$z_j^s \sim e^{i(k_x x + k_y y) - i\omega^s t}, \tag{5}$$

we obtain two linear modes

$$\omega^s = -s B_0 k_x, \quad s = \pm 1, \tag{6}$$

which propagate parallel to the background magnetic field (in both directions) with the group velocity, $\mathbf{v}_g^s = -s\mathbf{B}_0$. Let us suppose that our system is periodic in the physical space (with period L in both x and y directions) and let us introduce the Fourier series

$$z_j^s(\mathbf{x},t) = \sum_{\mathbf{k}} a_j^s(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{7}$$

where the wave vector $\mathbf{k}$ takes values on a 2D grid, $\mathbf{k} = (k_x,k_y) = (2\pi m_x/L, 2\pi m_y/L)$ where $m_x,m_y \in Z$. Then, by applying the divergence operation on both sides of Eq. (3) and by using Eq. (4), we find the expression for the Fourier coefficients of the pressure P_*:

$$\hat{P}_*(\mathbf{k}) = -k^{-2} \sum_{\mathbf{k}_1,\mathbf{k}_2} (\mathbf{k}_2 \cdot \mathbf{a}^{-s}(\mathbf{k}_1,t)) \times (\mathbf{k} \cdot \mathbf{a}^s(\mathbf{k}_2,t))\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}), \tag{8}$$

where $\delta(\mathbf{p})$ is the delta function [$\delta(\mathbf{p}) = 1$ for $\mathbf{p} = 0$ and zero otherwise]. Thus, Eq. (3) in Fourier space becomes

$$(i\partial_t - \omega^s)\mathbf{a}^s(\mathbf{k},t) = \sum_{\mathbf{k}_1,\mathbf{k}_2} [\mathbf{k} \cdot \mathbf{a}^{-s}(\mathbf{k}_1,t)] \left\{ \mathbf{a}^s(\mathbf{k}_2,t) - \frac{\mathbf{k}}{k^2}[\mathbf{k} \cdot \mathbf{a}^s(\mathbf{k}_2,t)] \right\} \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}). \tag{9}$$

Using the incompressibility condition

$$a_y^s = -a_x^s \frac{k_x}{k_y}, \tag{10}$$

we reduce the expression (9) to one scalar equation

$$(i\partial_t - \omega^s)a_x^s(\mathbf{k},t) = \sum_{\mathbf{k}_1,\mathbf{k}_2} k_y \frac{(\mathbf{k}\times\mathbf{k}_1)_z\,(\mathbf{k}\cdot\mathbf{k}_2)}{k_{1y}k_{2y}k^2} a_x^{-s}(\mathbf{k}_1,t) \times a_x^s(\mathbf{k}_2,t)\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}). \tag{11}$$

Let us now introduce the notation

$$c_{\mathbf{k}}^s = i\frac{k}{\epsilon k_y} a_x^s(\mathbf{k},t) e^{i\omega^s t}, \tag{12}$$

which represents the slowly varying wave amplitudes (the factor $e^{i\omega^s t}$ compensates for the fast-scale oscillations arising due to the linear dynamics). Then, the MHD equations in the interaction representation become

$$\partial_t c_{\mathbf{k}}^{\pm} = \epsilon \sum_{\mathbf{k}_1,\mathbf{k}_2} V_{12k} c_{\mathbf{k}_1}^{\mp} c_{\mathbf{k}_2}^{\pm} e^{2ik_{1x}t} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}), \tag{13}$$

with the interaction coefficient

$$V_{12k} = \frac{(\mathbf{k}\cdot\mathbf{k}_2)\,[\mathbf{k}_1 \times \mathbf{k}_2]_z}{k k_1 k_2}. \tag{14}$$

Note that up to now we have not used the smallness of ϵ and Eq. (13) is completely equivalent to the initial system (3), (4).

C. Wave kinetic equation

The standard weak-turbulence approach [42,54] exploits the smallness of the nonlinearity, the randomness of phases, and the infinite-box limit. In Appendix A, we apply this approach to Eq. (13) which gives the following kinetic equation for the wave spectrum $n_{\mathbf{k}}$:

$$\partial_t n_{\mathbf{k}}^{\pm} = \pi \int V_{12k}^2 n_{\mathbf{k}_1}^{\mp} [n_{\mathbf{k}_2}^{\pm} - n_{\mathbf{k}}^{\pm}] \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})\,\delta(2k_{1x}) d\mathbf{k}_1 d\mathbf{k}_2, \tag{15}$$

where the interaction coefficient is given by expression (14). In the following sections, we shall proceed with the detailed analysis of this equation.

D. Anisotropic limit

One remarkable property of MHD turbulence—which makes it very different from HD—is the development of strong anisotropy in the presence of a strong background magnetic field. This property was observed in direct numerical simulations in both 2D [11] and 3D [12]. The wave kinetic formalism confirms such an anisotropy through the form of the kinetic equation. In fact, for Alfvén waves the resonant three-wave interaction [11] is organized in such a way that one member of each triad must have its wave vector perpendicular to the external magnetic field $\mathbf{B_0}$ whereas the two other wave vectors have the same parallel wave numbers ($k_\parallel = k_{2\parallel}$). Formally, this property appears in both 2D and 3D through the delta function $\delta(2k_{1\parallel})$ in the kinetic equation as we see in Eq. (15) for the 2D case and in Eq. (26) in Ref. [36] for the 3D case. Using this delta function and integrating over $k_{1\parallel}$, we see that the parallel component of the wave number enters into the kinetic equation as an external parameter; then the spectrum dynamics is decoupled at each level of $k_\parallel$. In other words, there is no energy transfer in the parallel (to the external field $\mathbf{B_0}$) direction in the $\mathbf{k}$ space: the initial spectrum spreads out only on the transverse wave numbers $k_\perp$. Therefore, at large time such a spectrum becomes flat (pancake like). This two-dimensionalization means that at large times the energy spectrum is supported on a volume of wave numbers such that, for most of them, energy is in modes verifying $k_\perp \gg k_\parallel$.

We shall consider the anisotropic limit of the kinetic equation (15) which reads with our notation as $k_y \gg k_x$. Taking into account the resonant interaction conditions for the parallel wave numbers, we obtain a significant simplification of the interaction coefficient, namely

$$V_{k12} = -k_x, \tag{16}$$

which leads to the following kinetic equation:

$$\partial_t n^{\pm}(k_x,k_y) = \pi k_x^2 \int n^{\mp}(0,k_{1y})[n^{\pm}(k_x,k_{2y}) - n^{\pm}(k_x,k_y)] \times \delta(k_{1y} + k_{2y} - k_y) dk_{1y} dk_{2y}. \tag{17}$$

This equation describes the three-wave interactions of the PAWs in the anisotropic limit of 2D MHD turbulence.

One can immediately see that the energy is conserved separately in the "+" and "−" waves for each k_x

$$\partial_t \int n^{\mp}(k_x,k_y,t)dk_y = 0. \tag{18}$$

As we will see, the factor k_x^2 in the r.h.s. of relation (17) is very important. In the 3D case, there exists a similar term which corresponds to a subleading contribution. We remind the reader that in 3D at the leading order of the perturbation theory, the PAWs are scattered on SAWs and do not interact directly with each other. In 2D, there are no SAWs and, therefore, the r.h.s. of Eq. (17) becomes the leading-order contribution. Additionally, in the 3D problem, at the leading order there is no k_x^2 factor and the substitution $n(k_\perp,k_\parallel,t) = n_\perp(k_\perp,t)n_\parallel(k_\parallel)$ leads to an equation for $n_\perp(k_\perp,t)$ which does not involve $k_\parallel$. In 2D, one can also obtain an equation which does not involve k_x but, for this, one has to introduce an "effective time" variable $\tau = \pi k_x^2 t$:

$$\begin{aligned}\partial_\tau n^{\pm}(k_x,k_y;\tau) &= \int n^{\mp}(0,k_{1y};0)[n^{\pm}(k_x,k_{2y};\tau) - n^{\pm}(k_x,k_y;\tau)] \\ &\quad \times \delta(k_{1y}+k_{2y}-k_y)dk_{1y}dk_{2y}.\end{aligned} \tag{19}$$

Now, we can seek a solution with the following form:

$$n(k_x,k_y,\tau) = \mu(k_x)\eta(k_y,\tau), \tag{20}$$

where $\mu(k_x)$ represents the parallel (nonevolving) component of the energy spectrum and $\eta(k_y,\tau)$ is the perpendicular one. Without loss of generality we can assume $\mu(0) = 1$. Substituting expression (20) into Eq. (20), we find the following equation for η:

$$\begin{aligned}\partial_\tau \eta^{\pm}(k_y,\tau) &= \int \eta^{\mp}(k_{1y},0)[\eta^{\pm}(k_{2y},\tau) - \eta^{\pm}(k_y,\tau)] \\ &\quad \times \delta(k_{1y}+k_{2y}-k_y)dk_{1y}dk_{2y}.\end{aligned} \tag{21}$$

Like in 3D, we have an evolution equation for the perpendicular part of the spectrum which does not explicitly depend on the parallel one. However, the qualitative difference with the 2D case is that there is an implicit dependence on k_x via the effective time variable τ which leads to the fact that in the r.h.s. of Eq. (21) one of the ηs is taken at $\tau = 0$, making this equation linear and—as we will see below—integrable. Another distinct feature arising from such an implicit dependence on k_x via τ is the sharpening of the spectrum at small k_x leading to the breakdown of the wave kinetic description. This effect and its consequences will be investigated in the next section.

III. KOLMOGOROV-ZAKHAROV SPECTRA AND LOCALITY

As a first step in our investigation of the wave kinetic equation (21), we shall derive the exact stationary power-law solutions, $\eta(k_y;\infty)^{\pm} \propto k_y^{\nu^{\pm}}$. As usual in weak turbulence, we use the so-called Zakharov transformation

$$k'_{1y} = \frac{k_y k_{1y}}{k_{2y}}, \quad k'_{2y} = \frac{k_y^2}{k_{2y}}, \tag{22}$$

to obtain the power-law exponent. In practice, we have to split the integral into the r.h.s. of Eq. (21) in two and we perform the Zakharov transformation in the integrand of one them. This manipulation leads eventually to the following conditions on the exponents:

$$\nu^+ + \nu^- = -2. \tag{23}$$

The resulting power-law spectra represent relevant mathematical solutions if and only if the original (before the transformation) integral in the r.h.s. of Eq. (21) converges for these power-law solutions. In Appendix B, we perform such a convergence analysis and demonstrate that this integral never converges. Therefore, the Kolmogorov-Zakharov solutions do not exist in 2D MHD. Note that the balanced-turbulence spectrum (for which $\nu^+ = \nu^- = -1$) has a logarithmic divergence; thus, one could anticipate that such a marginal nonlocality could be "fixed" by a logarithmic corrections. We will demonstrate later that this not possible.

IV. INTEGRATION OF KINETIC EQUATION

A remarkable property of the kinetic equation (20) is its relative simplicity. In this section, we will show that in some physical situations it can be solved analytically. Let us introduce into Eq. (20) some sources and sinks for the waves:

$$\begin{aligned}\partial_\tau n^{\pm}(k_x,k_y,\tau) &= \int n^{\mp}(0,k_{1y},0)[n^{\pm}(k_x,k_{2y},\tau) - n^{\pm}(k_x,k_y,\tau)] \\ &\quad \times \delta(k_{1y}+k_{2y}-k_y)dk_{1y}dk_{2y} \\ &\quad + \mathcal{F}(k_y,k_x) - \sigma_d n(k_x,k_y,\tau).\end{aligned} \tag{24}$$

The function $\mathcal{F}(k_x,k_y)$ may represent a forcing or a dissipation (depending on the choice of the sign before it) and the constant σ_d introduces a uniform friction. In order to use the factorization (20) and eliminate $\mu(k_x)$ in both sides of the forced kinetic equation, we assume the following type of force-dissipation function, $\mathcal{F}(k_y,k_x) = \mathcal{F}_x(k_x)\mathcal{F}_y(k_y)$. Then the parallel component of (20) must be chosen as $\mu(k_x) = \mathcal{F}_x(k_x)$. Finally, we obtain the following forced-dissipated kinetic equation for the perpendicular component of the energy spectrum:

$$\begin{aligned}\partial_\tau \eta^{\pm}(k_y,\tau) &= \int \eta^{\mp}(k_{1y},0)[\eta^{\pm}(k_{2y},\tau) - \eta^{\pm}(k_y,\tau)] \\ &\quad \times \delta(k_{1y}+k_{2y}-k_y)dk_{1y}dk_{2y} \\ &\quad + \mathcal{F}_y(k_y) - \sigma_d \eta(k_y,\tau).\end{aligned} \tag{25}$$

A. Pseudophysical space

A considerable simplification of Eq. (26) may be obtained when performing an inverse Fourier transform on the variable $\eta(k_y,\tau)$:

$$\mathcal{E}^{\pm}(y,\tau) = \int \eta^{\pm}(k_y,\tau)e^{ik_y y}dk_y. \tag{26}$$

We call $\mathcal{E}^{\pm}(y,\tau)$ the pseudophysical-space energy, keeping in mind that what is transformed is the spectrum, not the original wave variable. Then, we obtain the following representation of Eq. (26) in the pseudophysical space:

$$\partial_\tau \mathcal{E}^{\pm}(y,\tau) = \mathcal{E}^{\mp}(y,\tau)\,[\mathcal{E}^{\pm}(y,0) - \mathcal{E}^{\pm}(0,0) - \sigma_d] + \widehat{\mathcal{F}}(y). \tag{27}$$

B. General solutions

Let us consider the balanced-turbulence case for which $\mathcal{E}^{+}(y,\tau) = \mathcal{E}^{-}(y,\tau)$. Then, the general solution of equation (27) can be written as

$$\mathcal{E}(y,\tau) = C(y)e^{[\mathcal{E}(y,0)-\mathcal{E}(0,0)-\sigma_d]\tau} - \frac{\widehat{\mathcal{F}}(y)}{\mathcal{E}(y,0) - \mathcal{E}(0,0) - \sigma_d}, \tag{28}$$

where the first term represents the general solution for the homogeneous equation and the second term is a particular (time independent) solution of the inhomogeneous equation. Function $C(y)$ has to be fixed by the initial condition

$$C(y) = \mathcal{E}(y,0) + \frac{\widehat{\mathcal{F}}(y)}{\mathcal{E}(y,0) - \mathcal{E}(0,0) - \sigma_d}. \tag{29}$$

Now let us consider two particular examples for the forcing and the dissipation. In both cases, we will assume a Gaussian shape forcing, $\widehat{\mathcal{F}}(y) = \sigma_f e^{-k_f^2 y^2/2}$, where the constants σ_f and k_f represent, respectively, the forcing strength and its characteristic wave vector (in k_y space the forcing is also Gaussian, centered at $k_y = 0$ and with a width k_f). In the first example, the dissipation will be represented by uniform friction: we can find the analytical solutions of the kinetic equation in the pseudophysical space. In the second case, we will consider a viscous dissipation for which a qualitative analysis of the stationary regime can be done; in order to illustrate this spectrum evolution a numerical solution will be used.

1. Uniform friction

For the uniform-friction case, we have

$$\mathcal{E}(y,\tau) = C(y)e^{[\mathcal{E}(y,0)-\mathcal{E}(0,0)-\sigma_d]\tau} - \frac{\sigma_f e^{-k_f^2 y^2/2}}{\mathcal{E}(y,0) - \mathcal{E}(0,0) - \sigma_d}. \tag{30}$$

For simplicity, let us use the single-wave initial condition

$$\mathcal{E}(y,0) = 2A\cos(k_0 y), \quad A = \text{const.} > 0, \tag{31}$$

which corresponds to two delta functions in k_y space (at $k_y = \pm k_0$). Then, we can find a function $C(y)$ using Eq. (29) and substitute it into our solution. It yields

$$\mathcal{E}(y,\tau) = \left[2A\cos(k_0 y) + \frac{\sigma_f e^{-k_f^2 y^2/2}}{2A\,[\cos(k_0 y) - 1] - \sigma_d}\right] \times e^{[2A(\cos k_0 y - 1)-\sigma_d]\tau} - \frac{\sigma_f e^{-k_f^2 y^2/2}}{2A\,[\cos(k_0 y) - 1] - \sigma_d}. \tag{32}$$

Let us examine the steady state which corresponds to the limit $t \to \infty$ (and therefore $\tau \to \infty$). The time needed for the formation of the steady state becomes longer as k_x decreases, and there always exist very small k_x values where the spectrum is evolving at any large time. In the limit $\tau \to \infty$, the solution is given by the second term in the r.h.s. of Eq. (33). Far from the initial and the forcing scales, at $k \gg k_0$ and $k \gg k_f$ which correspond to $y \ll 1/k_0$ and $y \ll 1/k_f$, we have $\cos(k_0 y) = 1 - (k_0 y)^2/2 + O((k_0 y)^4)$ and $\exp(-k_f^2 y^2/2) = 1 - O((k_f y)^2)$. Thus, for this range of scales we have the following expression for the steady-state solution in the pseudo-Fourier space:

$$\mathcal{E}(y,\infty) = \frac{\sigma_f}{\sigma_d + \lambda y^2}, \tag{33}$$

where $\lambda = Ak_0^2$. Performing the Fourier transform of $\mathcal{E}(y,\infty)$, we get the steady spectrum

$$\eta(k_y,\infty) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{E}(y,\infty)\,e^{-ik_y y}\,dy. \tag{34}$$

For wave numbers in the inertial range ($k_0, k_f \ll k \ll k_d = \sqrt{\lambda/\sigma_d}$), expression (33) becomes effectively a delta function in the integrand of Eq. (34):

$$\frac{\sigma_f}{\sigma_d + \lambda y^2} \approx \frac{\pi\sigma_f}{\sqrt{\sigma_d\lambda}}\delta(y), \tag{35}$$

and we have

$$\eta(k_y;\,\infty) = \frac{1}{2}\frac{\sigma_f}{\sqrt{\sigma_d\lambda}} = \frac{1}{2}\frac{\sigma_f}{\sqrt{\sigma_d A k_0^2}}. \tag{36}$$

Therefore, we may conclude that in the equilibrium state the energy spectrum of our system (in the inertial range) is flat. Formally, it is a power law with the exponent $\nu = 0$ which is very different from the Kolmogorov-Zakharov exponent ($\nu = -1$) found in Sec. III. Recall that the Kolmogorov-Zakharov spectrum in the balanced case was found to be marginally nonlocal and the common wisdom would suggest that it could be fixed by a logarithmic correction. As we see now this is not true: our exact solution has a completely different exponent and has no logarithmic factor. We also see that our exact solution is nonlocal: it not only depends on the energy flux but also contains information about both the sources and the sinks as well as about the initial conditions.

2. Viscous friction

Let us replace the uniform friction by a viscous dissipation keeping the same one-wave initial condition as before. Equation (27) becomes

$$\frac{\partial\mathcal{E}(y;\tau)}{\partial\tau} = 2A\,[\cos(k_0 y) - 1]\,\mathcal{E}(y;\tau) + \sigma_\nu\frac{\partial^2\mathcal{E}(y;\tau)}{\partial y^2} + \sigma_f e^{-\frac{k_f^2 y^2}{2}}, \tag{37}$$

where σ_ν denotes the viscosity coefficient [we have also used the initial conditions (31)]. To realize this estimation, we need first to get the expression for the steady-state solution. Let us examine the steady state in the uniform friction case for scales less than the forcing and the initial scales which, in terms of the pseudophysical space variables, means $y \ll 1/k_0$ and $y \ll 1/k_f$. Performing the same type of expansion as before in small y, we obtain

$$\sigma_\nu\frac{d^2\mathcal{E}(y;\infty)}{dy^2} - \lambda y^2\mathcal{E}(y;\infty) + \sigma_f = 0. \tag{38}$$

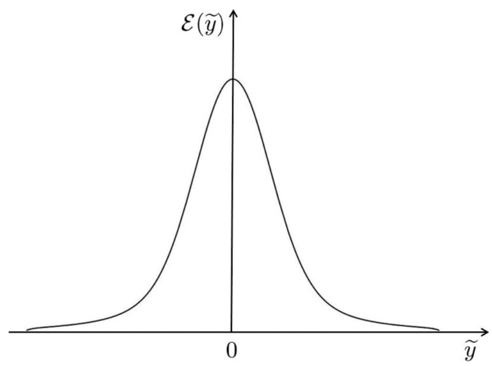

FIG. 1. Stationary solution in pseudophysical space.

By performing the following rescaling

$$\widetilde{y} = y\left(\frac{\lambda}{\sigma_\nu}\right)^{\frac{1}{4}}, \quad \widetilde{\mathcal{E}} = \frac{\sqrt{\lambda\sigma_\nu}}{\sigma_f}\mathcal{E}, \tag{39}$$

we obtain

$$\frac{d^2\widetilde{\mathcal{E}}(\widetilde{y},\infty)}{d\widetilde{y}^2} - \widetilde{y}^2\,\widetilde{\mathcal{E}}(\widetilde{y},\infty) + 1 = 0. \tag{40}$$

The homogeneous part is the equation of the *parabolic cylinder*: its solutions are the parabolic cylinder special functions, whose properties and asymptotics can be found, e.g., in Ref. [55]. Qualitatively, the behavior is similar to the one we found in the previous (friction-dissipation) example: it reaches a maximum at $\widetilde{y} = 0$ and it decays for $\widetilde{y} \to \infty$ (faster than in the previous example). In Fig. 1, we present such a solution in the pseudophysical space obtained using MATLAB in the interval $\widetilde{y} \in [-6.1, 6.1]$. In order to obtain the decaying solution we need to take $\widetilde{\mathcal{E}}(0) = 1.311\,028\,895\,9$ with high accuracy (to eliminate the contribution of the growing parabolic cylinder function).

In the k_y space, we have a flat spectrum in the inertial range

$$\eta(k_y;\infty) = C\sigma_f\left(\frac{\sigma_\nu}{\lambda^3}\right)^{\frac{1}{4}}, \tag{41}$$

for

$$k_0, k_f \ll k \ll k_\nu = (\lambda/\sigma_\nu)^{\frac{1}{4}}, \tag{42}$$

where C is an order-one constant. Once again, we see that the spectrum is nonlocal (i.e., it is dependent on the details of the forcing and the sink parameters rather than just the energy flux), and its exponent is zero (with no logarithmic correction to the Kolmogorov-Zakharov spectrum).

In order to illustrate the dynamical evolution of the spectrum in the viscous dissipation case, we perform numerical simulations of the kinetic equation (26) in the k_y space. In these simulations, we take $\eta^+ = \eta^-$, $\sigma_d = 0$, and $\mathcal{F}_y(k_y) = \mathcal{F}_{\rm force} - \sigma_\nu k_y^2$ with $\sigma_\nu = 10^{-6}$. The result is presented in Fig. 2: the black curve represents the initial spectrum with a Gaussian shape and with a large-scale forcing $\mathcal{F}_{\rm force} = 3 \times 10^{-4}/k$ realized for $k \in [3,9]$. We see that the system converges toward a steady state with a flat spectrum in the inertial range.

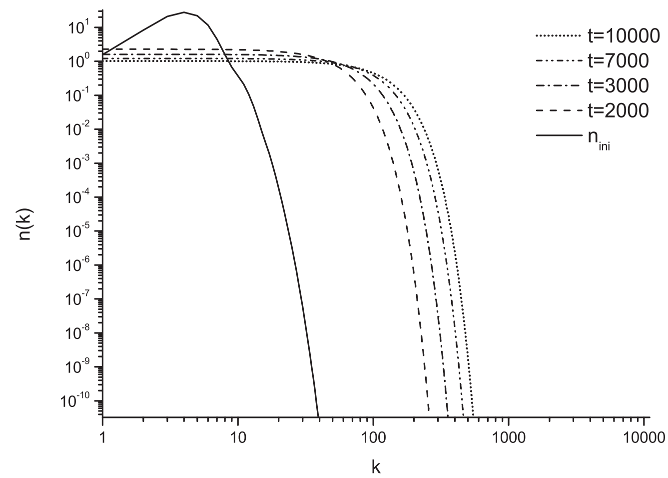

FIG. 2. Time evolution ($\tau \in [0,100\,00]$) of energy spectrum for viscous-dissipation case.

V. BEYOND WEAK TURBULENCE: A QUALITATIVE DESCRIPTION

A. Breakdown and reemergence of weak turbulence

In this section, we provide a qualitative description of the energy spectrum evolution at the late stage of the evolution and beyond the weak-turbulence regime. The wave kinetic equation is valid as long as the nonlinearity remains relatively weak, i.e., the nonlinear time scale is much longer than the linear wave period:

$$\frac{t_{\rm lin}}{t_{\rm nl}} = \frac{b_y k_y}{B_0 k_x} \ll 1. \tag{43}$$

Here, we have used the nonlinear time of hydrodynamics and the equipartition hypothesis between the velocity and the magnetic-field fluctuations. This applicability condition can be rewritten as the following condition for the parallel wave number

$$k_x \gg k_x^* = \frac{b_y k_y}{B_0}. \tag{44}$$

It means that for parallel wave numbers smaller than k_x^* the kinetic equation becomes invalid. The question of the applicability of the wave kinetic equation near $k_\parallel = 0$ has been frequently discussed in the literature in the context of 3D MHD turbulence; in particular, it was speculated in Ref. [36] that a sufficient spectrum smoothness for wave numbers near $k_\parallel = 0$ must be present. In the 2D case, the smoothness of the spectrum near $k_x = 0$ is asymptotically (in time) broken. Indeed, as we have mentioned above, the wave kinetic equation in 2D MHD is formulated in terms of a self-similar "time" variable $\tau = k_x^2 t$. Therefore, the dependence on the parallel wave number is still present in the perpendicular part of the energy spectrum $\eta(k_y,\tau)$ in an implicit way via τ. Such a self-similar dependence on k_x is manifested, at each fixed k_y in shrinking of the original k_x profile along the k_x axis as time grows (see Fig. 3). The spectrum is narrowing and its derivative is growing near small values of k_x. When it is so steep that a significant variation occurs over the range $\sim k_x^*$, the kinetic equation breaks down. The time estimate for such

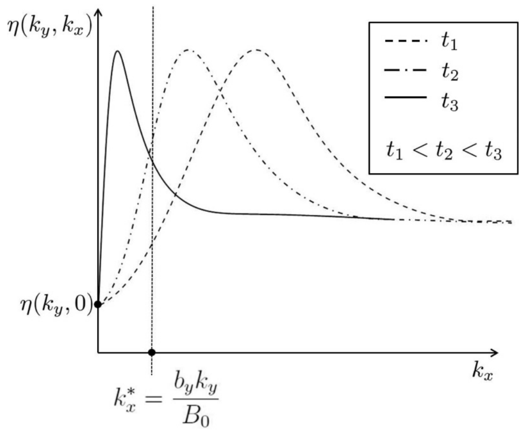

FIG. 3. Spectrum narrowing at large times.

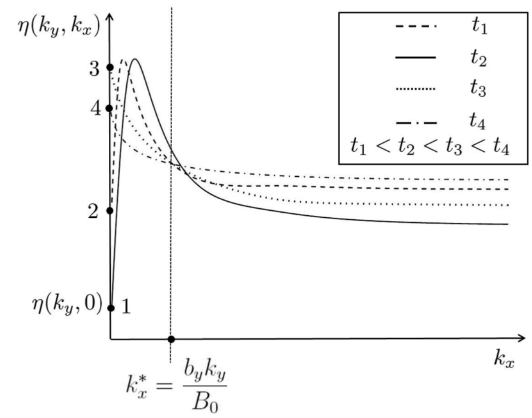

FIG. 4. Gradient smoothing process. Four iterations of the spectrum value stabilizations are presented. At the first stage, the gradient of the spectrum in the vicinity of $k_x = 0$ (point 1) is positive; the initial value of the spectrum $\eta(k_y,0)$ increases and reaches point 2; after crossing the maximum, it moves to point 3, which corresponds to a negative slope for the spectrum. Then, the initial value decreases and arrives at the position 4. This process will continue until the spectrum stabilizes at $\eta(k_y,0) = \eta(k_y,\infty)$.

a breakdown is $t \sim (k_x^*)^{-2}$. Therefore, the weak-turbulence description will break down at the late evolution stage and the wave kinetic equation will no longer apply. However, it is possible to amend this description to take into account the strongly nonlinear effects and develop a qualitative theory of subsequent evolution leading to a steady state. Below, we present a qualitative argument which allows us to obtain such a theory.

First of all, we note that the three-wave interaction is never exactly resonant: it involves all the quasiresonant frequencies within a certain small distance from the exact resonant frequency—the so-called nonlinear resonance broadening $\Gamma \sim t_{nl}^{-1}$. In other words, the delta function in the kinetic equation, $\delta(2k_{1x}) = \delta(\omega_{1k} + \omega_{2k} - \omega)$, should be substituted by a peaked function $f(k_{1x})$ with a small but finite width Γ. For sufficiently smooth spectra the difference from the delta function can be ignored but for sharp and narrow spectra the integrand in the kinetic equation becomes peaked and the delta function broadening becomes important. Its main effect is that of a filter $f(k_x)$ in the k_x variable which acts to smooth any sharp changes over the range $\Delta k \sim k_x^*$. Then, the energy is no longer conserved separately at each fixed k_x.

For large wave numbers $k_x \gg k_x^*$ (where the spectrum remains slowly varying even when it is steep at $k_x \sim k_x^*$), the kinetic equation could be easily amended by replacing $n^{\mp}(0,k_y,0) = \int n^{\mp}(k_{1x},k_{1y},\tau)\delta(k_{1x})dk_{1x}$ with $\langle n^{\mp}\rangle(k_y,\tau) = \int n^{\mp}(k_{1x},k_{1y},\tau) f(k_{1x})dk_{1x}$. For small wave numbers, $k_x \sim k_x^*$, the effect of the resonance broadening is not reduced to such a simple modification of just one function in the integral. It is clear that the spectrum at $k_x = 0$, which was fixed in the wave kinetic approximation, will suffer changes caused by the smoothing in the direction determined by the spectral slope at small k_x: if the gradient is positive (negative) the value will increase (decrease) as illustrated in Fig. 4. The details of the evolution at small wave numbers are not important because the combined action of the self-similar shrinking and smoothing will lead to a rapid wipe out of all the gradients in k_x and the formation of a steady state with η independent of k_x. Correspondingly, the values of η at $k_x = 0$ will adjust themselves to the values at $k_x = \infty$. After this moment, when the rapid dependencies on k_x disappear, the kinetic equation in its usual weak-turbulence form becomes valid once again and can be used for finding the final steady-state spectrum. Since η is now independent of k_x, the steady state could be readily obtained from the formal condition $\eta(k_y,0) = \eta(k_y,\infty)$ which simply means that our solution is independent of τ (it has nothing to do with the initial or final values of the spectrum in time or k_x).

The time evolution can be summarized as follows: At the early stage, $t \ll (k_x^*)^{-2}$, the evolution is described by the three-wave kinetic equation. Then, at the advanced stage, with characteristic times scales $t \sim (k_x^*)^{-2}$, the kinetic equation is broken down by its own evolution. Smoothing of strong gradients in k_x occurs, which results in spectrum stabilization and the reemergence of the kinetic equation description at the large time scales $t \gg (k_x^*)^{-2}$. This kinetic equation describes the spectrum evolution within the steady-state regime. We shall now consider the properties of such a steady state.

B. Spectrum of steady state

Let us analyze the steady state. Based on what was said above, we will look for a τ-independent solution of the pseudo-Fourier space equation (27)

$$\mathcal{E}^2(y) - \mathcal{E}(y)\,[\mathcal{E}(0) + \sigma_d] + \widehat{\mathcal{F}}(y) = 0 \tag{45}$$

formally coinciding with the condition $\mathcal{E}(y,0) = \mathcal{E}(y,\infty) = \mathcal{E}(y)$. Considering this equation at $y = 0$, we have $\mathcal{E}(0) = \widehat{\mathcal{F}}(0)/\sigma_d$. Also, we have $\mathcal{E}(0) = \int \eta(k_y)dk_y > 0$ (because $\eta(k_y) \geqslant 0$). Solving the quadratic equation, we obtain

$$\mathcal{E}(y) = \tfrac{1}{2}[\sigma_d + \widehat{\mathcal{F}}(0)/\sigma_d] \pm \tfrac{1}{2}\{[\sigma_d + \widehat{\mathcal{F}}(0)/\sigma_d]^2 - 4\widehat{\mathcal{F}}(y)\}^{1/2}. \tag{46}$$

To satisfy the condition $\mathcal{E}(0) = \widehat{\mathcal{F}}(0)/\sigma_d$, we must choose "+" if $\widehat{\mathcal{F}}(0) > \sigma_d^2$ and "−" otherwise.

We suppose that the forcing decays at infinity, $\lim_{y\to\infty} \widehat{\mathcal{F}}(y) \to 0$, which is the case, e.g., with the Gaussian forcing. Then, we see that $\lim_{y\to\infty} \mathcal{E}(y) \to 0$ if $\widehat{\mathcal{F}}(0) < \sigma_d^2$ and $\lim_{y\to\infty} \mathcal{E}(y) \to \sigma_d + \widehat{\mathcal{F}}(0)/\sigma_d$ if $\widehat{\mathcal{F}}(0) > \sigma_d^2$. In the second case we have a spectrum with a delta function at $k_y = 0$. Thus, we observe an interesting phenomenon of condensation into the $k_y = 0$ mode in the cases when the forcing prevails over dissipation at small-scale y (corresponding to high k_y). In the first case, $\mathcal{E}(y)$ is a monotonically decreasing (to 0) function of y whereas in the second case it is monotonically increasing (to an asymptotic constant). The first situation is physically more relevant because in most cases of interest, dissipation dominates over forcing at small scales. In this case, $\mathcal{E}(y)$ behaves qualitatively as in the two examples considered in Sec. IV. Namely, if we take the same Gaussian forcing as in these two examples, we find $\mathcal{E}(y)$ with a maximum at $y = 0$ is smooth everywhere (including $y = 0$) and is rapidly decaying to zero for $y \to \infty$. However, there is an important difference from the previous examples in that now the characteristic width of the function $\mathcal{E}(y)$, and respectively the width of the spectrum in the k_y variable, is of the same order as the width of the forcing function. Therefore, there is no inertial range in the final steady state considered here. This is an even stronger case of a nonlocal interaction than in the two examples considered before. Both the forcing and the dissipation parameters enter in the final answer but not the parameters of the initial condition: the steady state beyond the weak turbulence has already forgotten all the initial data.

VI. SUMMARY

We have shown that the three-wave interactions for PAWs in 2D MHD are nonempty and it is possible to obtain a three-wave kinetic equation within the weak turbulence formalism. These interactions take place at second order in the anisotropy parameter.

We have found the Kolmogorov-Zakharov power-law spectra for PAWs in 2D MHD and showed that they are not realizable due to the divergence of the collision integrals of the kinetic equation. In the balanced case this divergence is marginal. This is an indirect indication that the 2D PAW turbulence is nonlocal: it is dominated by the interaction of waves with very different wavelengths. Our full analytical solution of the kinetic equation confirms such a nonlocality. It also dispels the myth that all marginally nonlocal spectra can be "fixed" by a logarithmic correction.

The crucial technique for our analysis is passing to pseudophysical space via Fourier transformation of the kinetic equation and by using a self-similar effective time variable. This has allowed us to dramatically simplify the kinetic equation, solve it analytically in some important cases, and fully analyze it in the other important cases. The two main examples we analyze have a Gaussian-shaped forcing localized at small wave numbers and a dissipation represented by either a uniform friction or a viscosity. The first case is solvable analytically and the second one is shown to possess a similar behavior; namely, the spectrum evolves independently at each k_x and it tends to a flat steady-state spectrum in the inertial range (which is not a logarithmic correction to the Kolmogorov-Zakharov spectrum).

At each fixed k_y, the spectrum develops sharp gradients at small k_x which eventually leads to the breakdown of the weak-turbulence description. We present a qualitative argument about what follows after this moment. We argue that the effect of strong turbulence is to smooth the sharp gradients via the nonlinear resonance broadening effect. This leads to a steady state with no gradient in k_x for which the weak-turbulence kinetic equation formally works once again, and we present an analytical solution for such a steady state.

As an example of a practical implication of our results, let us consider an estimate for the two-particle diffusion caused by weak 2D MHD turbulence presented in Ref. [56] [see their formula (68)]. First of all, the authors of that paper correctly stated that the characteristic time in the 2D case will be longer than the one in 3D by a factor $k_\perp^2/k_\parallel^2$. This corresponds to the presence of the factor k_x^2 in our kinetic equation (17). It was also correctly assumed that the wave-wave interaction is a three-wave process. However, their estimate (68) assumes local-scale interactions and the presence of an inertial range with constant energy flux, where as we have established in the present paper that the interaction is nonlocal, and that there is no inertial range. In fact, our conclusion in the end of Sec. V is that the steady-state spectrum does not extend far beyond the forcing scales, i.e., the smaller scales are suppressed in the nonlocal 2D MHD turbulence. Assuming that only scales much smaller than the forcing scales contribute to the turbulent diffusion, we are led to conclude that there can be no turbulent diffusion in weak 2D MHD turbulence. In other words, we could also say that the magnetic reconnection process cannot be facilitated.

Overall, one should derive from our work a warning that 2D and 3D MHD turbulence are dramatically different, and one should be careful when extrapolating the 2D results, e.g., numerical ones, to the 3D case. Indeed, in contrast to 2D, in 3D there is no gradient sharpening at small parallel wave numbers, and the Kolmogorov-Zakharov spectrum is a local and well-behaved solution.

ACKNOWLEDGMENT

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APPENDIX A: DERIVATION OF WAVE KINETIC EQUATION

In Sec. II A, we wrote the 2D MHD equations in the interaction representation (13) which is a starting point for the derivation of the wave kinetic equation. Let us define the wave spectrum as

$$n_k^\pm = L^2 \epsilon^2 \langle |c_k^\pm|^2 \rangle,$$

where the average is taken over the random initial conditions and L^2 is the area of the periodic box. With this normalization

n_k tends to a finite limit $\sim\epsilon^2$ as $L \to \infty$ provided that the wave density is finite and uniform in the 2D physical space.

The next step consists of making use of the time-scale separation. We are introducing an intermediate time scale T which should be much smaller than the typical nonlinear time, $t_{\rm nl} = 2\pi/(\epsilon^2\omega)$ and much greater than the linear wave period, $t_{\rm lin} = 2\pi/\omega$. Taking $T = 2\pi/(\epsilon\omega)$ will satisfy these conditions, $t_{\rm lin} \ll T \ll t_{\rm nl}$. Then, we are looking for solutions at time $t = T$ in the form of series in small ϵ

$$c_k^{\pm}(T) = c_k^{(\pm,0)} + \epsilon c_k^{(\pm,1)} + \epsilon^2 c_k^{(\pm,2)} + \cdots, \tag{A1}$$

where we suppose that the lowest-order amplitudes $c_k^{\pm,(0)} = c_k^{\pm}(0)$ correspond to the linear regime. For the spectrum we have

$$\frac{[n_k^{\pm}(T) - n_k^{\pm}(0)]}{(\epsilon^4 L^2)} = \left\langle \left|c_k^{(\pm,1)}\right|^2 \right\rangle + \left\langle c_k^{(\pm,0)*} c_k^{(\pm,2)} \right\rangle + \left\langle c_k^{(\pm,0)} c_k^{(\pm,2)*} \right\rangle. \tag{A2}$$

After substituting expansion (A1) into Eq. (13) at first order, we obtain

$$c_k^{(\pm,1)}(T) = \sum_{1,2} V_{12k} \Delta_T\,(\pm 2k_{1x})\, c_1^{(\mp,0)} c_2^{(\pm,0)} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}), \tag{A3}$$

where

$$\Delta_T\,(\pm 2k_{1x}) = \int_0^T e^{\pm 2ik_{1x}t} dt = \frac{e^{\pm i2k_{1x}T} - 1}{\pm 2ik_{1x}}. \tag{A4}$$

For the second order we can write

$$\begin{aligned} c_k^{(\pm,2)} = \sum_{1,2,3,4} & V_{12k} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \\ & \times \left[V_{342} \delta(\mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_2) c_1^{(\mp,0)} c_3^{(\mp,0)} c_4^{(\pm,0)} \right. \\ & \times E(\pm 2k_{1x}, \pm 2k_{3x}) + V_{341}\, \delta(\mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_1) \\ & \left. \times c_2^{(\pm,0)} c_3^{(\pm,0)} c_4^{(\mp,0)} E(\pm 2k_{1x}, \mp 2k_{3x}) \right], \end{aligned} \tag{A5}$$

with

$$E(x,y) = \int_0^T e^{ixt} \Delta_t\,(y)\, dt. \tag{A6}$$

Next, we are going to assume that the initial amplitudes $c_k^{(\pm,0)}$ are Gaussian random variables which are statistically independent at each $\mathbf{k}$ and use Wick's rule:

$$\begin{aligned} & \left\langle c_1^{(\pm,0)} c_2^{(\pm,0)} c_3^{(\mp,0)} c_4^{(\mp,0)} \right\rangle \\ & \quad = \delta(\mathbf{k}_1 + \mathbf{k}_2) \delta(\mathbf{k}_3 + \mathbf{k}_4) \left\langle \left|c_1^{(\pm,0)}\right|^2 \right\rangle \left\langle \left|c_3^{(\mp,0)}\right|^2 \right\rangle. \end{aligned} \tag{A7}$$

We also remember that because the physical-space amplitudes are real functions, we have $(c^{(\pm,0)}(\mathbf{k}))^* = c^{(\pm,0)}(-\mathbf{k})$. We obtain

$$\begin{aligned} \left\langle \left|c_k^{(\pm,1)}\right|^2 \right\rangle &= \sum_{1,2,3,4} V_{12k} V_{34k} \Delta_T(\pm 2k_{1x}) \Delta_T^*(\pm 2k_{3x}) \\ & \quad \times \left\langle c_1^{(\mp,0)} c_2^{(\pm,0)} \left(c_3^{(\mp,0)}\right)^* \left(c_4^{(\pm,0)}\right)^* \right\rangle \\ & \quad \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \delta(\mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}) \\ &= \frac{1}{L^4 \epsilon^4} \sum_{12} |V_{12k}|^2 |\Delta_T(\pm 2k_{1x})|^2 n_1^{\mp} n_2^{\pm} \\ & \quad \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}), \end{aligned} \tag{A8}$$

and

$$\begin{aligned} \left\langle \left(c_k^{(\pm,0)}\right)^* c_k^{(\pm,2)} \right\rangle &= \sum_{1234} V_{12k} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \\ & \quad \times \left[V_{342} \delta(\mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_2) E(\pm 2k_{1x}, \pm 2k_{3x}) \right. \\ & \quad \left. \times \left\langle \left(c_k^{(\pm,0)}\right)^* c_1^{\mp,0} c_3^{\mp,0} c_4^{\pm,0} \right\rangle \right] \\ &= -\frac{1}{L^4 \epsilon^4} \sum_{12} |V_{12k}|^2 E(\pm 2k_{1x}, \mp 2k_{1x}) \\ & \quad \times n_1^{\mp} n_k^{\pm} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}), \end{aligned} \tag{A9}$$

where we have used abbreviations $n_1^{\mp} = n^{\mp}(\mathbf{k}_1,t)$, $n_2^{\mp} = n^{\mp}(\mathbf{k}_2,t)$. Next we note that

$$\mathrm{Im} E(\pm 2k_{1x}, \mp 2k_{1x}) = -\mathrm{Im} E(\mp 2k_{1x}, \pm 2k_{1x}), \tag{A10}$$

and

$$\begin{aligned} \mathrm{Re} E(\pm 2k_{1x}, \mp 2k_{1x}) &= \mathrm{Re} E(\mp 2k_{1x}, \pm 2k_{1x}) \\ &= \frac{\sin^2(k_{1x}T)}{2\, k_{1x}^2}. \end{aligned} \tag{A11}$$

Let substitute expressions (A8) and (A9) into Eq. (A2):

$$\begin{aligned} n_k^{\pm}(T) - n_k^{\pm}(0) &= \frac{1}{L^2} \sum_{1,2} |V_{12k}|^2 n_1^{\mp} (n_2^{\pm} - n_k^{\pm}) \\ & \quad \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \frac{\sin^2(k_{1x}T)}{k_{1x}^2}, \end{aligned} \tag{A12}$$

where we have used that $|\Delta_T(2k_{1x})|^2 = \sin^2(k_{1x}T)/k_{1x}^2$. Now, we take the infinite-box limit, $L \to \infty$, and pass to the continuous description in the $\mathbf{k}$ space using the rule

$$\frac{1}{L^2} \sum_{1,2} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \to \int \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d\mathbf{k}_1 d\mathbf{k}_2,$$

where δ in the integrand means the delta function (recall that it is Kronecker delta in the sum).

At the next stage of the wave kinetic procedure we need to use the weakness of the nonlinearity in our system by taking the limit $\epsilon \to 0$, which is equivalent to $T \to \infty$. For the r.h.s. of Eq. (A12), we obtain

$$\lim_{T\to\infty} \frac{\sin^2(k_{1x}T)}{k_{1x}^2} = \pi T \delta\,(k_{1x}). \tag{A13}$$

Then, after multiplying both parts of Eq. (A12) by $1/T$, its left-hand side (l.h.s.) becomes

$$\frac{n(T) - n(0)}{T} \to \partial_t n(T), \tag{A14}$$

where we take into account that T is much less than the nonlinear time at which the spectrum evolves. After these steps, we can finally write down the kinetic equation

$$\partial_t n_k^{\pm} = \pi \int V_{12k}^2 n_1^{\mp}[n_2^{\pm} - n_k^{\pm}]\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \times \delta(2\, k_{1x}) d\mathbf{k}_1 d\mathbf{k}_2. \quad \text{(A15)}$$

APPENDIX B: LOCALITY OF KOLMOGOROV-ZAKHAROV SPECTRA

In order to explore the realizability of the Kolmogorov-Zakharov spectra, $\eta(k_y;\infty)^{\pm} \propto k_y^{\nu^{\pm}}$, we need to proceed with a convergence study of the collisional integrals

$$\int_{-\infty}^{\infty} \delta(k_{1y} + k_{2y} - k_y)|k_{1y}|^{\alpha_{\pm}}(|k_{2y}|^{\alpha_{\mp}} - |k_y|^{\alpha_{\mp}}) dk_{1y} dk_{2y}.$$

Let us consider the first one, choosing α_+ at the exponent of $|k_{1y}|$. There are three singular points:

(1) $k_{1y}, k_{2y} \to \infty$,
(2) $k_{1y} \to 0, k_{2y} \to k_y$,
(3) $k_{2y} \to 0, k_{1y} \to k_y$.

At the first point, we should use the fact that the integral $\int_1^{\infty} |x|^{\nu} dx$ converges when $\nu < -1$. After substituting $k_{2y} = k_y - k_{1y}$, we have

$$\int_1^{\infty} |k_{1y}|^{\alpha_+}(|k_y - k_{1y}|^{\alpha_-} - |k_y|^{\alpha_-}) dk_{1y}.$$

Then two cases are possible:

(i) When $\alpha_- > 0$, the main contribution is made by

$$\int_1^{\infty} |k_{1y}|^{\alpha_+ + \alpha_-} dk_{1y},$$

which is convergent for $\alpha_+ + \alpha_- < -1$.

(ii) When $\alpha_- < 0$ the expression for the main contribution is made by

$$\int_1^{\infty} |k_{1y}|^{\alpha_+}|k_y|^{\alpha_-} dk_{1y},$$

and is convergent for $\alpha_+ < -1$.

At the second singular point, after integration over k_{2y} using the delta function in Eq. (B1), we have

$$\int_0^{\epsilon} |k_{1y}|^{\alpha_+}(|k_y - k_{1y}|^{\alpha_-} - |k_y|^{\alpha_-}) dk_{1y} \sim \int_0^{\epsilon} k_{1y}^{\alpha_+ + 1} dk_{1y},$$

and we get the convergence condition $\alpha_+ > -2$. To get this condition, we performed the series expansion $|k_y - k_{1y}|^{\alpha_-} = |k_y|^{\alpha_-}(1 + \alpha_- k_{1y}/k_y) + \cdots$, and we have used the fact that the integral $\int_0^{\epsilon} x^{\nu} dx$ is convergent for $\nu > -1$. To obtain the convergence condition for the last singular point, we integrate over k_{1y} using the delta function

$$\int_0^{\epsilon} |k_y - k_{2y}|^{\alpha_+}(|k_{2y}|^{\alpha_-} - |k_y|^{\alpha_-}) dk_{2y} \sim \int_0^{\epsilon} |k_y|^{\alpha_+}|k_{2y}|^{\alpha_-} dk_{2y} - \int_0^{\epsilon} |k_y|^{\alpha_+ + \alpha_-} dk_{2y}\,.$$

The second integral is always convergent, and the first one is convergent for $\alpha_- > -1$. Finally, the convergence region for the first collisional integral (B1) in the space of indices is

$$\{\{(\alpha_+ + \alpha_- < -1) \cap (\alpha_- > 0)\} \cup \{(\alpha_+ < -1) \cap (\alpha_- < -0)\}\} \cap (\alpha_+ > -2) \cap (\alpha_- > -1). \quad \text{(B1)}$$

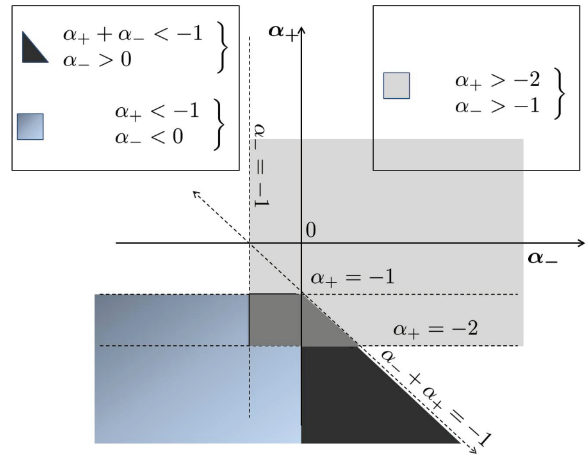

FIG. 5. (Color online) Locality study for Kolmogorov-Zakharov spectrum.

It is represented by the gray trapezoid in Fig. 5. To find the convergence zone for the second integral of Eq. (B1) (with α_- in the exponent of k_{1y}) we just take the reflection of the convergence zone for the first integral with respect to the line $\alpha_- = \alpha_+$. Finally, to get the convergence conditions for both collisional integrals one should take the intersection of both zones. As we can see in Fig. 5, such an intersection produces a zero set. There are no power-law exponents α^- and α^+ for which both collision integrals would be convergent and there is a single point which corresponds to marginal (logarithmic) divergence, $\alpha^- = \alpha^+ = -1$. This point corresponds to the Kolmogorov-Zakharov spectrum in the balanced-turbulence case. Common wisdom [57] is that such marginally nonlocal spectra can be fixed by a logarithmic correction. However, in the main text of this paper we show that this is not the case for this problem.

[1] E. Marsch and C. Tu, Ann. Geophys. **12**, 1127 (1994).
[2] E. Priest, *Solar Magnetohydrodynamics* (Reidel, Dordrecht, 1982).
[3] C. Ng, A. Bhattacharjee, K. Germashewski, and S. Galtier, Phys. Plasmas **10**, 1954 (2003).
[4] E. Zweibel and C. Heiles, Nature (London) **385**, 131 (1997).
[5] J. Goedbloed, R. Keppens, and S. Poedts, *Advanced Magnetohydrodynamics* (Oxford University Press, 2010).
[6] P. Iroshnikov, Soviet Astron. **7**, 566 (1963).
[7] R. Kraichnan, Phys. Fluids **8**, 1385 (1965).
[8] A. Kolmogorov, Dokl. Akad. Nauk **30**, 229 (1941).
[9] D. Biskamp, *Magnetohydrodynamic Turbulence* (Cambridge University Press, Cambridge, 2003).

[10] D. Montgomery and L. Turner, Phys. Fluids **24**, 825 (1981).
[11] J. Shebalin, W. Matthaeus, and D. Montgomery, J. Plasma Phys. **29**, 525 (1983).
[12] S. Oughton, E. Priest, and W. Matthaeus, J. Fluid Mech. **280**, 95 (1994).
[13] C. Ng and A. Bhattacharjee, Phys. Plasmas **4**, 605 (1997).
[14] B. Bigot, S. Galtier, and H. Politano, Phys. Rev. Lett. **100**, 074502 (2008).
[15] M. Verma, Phys. Rep. **401**, 229 (2004).
[16] B. D. G. Chandran, Phys. Rev. Lett. **95**, 265004 (2005).
[17] S. Boldyrev, Phys. Rev. Lett. **96**, 115002 (2006).
[18] B. Bigot and S. Galtier, Phys. Rev. E **83**, 026405 (2011).
[19] H. Politano, A. Pouquet, and V. Carbone, Europhys. Lett. **43**, 516 (1998).
[20] T. Gomez, H. Politano, and A. Pouquet, Phys. Fluids **11**, 2298 (1999).
[21] Nils Erland L. Haugen, A. Brandenburg, and W. Dobler, Phys. Rev. E **70**, 016308 (2004).
[22] D. Biskamp and W. Müller, Phys. Plasmas **7**, 4889 (2000).
[23] J. Merrifield, C. Chapman, and R. Dendy, Phys. Plasmas **14**, 012301 (2007).
[24] J. Merrifield, T. Arber, C. Chapman, and R. Dendy, Phys. Plasmas **13**, 012305 (2006).
[25] B. Bigot, S. Galtier, and H. Politano, Phys. Rev. E **78**, 066301 (2008).
[26] D. Dritchel and S. Tobias, J. Fluid Mech. **703**, 85 (2012).
[27] D. Biskamp and E. Schwarz, Phys. Plasmas **8**, 3282 (2001).
[28] S. Galtier, H. Politano, and A. Pouquet, Phys. Rev. Lett. **79**, 2807 (1997).
[29] S. Galtier, E. Zienicke, H. Politano, and A. Pouquet, J. Plasma Phys. **61**, 507 (1999).
[30] P. Sweet, in *Electromagnetic Phenomena in Cosmical Physics*, IAU Symposium, Vol. 6, edited by B. Lehnert (Cambridge University Press, 1958) pp. 123–134.
[31] E. Parker, J. Geophys. Res. **62**, 509 (1957).
[32] A. Lazarian and E. Vishniac, Astrophys. J. **517**, 700 (1999).
[33] S. Servidio, W. H. Matthaeus, M. A. Shay, P. A. Cassak, and P. Dmitruk, Phys. Rev. Lett. **102**, 115003 (2009).
[34] G. Einaudi, M. Velli, H. Politano, and A. Pouquet, Astrophys. J. **457**, L113 (1996).
[35] D. Biskamp, *Nonlinear Magnetohydrodynamics* (Oxford University Press, 1993).
[36] S. Galtier, S. Nazarenko, A. Newell, and A. Pouquet, J. Plasma Phys. **63**, 447 (2000).
[37] A. Achterberg, Astron. Astrophys. **76**, 276 (1979).
[38] A. Akhiezer, I. Akhiezer, R. Polovin, A. Sitenko, and K. Stepanov, *Plasma Electrodynamics II: Non-Linear Theory and Fluctuations* (Pergamon Press, Oxford, 1975).
[39] I. McIvor, Mon. Not. R. Astron. Soc. **178**, 85 (1974).
[40] V. Zakharov, Izv. Vyssh. Uchebn. Radiophyz. **17**, 431 (1974).
[41] V. Zakharov, *Kolmogorov Spectra in Weak Turbulence Problems*, in Basic Plasma Physics II, edited by A. A. Galeev and R. Sudan (North-Holland, Amsterdam, 1984), p. 48.
[42] V. Zakharov, V. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence 1. Wave turbulence* (Springer, Berlin, 1992).
[43] E. Kuznetsov, Sov. Phys. JETP **35**, 310 (1972).
[44] E. Kuznetsov, Inst. Nucl. Phys. Preprint **81**, 73 (1972).
[45] V. Tsytovich, *Nonlinear Effects in Plasma* (Plenum Press, New York, 1970).
[46] R. Sagdeev and A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969).
[47] A. Vedenov, *Theory of Turbulent Plasma* (Iliffe Books, London/American Elsevier, New York, 1968).
[48] S. Galtier, S. Nazarenko, and A. Newell, Nonlinear Proc. Geophys. **8**, 1 (2001).
[49] S. Galtier and S. Nazarenko, J. Turbul. **9**, 1 (2008).
[50] S. Galtier, Nonlinear Proc. Geophys. **16**, 83 (2009).
[51] S. Shridar and P. Goldreich, Astrophys. J. **432**, 612 (1994).
[52] D. Montgomery and W. Matthaeus, Astrophys. J. **447**, 706 (1995).
[53] S. Galtier, S. Nazarenko, A. Newell, and A. Pouquet, Astrophys. J. **564**, L49 (2002).
[54] S. Nazarenko, *Wave Turbulence*, in Lecture Notes in Physics, Vol. 825 (Springer, Berlin, 2011).
[55] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
[56] G. Eyink, A. Lazarian, and E. Vishniac, Astrophys. J. **743**, 51 (2011).
[57] R. Kraichnan, J. Fluid Mech. **47**, 525 (1971).