Aging generates regular motions in weakly chaotic systems

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Using intermittent maps with infinite invariant measures, we investigate the universality of time-averaged observables under aging conditions. According to Aaronson's Darling-Kac theorem, in non-aged dynamical systems with infinite invariant measures, the distribution of the normalized time averages of integrable functions converges to the Mittag-Leffler distribution. This well known theorem holds when the start of observations coincides with the start of the dynamical processes. Introducing a concept of the aging limit where the aging time t_a and the total measurement time t goes to infinity while the aging ratio t_a/t is kept constant, we obtain a novel distributional limit theorem of time-averaged observables integrable with respect to the infinite invariant density. Applying the theorem to the Lyapunov exponent in intermittent maps, we find that regular motions and a weakly chaotic behavior coexist in the aging limit. This mixed type of dynamics is controlled by the aging ratio and hence is very different from the usual scenario of regular and chaotic motions in Hamiltonian systems. The probability of finding regular motions in non-aged processes is zero, while in the aging regime it is finite and it increases when system ages.

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I. INTRODUCTION

Aging is a concept describing slow relaxation phenomena in spin glasses [1], interface fluctuations in liquid-crystal turbulence [2], blinking quantum dots [3,4], and transports in cells [5]. In nature dynamical processes may start at time $-t_a$ long before the actual measurement of the process begins at $t = 0$. In aging systems, statistical quantities measured in the time interval $[0,t]$ are crucially affected by the aging time t_a [6,7]. For example, the distribution of time-averaged mean square displacement was considered in [7] using stochastic tools relevant to diffusion of biomolecules in the cell. In contrast, for stationary and ergodic processes, statistical properties of time-averaged observables do not depend on the aging time t_a if the measurement time is large enough ($t \to \infty$). Here, we introduce the aging limit where the aging time t_a and the total measurement time t goes to infinity while the ratio $T_a \equiv t_a/t$ (the aging ratio) is kept constant. We proceed to show universal statistical properties of time-averaged observables in a class of dynamical systems, extending infinite ergodic theory [8] to the aging regime.

Exponential separation of nearby trajectories, i.e., chaos, is a feature found in many dynamical systems. In many generic Hamiltonian systems, the phase space is either chaotic or regular. A closer look at dynamics many times reveals a mixed phase space. This means that parts of the trajectories are rather regular, for example Kolmogorov-Arnold-Moser tori in phase space, while others are chaotic [9]. Determining which type of motion depends ultimately on the choice of initial conditions. Here we investigate a completely new mechanism for dual structure of dynamics. We show how, for a class of dynamical systems possessing an infinite invariant measure, the dynamics are generically split into regular and weakly chaotic motions. This splitting is controlled by the age of the process, and hence is completely different from usual scenarios.

In weakly chaotic systems the separation of nearby trajectories is subexponential [10]. In many cases such systems have an infinite invariant density [11], i.e., a non-normalizable density (see details below). It is known that some dynamical systems with infinite invariant measures show an aging behavior [12]. Here we consider a dynamical system whose evolution started at time $-t_a$ with $t_a > 0$, while an observation of the dynamical process starts at time $t = 0$. Within the observation time $(0,t)$ an observer evaluates time averages of an observation function and we are interested in the ergodic properties of these time averages. From the viewpoint of dynamical systems, aging means that a density at $t = 0$ strongly depends on the aging time t_a, even when the latter is long.

Let $\rho_t(x;t_a)$ be the density at time t for a process whose aging time is t_a. If a dynamical system has an invariant *probability* measure, a smooth initial density $\rho_0(x;t_a)$ converges to the invariant density as $t_a \to \infty$, indicating that the aging ratio does not affect statistical properties of time-average observables. In contrast, in an infinite measure system, at the start of the measurement $t = 0$ the density $\rho_0(x;t_a)$ does not converge to an invariant measure which is t_a independent. Namely, $\rho_0(x;t_a)$ does not converge to an invariant density absolutely continuous with respect to the Lebesgue measure as $t_a \to \infty$.

In the mathematical literature, a transformation T is called ergodic if $\mu(A) = 0$ or $\mu(A^c) = 0$ for all invariant sets $A = T^{-1}A$, where A^c is the complement of A and μ is an invariant measure, which satisfies $\mu(B) = \mu(T^{-1}B)$ for every measurable set B in the dynamical system. This definition is applicable even when the invariant measure μ cannot be normalized (infinite invariant measure). Birkhoff's ergodic theorem states that the time averages of integrable functions converge to constant values (ensemble averages) for almost all initial conditions if the dynamical system has an absolutely continuous invariant probability measure [13]. If an invariant measure cannot be normalized, $\sum_{k=0}^{t-1} f(x_k)/t^{\alpha}$ does not converge to a nontrivial constant ($\neq 0$ and $\pm\infty$) where

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$0 < \alpha < 1$. Instead it remains random (see details below). Here $f(x)$ is an integrable function with respect to an invariant measure, and $\{x_k\}_{k=0,\dots}$ is a trajectory. Aaronson's Darling-Kac (ADK) theorem [14,15] gives the distribution of such sums along a path. In particular (for $t_a = 0$), for integrable functions there exists a sequence $|a_t| \propto t^\alpha$ such that

$$\Pr\left(\frac{1}{a_t}\sum_{k=0}^{t-1} f(x_k) < x\right) \to \int_0^x d_\alpha(\xi)d\xi \quad \text{as } n\to\infty, \qquad (1)$$

where $d_\alpha(x)$ is the Mittag-Leffler density of order α [8]. In other words, $\sum_{k=0}^{t-1} f(x_k)/t^\alpha$ depends strongly on an initial position but its distribution converges to a universal distribution for almost all smooth initial densities. Here we investigate ergodic theory under the conditions of aging. We show large differences from the ADK theorem in the aging regime where both t_a and t are large, and propose a limit theorem describing distributions of $\sum_{k=0}^{t} f(x_k)/t^\alpha$.

Recently, it was shown that infinite ergodic theory plays an important role in elucidating an intrinsic randomness of time-averaged observables in dichotomous processes modeling blinking quantum dots [16] and anomalous diffusion [17,18]. Since aging appears in infinite measure dynamical systems, it is an interesting and important problem to clarify whether distributional behaviors of time-averaged observables are affected by the aging ratio. Here, we provide the evidence that the aging limit, where $t \to \infty$ and $t_a \to \infty$ but their ratio $T_a \equiv t_a/t$ remaining finite, plays a crucial role in characterizing behaviors of time-averaged observables. In particular, we show that the distribution of sums of integrable functions is determined by the aging ratio T_a for weakly chaotic systems with an indifferent fixed point.

II. AGING DYNAMICAL SYSTEMS

We consider maps $T:[0,1] \to [0,1]$ which satisfy the following conditions for some $\gamma_1 \in (0,1)$: (i) the restrictions $T:(0,\gamma_1) \to (0,1)$ and $T:(\gamma_1,1) \to (0,1)$ are C^2 and onto, and have C^2-extensions to the respective closed intervals; (ii) $T'(x) > 1$ on $(0,\gamma_1] \cup [\gamma_1,1]$; $T'(0) = 1$; and (iii) $T(x) - x$ is regularly varying at zero with index $1 + 1/\alpha$, $T(x) - x \sim a_0 x^{1+1/\alpha}$ ($\alpha > 0$). We note that the map has only one indifferent fixed point. These maps are related to number theory [19], intermittency [20–22], and anomalous diffusions [23–26]. One of the best known examples is the Pomeau-Manneville map [20,21]:

$$x_{t+1} = T(x_t) = x_t + x_t^{1+1/\alpha} \mod 1. \qquad (2)$$

In what follows, we use the map (2) for numerical simulations. This famous map has an indifferent fixed point at $x = 0$ and hence a trajectory is trapped in its vicinity, escapes slowly, and then is reinjected back. According to Thaler's estimation [27], an invariant density $\tilde{\rho}(x)$ is given by $\tilde{\rho}(x) \sim \tilde{h}(x)x^{-1/\alpha}$ for $x \in (0,1]$, where $\tilde{h}(x)$ is a positive bounded continuous function on $[0,1]$. Notice that when $\alpha < 1$ this invariant density cannot be normalized, due to its divergence close to $x = 0$. Since $\tilde{\rho}(x)$ is non-normalizable, it is determined up to a multiplicative constant, whereas we use Eq. (27) as a specific invariant density. In what follows, we consider an infinite measure system ($0 < \alpha < 1$). In aging systems, the dynamics start at time $-t_a$ before the measurement is started at $t = 0$, where t_a is called the aging time (see Fig. 1). For our numerical simulation we assume that initial points at time $-t_a$ are uniformly distributed on $[0,1]$. If the initial density is absolutely continuous with respect to the Lebesgue measure, the choice of the initial density does not affect our results. This is because all scaled densities at time $-t_a$ converge to an invariant density at the start of the measurement $t = 0$, provided that densities are Riemann-integrable functions and $t_a \to \infty$ [28,29] (see also Appendix A). Notice that the density at $t = 0$, $\rho_0(x;t_a)$, is given by the density at time t_a starting with the same initial density (e.g., uniform density) at $t = 0$, namely $\rho_{t_a}(x;0) = \rho_0(x,t_a)$.

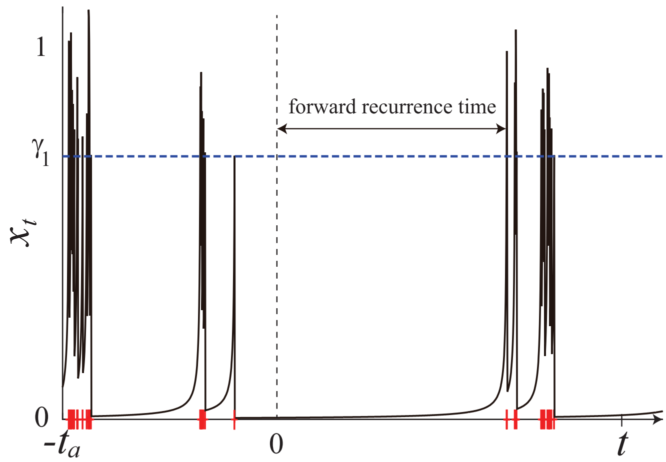

FIG. 1. (Color online) Schematic view of an aging process in the map (2) ($\alpha = 0.8$). A trajectory is given by a solid line while renewals, $x_t \geqslant \gamma_1$, i.e., $\sigma(x_t) = 1$, are depicted as crosses on the time axis. $t = 0$ is the time at which a measurement starts.

III. RENEWAL PROCESSES

Renewal processes are point processes where interevent times of points are independent and identically distributed (IID) random variables [30]. Therefore, renewal processes are characterized by the distribution of interevent times of renewals. Let us consider the following observation function

$$\sigma(x) = \begin{cases} 0 & (x < \gamma_1), \\ 1 & (x \geqslant \gamma_1), \end{cases} \qquad (3)$$

where $\gamma_1 < 1$ attains $T(\gamma_1) = 1$. We call the state $\sigma = 0$ the laminar phase while $\sigma = 1$ is the chaotic phase. In the chaotic phase, trajectories x_t show usual chaotic behavior because of the condition (ii). Moreover, the jump transformation $T^{n(x)}$ on $x \in [\gamma_1,1]$, which is a transformation restricted to $[\gamma_1,1]$, has an absolutely continuous invariant *probability* measure whereas the original map T does not, where $n(x) \equiv \min\{k \geqslant 1 : T^k(x) \in [\gamma_1,1]\}$ [27]. With the aid of these chaotic properties in the jump transformation, trajectories $\sigma_t = \sigma(x_t)$ can be regarded as a renewal process because the interevent times between the events $\sigma(x_t) = 1$ are considered to be IID random variables. The probability density function (PDF) of the interevent times (or residence times of laminar phase) is given by [23]

$$\psi(\tau) \sim \alpha A^\alpha \tau^{-1-\alpha}, \quad \tau \to \infty, \qquad (4)$$

where the exponent α describing the power-law tail of the PDF $\psi(\tau)$ is controlled by the nonlinearity of the map in the vicinity of the indifferent fixed. A is a constant, which depends on the map.

Let N_t be the number of renewals in the time interval $(0,t)$ for a process which started on $-t_a$, and with IID interevent times according to Eq. (4). In non-aged renewal theory, $t_a = 0$, the distribution of the number of jumps N_t/t^α obeys the Mittag-Leffler distribution of order α (<1) [31]. Here, we review a derivation of the distribution and extend it to the aging regime [32]. Let S_n be the sum of the interevent times ($S_n = \tau_1 + \cdots + \tau_n$), then we have the following relation:

$$\Pr(N_t < n) = \Pr(S_n > t). \tag{5}$$

We notice that the interevent time PDF (4) belongs to the domain of attraction of stable laws [31]. In what follows, we use the notation $\Pr(\cdot;0)$ for non-aged processes, and $\Pr(\cdot;t_a)$ for aged processes. By the generalized central limit theorem [31] and setting $n = t^\alpha x$, we have

$$\Pr(N_t/t^\alpha < x;0) = \Pr(S_n/n^{1/\alpha} > x^{-1/\alpha};0) \tag{6}$$

$$\to \int_{x^{-1/\alpha}}^\infty l_\alpha(y)dy \quad \text{as } t\to\infty, \tag{7}$$

where $l_\alpha(x)$ is the one-sided stable density with index α, which depends on A, and its Laplace transform is given by

$$\int_0^\infty l_\alpha(x)e^{-sx}dx = \exp[-\Gamma(1-\alpha)(As)^\alpha]. \tag{8}$$

Let τ_F be the forward recurrence time, that is the time between the start of an observation $t = 0$ and the first renewal event (see Fig. 1). The PDF $\psi_0(\tau_F;t_a)$ of the forward recurrence time is different from $\psi(\tau)$, Eq. (4). According to [32,33], the double Laplace transform of $\psi_0(\tau_F;t_a)$,

$$\hat{\psi}_0(s,s_a) \equiv \int_0^\infty\int_0^\infty \psi_0(\tau_F,t_a)e^{-t_a s_a-\tau_F s}d\tau_F dt_a, \tag{9}$$

is, in the small s_a and s limit, given by

$$\hat{\psi}_0(s,s_a) \sim \frac{s_a^\alpha - s^\alpha}{s_a^\alpha(s_a - s)}. \tag{10}$$

Furthermore, by Dynkin's limit theorem [34], the limit PDF ($t_a \to \infty$) reads

$$\psi_0(\tau_F;t_a) \sim \frac{\sin(\pi\alpha)}{\pi}\frac{t_a^\alpha}{(\tau_F)^\alpha(t_a+\tau_F)}. \tag{11}$$

The probability of $N_t = 0$ is given by $\int_t^\infty \psi_0(\tau_F;t_a)d\tau_F$, while, for $N_t \geqslant 1$, the probability $\Pr(N_t < n;t_a)$ is represented by the convolution of $\psi_0(\tau_F;t_a)$ and $\Pr(N_\tau < n-1;0)$:

$$\Pr(N_t/t^\alpha < x;t_a) = 1 - m_\alpha(T_a) + \int_0^t \Pr(N_{t-\tau_F} < xt^\alpha - 1;0)\psi_0(\tau_F;t_a)d\tau_F, \tag{12}$$

where $m_\alpha(T_a) \equiv \int_0^t \psi_0(\tau_F;t_a)d\tau_F$, which is represented by the incomplete beta function,

$$m_\alpha(T_a) = \frac{\sin(\pi\alpha)}{\pi}B\left(\frac{1}{T_a+1};1-\alpha,\alpha\right). \tag{13}$$

The $1 - m_\alpha(T_a)$ term in Eq. (12) describes trajectories with no renewal events in $(0,t)$, or in the context of dynamics of maps trajectories which did not escape from the vicinity of an indifferent fixed point in the observation interval. We note that T_a is the aging ratio, defined in the Introduction. Similar to the calculation of the probability of N_t/t^α in non-aged renewal processes, we have

$$\Pr(N_t/t^\alpha < x;t_a) \cong 1 - m_\alpha(T_a) + \int_0^t \Pr\left(\frac{S_n}{n^{1/\alpha}} > \frac{t-\tau}{n^{1/\alpha}};0\right)\psi_0(\tau;t_a)d\tau$$

$$\to 1 - m_\alpha(T_a) + \int_0^t \psi_0(\tau;t_a)\int_{\frac{t-\tau}{x^{1/\alpha}t}}^\infty l_\alpha(y)dy\,d\tau, \tag{14}$$

for $t\to\infty$. As a result, the PDF $p_\alpha(\xi;T_a)$ of $\xi \equiv N_t/t^\alpha$ in the aging limit, $t_a/t \to T_a$ ($t,t_a \to \infty$), is written as

$$p_\alpha(\xi;T_a) = \delta(\xi)[1-m_\alpha(T_a)] + \frac{\sin(\pi\alpha)}{\pi\alpha\xi^{1+1/\alpha}} \times \int_0^{1/T_a}\frac{1-T_a y}{y^\alpha(1+y)}l_\alpha\left(\frac{1-T_a y}{\xi^{1/\alpha}}\right)dy. \tag{15}$$

This result is consistent with [7,32]. We note that the distribution depends on T_a even when the total measurement time goes to infinity. For $T_a \to 0$ (non-aging limit), $p_\alpha(\xi;T_a)$ converges to the Mittag-Leffler density as expected.

One can show that the mean $\langle N_t\rangle$ [7] is

$$\langle N_t\rangle = \frac{\{(t+t_a)/A\}^\alpha}{\Gamma(\alpha)}B\left(\frac{1}{T_a+1};1,\alpha\right). \tag{16}$$

It will soon be useful to define a normalized variable $\chi = N_t/\langle N_t\rangle$. Using Eq. (15) and $C_A \equiv t^\alpha/\langle N_t\rangle$, we have

$$P_\alpha(\chi;T_a) = \frac{1}{C_A}p_\alpha(\chi/C_A;T_a). \tag{17}$$

We note that PDF $P_\alpha(\chi;T_a)$ has the advantage of being A independent [PDF $p_\alpha(\xi;T_a)$ depends on A].

IV. RESULTS

A. Distributional limit theorem in the aging limit

The distributional limit theorem (14) in aging renewal processes implies that the properly scaled sum of $\sigma(x_k)$ converges in distribution:

$$\Pr\left(\frac{1}{t^\alpha}\sum_{k=0}^{t-1}\sigma(x_k) < x;t_a\right) = \Pr(N_t/t^\alpha < x;t_a) \tag{18}$$

$$\to \int_0^x p_\alpha(\xi;T_a)d\xi. \tag{19}$$

Because $\sigma(x)$ is an integrable function with respect to an invariant measure, this distributional limit theorem is a generalization of ADK theorem (1). By Hopf's ergodic theorem [35], the ratio of the sums of arbitrary integrable observation functions $f(x)$ and $\sigma(x)$ converges to a constant for almost all initial points:

$$\frac{\sum_{k=0}^n f(x_k)}{\sum_{k=0}^n \sigma(x_k)} \to \frac{\int_0^1 f d\mu}{\int_0^1 \sigma d\mu} \quad \text{as } n\to\infty, \tag{20}$$

where μ is an absolutely continuous invariant measure. Therefore, we have the following proposition. *In the aging limit $t_a/t \to T_a$ as t_a and $t \to \infty$, for all integrable functions $f(x)$ with respect to an invariant measure μ, the properly scaled sum of $f(x)$ converges in distribution:*

$$\Pr\left(\frac{C_A}{C_f t^\alpha}\sum_{k=0}^{t-1} f(x_k) < x; t_a\right) \to \int_0^x P_\alpha(\chi; T_a) d\chi, \qquad (21)$$

where $C_f = \int_0^1 f d\mu/\mu([\gamma_1,1])$. We note that the distribution depends on the aging ratio T_a and α.

B. From Dynkin's limit theorem to evolution of density

Here, we give an explicit representation of the density at $t = 0$ in aging processes. In aging renewal processes, the probability that there is no renewal until time t is given by $\int_t^\infty \psi_0(\tau_F; t_a) d\tau_F$. The corresponding probability in the map (2) is the probability that trajectories do not escape from the interval $[0,\gamma_1)$, which is given by $\int_0^{\gamma_t} \rho_{t_a}(x;0)dx$, where $T^t(\gamma_t) = 1$ for $\gamma_t < 1$. Using a continuous approximation, $T'(\gamma_t) \sim \frac{T(\gamma_t)-T(\gamma_{t+1})}{\gamma_t-\gamma_{t+1}}$, near $x \cong 0$ and $T(\gamma_t) = \gamma_{t-1}$, we have

$$\frac{\gamma_{t-1}-\gamma_t}{\gamma_t-\gamma_{t+1}} - 1 \sim a_0\left(1+\frac{1}{\alpha}\right)\gamma_t^{1/\alpha}. \qquad (22)$$

It follows that $\gamma_t \sim \alpha^\alpha (a_0 t)^{-\alpha}$ (the rigorous proof is given in [27]). Therefore, we have the following relation:

$$\int_0^{\alpha^\alpha (a_0 t)^{-\alpha}} \rho_{t_a}(x;0)dx \sim \int_t^\infty \psi_0(\tau_F;t_a)d\tau_F. \qquad (23)$$

Differentiating both sides of (23) with respect to t and using (11), we have

$$\rho_{t_a}(\alpha^\alpha a_0^{-\alpha} t^{-\alpha};0)\frac{\alpha^{\alpha+1}}{a_0^\alpha t^{\alpha+1}} \sim \frac{\sin(\pi\alpha)}{\pi}\frac{t_a^\alpha}{t^\alpha(t_a+t)}. \qquad (24)$$

As a result, we obtain the density at $t = 0$ in the aging process:

$$\rho_0(x;t_a) = \rho_{t_a}(x;0) \sim C\frac{1}{1+a_0 t_a x^{1/\alpha}/\alpha} \qquad (25)$$

for $x \ll 1$. The constant C is the normalization constant which depends on t_a, for $t_a \gg 1$:

$$C = \left(\int_0^1 \frac{1}{1+a_0 t_a x^{1/\alpha}/\alpha}dx\right)^{-1} \sim \frac{\sin(\pi\alpha)}{\pi\alpha}\left(\frac{a_0 t_a}{\alpha}\right)^\alpha. \qquad (26)$$

Surprisingly, evolutions of the density are in good agreement with the above estimation even for a small number of iterations and on whole space [0,1] (see Fig. 2). The density cannot converge to an absolutely continuous invariant density (equilibrium density) and will converge to the delta function $\delta(x)$ as t_a goes to infinity. This is a direct evidence of aging in dynamical systems.

As shown by Thaler [28], a scaled density converges to an infinite invariant density $\tilde{\rho}(x)$ (see Appendix A):

$$\lim_{t_a\to\infty} t_a^{1-\alpha}\rho_0(x;t_a) = \tilde{\rho}(x) \qquad (27)$$

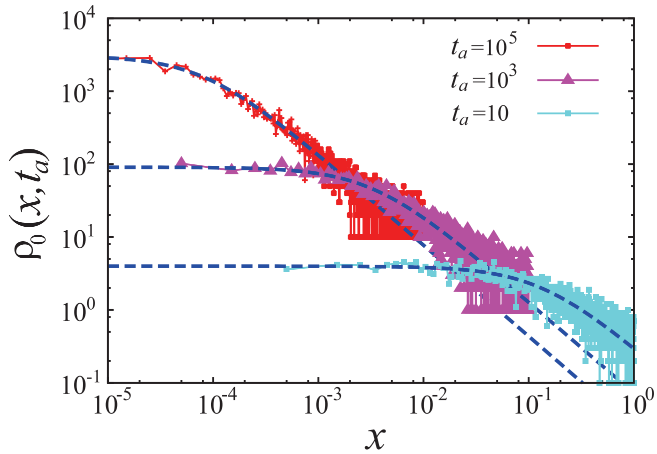

FIG. 2. (Color online) Densities at the start of the measurement $t = 0$ with different aging times $t_a = 10, 10^3$, and 10^5 ($\alpha = 0.8$). Symbols with lines are the results of numerical simulations. Dashed curves are the theoretical curves without fitting parameters. For all aging times t_a, the densities are in excellent agreement with the theory.

and

$$\tilde{\rho}(x) \sim \frac{\sin(\pi\alpha)}{\pi\alpha}\left(\frac{a_0}{\alpha}\right)^{\alpha-1} x^{-1/\alpha} \quad (x \to 0). \qquad (28)$$

We note that $ct_a^{1-\alpha}\rho_0(x;t_a) \to c\tilde{\rho}(x)$ ($t_a \to \infty$) is also an invariant density ($c > 0$). While we use $c = 1$, the choice of a multiplicative constant does not affect our results.

By the change of variable $x = \alpha^\alpha y/(a_0 t_a)^\alpha$, the scaled density $q(y) = \alpha^\alpha \rho_0(\alpha^\alpha a_0^{-\alpha} t_a^{-\alpha} y; t_a)/(a_0 t_a)^\alpha$ gives a master curve:

$$q(y) = \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{1}{1+y^{1/\alpha}}. \qquad (29)$$

This result is consistent with a rigorous result in general intermittent maps by Thaler [29]. Figure 3 shows a convergence of the scaled density to the master curve. We note that the master curve does not depend on details of the map except

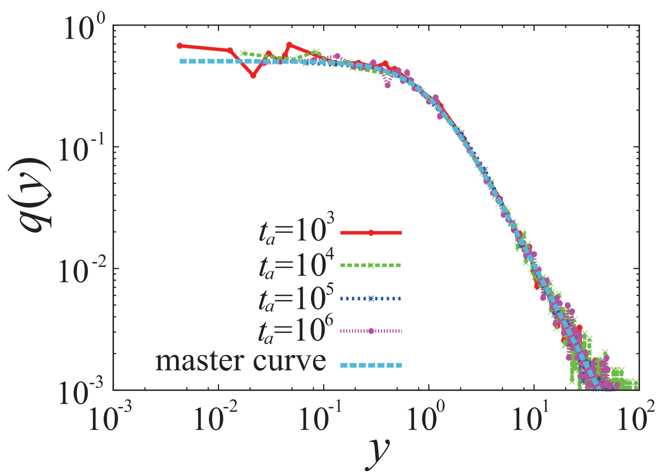

FIG. 3. (Color online) Scaled density ($\alpha = 0.6$). Symbols with lines are the results of numerical simulations. The dashed curve is the master curve, Eq. (29), for $\alpha = 0.6$. Scaled densities approach the master curve as $t_a \to \infty$.

near the fixed point $x = 0$, and also appears in the evolution of Riemann-integrable functions [29].

C. Dynamical instability

To investigate an effect of the aging on the dynamical instability, we consider the Lyapunov exponent. In general, weakly chaotic systems with infinite invariant measures have zero Lyapunov exponent [10,11,36,37]. However, these dynamical instabilities are known as a subexponential instability quantified by the generalized Lyapunov exponent Λ_α [11,37], which is defined as the average of the normalized Lyapunov exponent, $\Lambda_\alpha \equiv \langle \lambda_\alpha \rangle$, where $\langle \cdot \rangle$ is an average with respect to an initial density and

$$\lambda_\alpha \equiv \lim_{t\to\infty} \frac{1}{t^\alpha} \sum_{k=0}^{t-1} \ln |T'(x_k)| \quad (t_a = 0). \tag{30}$$

In non-aged systems, Λ_α does not depend on an initial density with the aid of ADK theorem. To investigate an effect of aging, we consider the generalized Lyapunov exponent in the aging limit $t_a/t \to T_a$, $\Lambda_\alpha(T_a) \equiv \langle \lambda_\alpha(T_a) \rangle$, where

$$\lambda_\alpha(T_a) \equiv \lim_{t\to\infty} \frac{1}{t^\alpha} \sum_{k=0}^{t-1} \ln |T'(x_k)| \quad (t_a/t \to T_a). \tag{31}$$

The aging generalized Lyapunov exponent is represented as

$$\Lambda_\alpha(T_a) \cong \frac{1}{t^\alpha} \int_0^t \int_0^1 g(x)\rho_{t'}(x;t_a)dx\, dt', \tag{32}$$

where $g(x) = \ln |T'(x)|$. Using the infinite invariant density $\tilde{\rho}(x)$ defined by Eq. (27), we obtain

$$\Lambda_\alpha(T_a) \cong \frac{(t + t_a)^\alpha - t_a^\alpha}{\alpha t^\alpha} \int_0^1 g(x)\tilde{\rho}(x)dx \tag{33}$$

for $t \gg 1$ and $t_a \gg 1$ with T_a fixed. According to ADK theorem, the non-aging generalized Lyapunov exponent ($t_a = 0$) is given by [37]

$$\Lambda_\alpha(T_a = 0) = \frac{1}{\alpha} \int_0^1 g(x)\tilde{\rho}(x)dx. \tag{34}$$

In the aging limit, this relation is generalized as

$$\Lambda_\alpha(T_a) = \frac{(1 + T_a)^\alpha - T_a^\alpha}{\alpha} \int_0^1 g(x)\tilde{\rho}(x)dx. \tag{35}$$

From Eq. (35) we see a novel relation between the separation under aging conditions and separation in the absence of aging:

$$\Lambda_\alpha(T_a)/\Lambda_\alpha(0) = (1 + T_a)^\alpha - (T_a)^\alpha. \tag{36}$$

This relation implies that knowledge of the separation in the non-aged case (e.g., numerically or analytically) is all that is needed to predict the separation under aging conditions. In the Conclusion section below we briefly explain why such relations are general beyond the observable under investigation. This aging effect on the generalized Lyapunov exponent has been confirmed numerically (Fig. 4). The result means that the dynamical instability becomes weak as the system ages. The $1/\alpha$ factor on the right-hand side of Eq. (35) is obviously related to our working definition of the invariant density, Eq. (27).

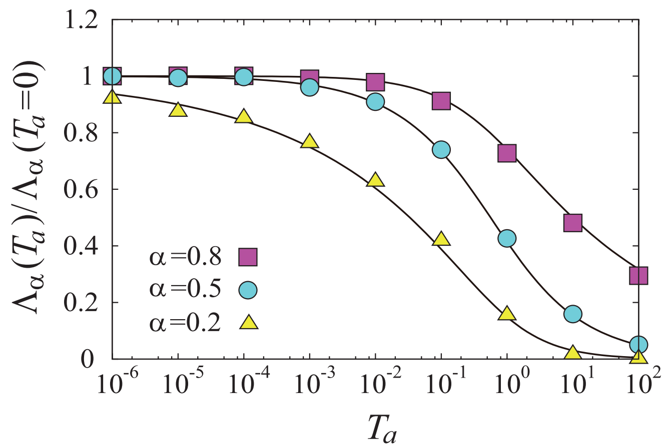

FIG. 4. (Color online) Generalized Lyapunov exponent as a function of the aging ratio T_a ($\alpha = 0.8$, 0.5, and 0.2). Different symbols are the results of numerical simulations for different α, where the total measurement time t is fixed as 10^6. Curves are theoretical ones without fitting parameter [$\Lambda_\alpha(T_a = 0)$ is obtained numerically]. Aging strongly affects the dynamical instability when the aging time t_a is larger than the measurement time t.

Using the distributional limit theorem (21), we obtain the PDF of the normalized Lyapunov exponent $\chi = \lambda_\alpha(T_a)/\Lambda_\alpha(T_a)$, given by the PDF (17), $P_\alpha(\chi;T_a)$. As shown in Fig. 5, the PDF does not depend on the measurement time t if the aging ratio T_a is fixed. Further, the strength of the delta peak at $\chi = 0$ increases as the aging ratio T_a is made larger (see Fig. 5). This delta peak corresponds to trajectories that

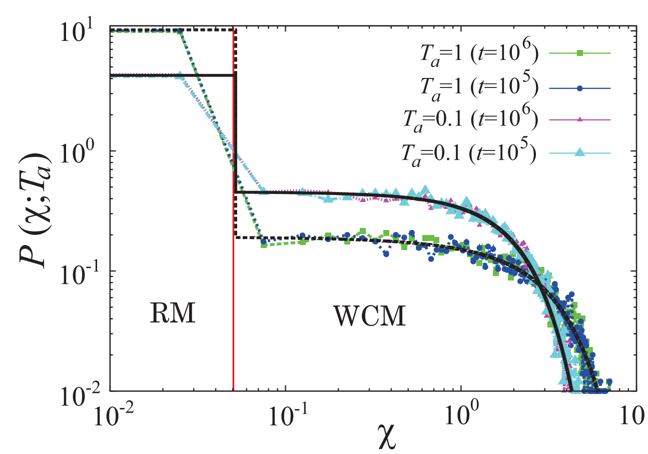

FIG. 5. (Color online) Probability density function $P(\chi;T_a)$ of the normalized Lyapunov exponent ($T_a = 0.1$ and 1, $t = 10^5$ and 10^6, and $\alpha = 0.5$). Symbols with lines are the results of numerical simulations. Solid curves are the theoretical ones calculated by a numerical integration of the convolution in (15) using the stable density with index 1/2, i.e., $l_{1/2}(x) = c_1 \exp[-c_1^2/(2x)]/\sqrt{2\pi x^3}$, where c_1 is a scaling parameter depending on T_a. The probability of finding regular motions is increased when the system ages. Almost all trajectories to the left of the red line exhibit regular motions (RMs) while trajectories to the right are weakly chaotic motions (WCMs), where the value of the red line ($\chi = 0.05$) represents the bin size of the PDF with histogram (the bin is needed to graphically represent a delta function). Theory is in good agreement with the numerical results including the delta distribution.

do not escape from $[0,\gamma_1]$ until time t, which weakens the dynamical instability. These trajectories are regular rather than chaotic (not even weakly chaotic). In fact, these trajectories can be treated by a continuous approximation. For $x_t \cong 0$, we use an ordinary differential equation [21,23] to describe the dynamics:

$$\frac{dx_t}{dt} = a_0 x_t^{1+1/\alpha} \quad (x_t \cong 0). \tag{37}$$

The solution of this continuous approximation is given by

$$x_t = x_0\big(1 - \alpha^{-1} x_0^{1/\alpha} a_0 t\big)^{-\alpha} \tag{38}$$

for $x_t \ll 1$ or $t \ll \alpha x_0^{-1/\alpha}/a_0$. The difference of nearby trajectories x_t and x_t' such that x_0 and $x_0' = x_0 + \delta x$ ($\delta x \ll x_0$) is described as

$$|x_t - x_t'| \cong \delta x\big[1 + x_0^{1/\alpha}(1+\alpha^{-1})a_0 t\big]. \tag{39}$$

In the aging limit, these regular motions appear even when time t goes to infinity. Therefore, in aging dynamical systems, regular motions (RMs) and weak chaotic motions (WCMs) intrinsically coexist in the aging limit. This implies that when $T_a \gg 1$ a large fraction of particles are located close to the indifferent fixed point and hence their motion is regular.

V. DISCUSSION

We have shown the distribution of sums of integrable functions in the aging limit, which is a generalization of Aaronson's Darling-Kac theorem. Although we use intermittent maps, this generalization will be valid for all weakly chaotic maps with infinite invariant measures (more precisely, a conservative, ergodic, measure preserving transformation) because aging in dynamical systems implies that the density does not converges to an equilibrium density even when time goes to infinity (nonequilibrium nonstationary density). Using numerical simulations, we confirmed the distributional aging limit theorem in the Boole transformation [8] (not shown).

VI. CONCLUSION

The aging ratio T_a plays an important role in characterizing aging systems. In the aging limit, the distribution of sums of integrable observables converges to a universal distribution, which is determined by the aging ratio T_a and the exponent α of infinite invariant measures in dynamical systems. The mathematical basis of this universal distribution is both the generalized central limit theorem and Dynkin's theorem for the forward recurrence time. We have also shown how to use the infinite invariant density to calculate statistical averages such as the measure of separation Λ_α within the aging regime. From Eq. (35) we have a novel relation under aging conditions (36). Thus, knowledge of the non-aging observable $\Lambda(0)$ via measurement or with the infinite invariant density is sufficient for the determination of the aged observable $\Lambda(T_a)$. Similar relations between aged and non-aged averages hold for a large class of observation functions which are integrable with respect to the infinite invariant density. Thus the infinite invariant density plays an important role for determination of ergodic properties of aging processes.

We have found that the dynamical instability is clearly divided into two different instabilities, i.e., regular motions and weakly chaotic motions, in the aging limit. Coexistence of regular and chaotic motions is reminiscent of generic Hamiltonian systems. However, the meaning of the coexistence is completely different. In aging dynamical systems, the probability of finding regular motions is increased according to the aging ratio T_a, whereas regular and chaotic phase spaces do not depend on T_a in generic Hamiltonian systems.

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APPENDIX A: DERIVATION OF EQS. (27) AND (28) BY THALER'S THEOREM

In [28], Thaler showed that, for all Riemann-integrable functions $u(x)$ on $[0,1]$,

$$w_n P^n u(x) \to \left(\frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\int_0^1 u(x)dx\right) h(x), \tag{A1}$$

uniformly on compact subsets of $(0,1]$ as $n \to \infty$, where w_n is the wandering rate, P is the Perron-Frobenius operator, and $h(x)$ is an invariant density. The wandering rate w_n is defined using the invariant measure μ corresponding to $h(x)$, i.e., $\mu([0,x]) = \int_0^x h(y)dy$,

$$w_n(T) \equiv \mu\left(\bigcup_{k=0}^{n-1} T^{-k}([\gamma_1,1])\right). \tag{A2}$$

Note that the wandering rate depends on μ or $h(x)$. It is known that the wandering rate for the map (2) and an invariant density $h(x) = \tilde{h}(x)x^{-1/\alpha}$ is given by

$$w_n \sim \tilde{h}(0)\frac{\alpha}{1-\alpha}\alpha^{\alpha-1}a_0^{1-\alpha}n^{1-\alpha}. \tag{A3}$$

Because an initial density is a Riemann-integrable function, the left-hand side of Eq. (A1) is a scaled density $\rho(x,n)$ at time n if $u(x) = \rho_0(x)$. It follows that a scaled density converges to an invariant density as $n \to \infty$. Applying Thaler's theorem to the map (2) and $u(x) = \rho_0(x)$, we have

$$n^{1-\alpha}\rho_0(x,n) \to \frac{\sin\pi\alpha}{\pi}\alpha^{-\alpha}a_0^{\alpha-1}\frac{\tilde{h}(x)}{\tilde{h}(0)}x^{-1/\alpha}, \tag{A4}$$

as $n \to \infty$. Our notation in the text for infinite invariant density, namely the right-hand side, is equal to $\tilde{\rho}(x)$. For $x \to 0$ and $n \to \infty$,

$$n^{1-\alpha}\rho_0(x,n) \sim \frac{\sin\pi\alpha}{\pi\alpha}\left(\frac{a_0}{\alpha}\right)^{\alpha-1}x^{-1/\alpha}. \tag{A5}$$

Therefore, Eqs. (27) and (28) are derived by Thaler's theorem. We note that the right-hand sides of Eqs. (27) and (28) do not depend on $\rho_0(x)$ if it is a Riemann-integrable function (not the delta function).

APPENDIX B: ANOTHER REPRESENTATION OF THE AGING DISTRIBUTIONAL LIMIT THEOREM

We presented a main result, the distributional limit theorem under aging conditions, by Eq. (21). Here, we give another representation of the aging distributional limit theorem using a specific invariant measure. Because the constant C_f in Eq. (21) does not depend on the multiplicative constant of an invariant measure μ, one can choose the constant.

However, the constant C_A should be represented by a specific invariant measure. For all integrable observation functions $f(x)$, the ensemble average of $\sum_{k=0}^{t-1} f(x_k)/t^\alpha$ can be represented by a specific invariant measure denoted by μ':

$$\left\langle \frac{1}{t^\alpha} \sum_{k=0}^{t-1} f(x_k) \right\rangle \to \int_0^1 f(x) d\mu' \quad \text{as } t \to \infty. \tag{B1}$$

We note that the specific invariant measure μ' does not depend on an observation function $f(x)$. This is a result from the ADK theorem, whereas the return sequence is used to represent the distributional limit theorem without specifying an invariant measure in the ADK theorem [8]. In other words, an invariant measure in Eq. (B1) is specified because the return sequence is fixed as t^α. Because $\sigma(x)$ is an integrable function with respect to an invariant measure, the ensemble average of $\sum_{k=0}^{t-1} \sigma(x_k)/t^\alpha$, which is $\langle N_t \rangle / t^\alpha$, converges to the constant $\int_0^1 \sigma(x) d\mu'$. It follows that one can represent the constant $C_A = t^\alpha / \langle N_t \rangle$ by the specific invariant measure μ', i.e., $C_A^{-1} = \int_0^1 \sigma(x) d\mu' = \mu'([\gamma_1, 1])$. If we choose an invariant measure such that $\mu'([\gamma_1, 1]) = C_A^{-1}$, the aging distributional limit theorem (21) can be represented by

$$\Pr\left(\frac{1}{t^\alpha} \sum_{k=0}^{t-1} f(x_k) < x; t_a \right) \to \int_0^{x/\int_0^1 f d\mu'} P_\alpha(\chi; T_a) d\chi. \tag{B2}$$

We note that the specific invariant density $d\mu/dx$ is given by $\tilde{\rho}(x)/\alpha$ because the ensemble average of $\sum_{k=0}^{t-1} \ln T'(x_k)/t^\alpha$, i.e., the generalized Lyapunov exponent, is represented by Eq. (34).

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