Elementary but accurate analytical approximation for a one-dimensional soliton, conservative or not

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Well-known rigorous methods have been elaborated long ago to exactly solve conservative soliton equations. Mainly, there are the inverse scattering transform, Hirota's direct method, and Sato's formalism. These methods are fully satisfactory: analytical expressions for solitons are obtained in any spatial dimension as well as multisoliton solutions. Therefore, there is no need for an additional approach, especially if it is an approximate one, restricted to a one-dimensional situation and to a single soliton solution. Except, our approach is really elementary, straightforward, and unexpectedly accurate. It provides a physical background to the newfound exp-function method and, most importantly, it furnishes an analytical description of front solution in a nonconservative equation for which no other rigorous methods exist.

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I. INTRODUCTION

The basic idea of the inverse scattering transform is to represent a nonlinear evolution equation as the compatibility condition between two linear operators, the so-called Lax pairs. The method was first introduced in the famous Gardner *et al.* paper [1]. Further advances were made by Lax [2], Zakharov and Shabat [3,4], and Ablowitz *et al.* [5,6]. Today, the method is well elaborated and the algebraic structure of Lax pairs was elucidated. The underpinnings of the early Hirota's direct method [7] was explained. Especially, works by Sato [8] and others in his wake [9] make it possible to understand the soliton theory from a unified point of view. Nevertheless, finding the Lax pair in the first place may still remain a problem even if methods based on symmetry considerations are developing [10].

In one dimension, provided that one restricts himself to solutions with simple time behavior (periodic or propagation without deformation), solitons can then be understood as homoclinic or heteroclinic connections of the associated dynamical system where the space coordinate is viewed as an effective evolution variable [11–13]. In this small way, this point of view is more general because it is not restricted to a conservative equation but is also valid for a nonconservative partial differential equation. It leads to important qualitative predictions [11–14] although it does not furnish any analytical expressions of the localized solutions.

Here, we adopt this latter point of view and try to carry the argument to its limit. In a first part we detail our analytic approach for the classical nonlinear Shrödinger equation (NLS) and investigate its convergence and accuracy. The cases of the Korteweg–de Vries (KDV) and sine-Gordon (SG) equations are reported in the Appendix. Then, we apply our method to the still unsolved problem of the computation of the front velocity in a modified Ginzburg-Landau (MGL) equation. In conclusion, we discuss the limits and possible extensions of the method.

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II. NLS SOLITON AS A HOMOCLINIC CONNECTION

The NLS equation is expressed as

$$i\partial_t\psi = \partial_{zz}\psi + |\psi|^2\psi. \tag{1}$$

Restricting ourselves to solutions of the form

$$\psi(t,z) = e^{-i\omega t}\phi(\xi) \qquad \text{with} \qquad \xi = z - ct, \tag{2}$$

where ω and c are real, we are left with the ordinary differential equation

$$\triangle(\phi) = -\omega\phi + ic\partial_\xi\phi + \partial_{\xi\xi}\phi + |\phi|^2\phi = 0. \tag{3}$$

$\phi = 0$ is a fixed point whose linear stability is given by the real part of k:

$$\begin{aligned} \phi = e^{kz} &\Longrightarrow -\omega + ick + k^2 = 0 \\ &\Longrightarrow k_\pm = \tfrac{-ic}{2} \pm \sqrt{\omega - \tfrac{c^2}{4}}. \end{aligned} \tag{4}$$

In what follows, we consider situations where $\omega > \frac{c^2}{4}$ and look for the soliton solution as a homoclinic connection biasymptotic to the vanishing hyperbolic fixed point. It is convenient to define $k_\pm = -iq \pm b$ where q and b are real and $b > 0$.

Because of the translation invariance of (3), there is no loss of generality to assume the soliton core to be localized at $\xi = 0$. Far from the core, on the left side ($\xi \longrightarrow -\infty$), we look for a solution of (3) as a power expansion in $e^{b\xi}$:

$$\phi_L(\xi) = L_1 e^{-iq\xi}e^{b\xi} + L_3(\xi)(e^{b\xi})^3 + L_5(\xi)(e^{b\xi})^5 + \dots, \quad \xi \longrightarrow -\infty. \tag{5}$$

Of course, L_1 is a complex constant. The $L_{i>1}(\xi)$ are complex functions to be determined and whose variations are intended to be slower than exponential. Note that the expansion does not contain even contribution because of the cubic nonlinearity.

Substitution of (5) in (3) leads to a hierarchy of linear differential equations. There is no contribution to the first order $(e^{b\xi})^1$ because of (4). At order $(e^{b\xi})^3$, we get

$$\mathcal{L}_{3b}L_3 = -|L_1|^2 L_1 e^{-iq\xi}, \tag{6}$$

where $\mathcal{L}_p$ is the linear operator defined as

$$\mathcal{L}_p U = [\partial_{\xi\xi} + (ic + 2p)\partial_\xi + (p^2 + icp - \omega)]U. \tag{7}$$

Then,

$$L_3(\xi) = -\frac{|L_1|^2 L_1}{8\left(\omega - \frac{c^2}{4}\right)} e^{-iq\xi} = G_3 e^{-iq\xi}. \tag{8}$$

At order $(e^{b\xi})^5$ we obtain

$$\mathcal{L}_{5b} L_5 = -\left(2|L_1|^2 G_3 + L_1^2 G_3^*\right) e^{-iq\xi}, \tag{9}$$

which leads to

$$\begin{aligned} L_5 &= -\frac{2|L_1|^2 G_3 + L_1^2 G_3^*}{24\left(\omega - \frac{c^2}{4}\right)} e^{-iq\xi} \\ &= \frac{|L_1|^4 L_1}{\left[8\left(\omega - \frac{c^2}{4}\right)\right]^2} e^{-iq\xi} \\ &= G_5 e^{-iq\xi}. \end{aligned} \tag{10}$$

The computation does not display any difficulty and can go on to any order. Programs to do mathematics such as MATHEMATICA or MAPLE are helpful. For the sake of clarity, we will stop here, but expansions up to much more higher orders have been performed, especially when dealing with other equations than NLS.

For the right side ($\xi \longrightarrow +\infty$) we look for a solution of (3) as a power expansion in $e^{-b\xi}$:

$$\begin{aligned} \phi_R(\xi) \;&=\; R_1 e^{-iq\xi} e^{-b\xi} + R_3(\xi)(e^{-b\xi})^3 \\ &\quad + R_5(\xi)(e^{-b\xi})^5 + \dots, \\ \xi &\longrightarrow +\infty, \end{aligned} \tag{11}$$

where R_1 is a new complex constant. Following the same steps as for the left side, we obtain

$$R_3(\xi) = -\frac{|R_1|^2 R_1}{8\left(\omega - \frac{c^2}{4}\right)} e^{-iq\xi} = D_3 e^{-iq\xi} \tag{12}$$

and

$$R_5 = \frac{|R_1|^4 R_1}{\left[8\left(\omega - \frac{c^2}{4}\right)\right]2} \quad e^{-iq\xi} = D_5 e^{-iq\xi}. \tag{13}$$

We are now in possession of the two analytic asymptotic expressions of the soliton. At the core, they are expected to connect each other, the two unknowns R_1 and L_1 being precisely determined by the corresponding matching conditions. However, it turns out that all direct attempts to connect (5) to (11) results in failure. Indeed, let us call N the order of the asymptotic expansions. Then, we observe that $\triangle[\phi_R(\xi)]$ and $\triangle[\phi_L(\xi)]$, which constitute a measure of the inaccuracy, do increase with N (values up to $N = 100$ have been tested). This is an indication that the asymptotic expansions have a finite convergence radius and diverge for $\xi = 0$, i.e., $e^{\pm b\xi} = 1$. Fortunately, Padé approximants are known to give better approximation than the associated truncated Taylor expansion, and possibly still work even where these latter do not converge. Therefore, we first construct the [3,3] Padé approximant of ϕ_L (resp. ϕ_R) as a function of the variable of $e^{b\xi}$ (resp. $e^{-b\xi}$). Note that we use a [3,3] approximant because (5) and (11) are valid up to order 6 ($6 = 3 + 3$):

$$\begin{aligned} \phi_L \;&=\; e^{-iq\xi}[L_1(e^{b\xi}) + G_3(e^{b\xi})^3 + G_5(e^{b\xi})^5] \\ &\Longrightarrow \phi_L^p = e^{-iq\xi}\frac{G_3 L_1 e^{b\xi} + \left(G_3^2 - L_1 G_5\right)(e^{b\xi})^3}{G_3 - G_5(e^{b\xi})^2}, \\ \phi_R \;&=\; e^{-iq\xi}[R_1(e^{-b\xi}) + D_3(e^{-b\xi})^3 + D_5(e^{-b\xi})^5] \\ &\Longrightarrow \phi_R^p = e^{-iq\xi}\frac{D_3 R_1 e^{-b\xi} + \left(D_3^2 - R_1 D_5\right)(e^{-b\xi})^3}{D_3 - D_5(e^{-b\xi})^2} \end{aligned} \tag{14}$$

and only after we apply the matching conditions. Because (3) is second order in space, there are two equations

$$\begin{aligned} \phi_L^p(0) &= \phi_R^p(0), \\ \partial_z \phi_L^p(0) &= \partial_z \phi_R^p(0). \end{aligned} \tag{15}$$

The previous system consists in two complex algebraic equations for the two complex variables (L_1 and R_1). In the most general way, it can be numerically efficiently solved with Newton's method. For the present case, it is easy to remark that the first equation is obviously satisfied for $L_1 = A = R_1$. The second equation is then expressed as

$$|A|^2 = 8\left(\omega - \frac{c^2}{4}\right) = 8b^2. \tag{16}$$

Substituting the previous results (16) in the Padé approximants (14) leads to

$$\phi_R^p(\xi) = \frac{\sqrt{2}b e^{-iq\xi}}{\cosh(b\xi)} = \phi_L^p(\xi), \tag{17}$$

which, miraculously, turns out to be the exact analytical soliton solution!

Important remarks are now in order:

(1) We chose to present the computation up to order 5, but the order 3 expansion leads already to the exact analytical result.

(2) The defocusing regime of the NLS equation corresponds to

$$i\partial_t \psi = \partial_{zz}\psi - |\psi|^2\psi \tag{18}$$

and is known to not sustain solitons. As the change of sign of the nonlinear term prevents neither the computation of the asymptotic behaviors nor the computation of the Padé approximants, our approach is still available. However, there is no contradiction with the predicted absence of solitons because it can be rigorously proved that in the defocusing regime, the matching conditions (15) do not possess solutions, but the vanishing one $R_1 = L_1 = 0$.

Applications of the method to the KDV and SG equations are reported in the Appendix.

III. FRONT IN THE MODIFIED GINZBURG-LANDAU EQUATION

A. Comparison with the exp-function method

Recently, a new method called *exp-function method* has been proposed to seek solitary solutions, periodic solutions, and compactonlike solutions of nonlinear differential equations [15]. The exp-function method is very simple. It is based on the assumption that traveling wave solutions $U(z,t)$ can be expressed as

$$U(\eta) = \frac{\sum_{n=-d}^{c} a_n e^{n\eta}}{\sum_{m=-q}^{p} b_m e^{m\eta}}, \quad (19)$$

where c, d, p, and q are unknown positive integers, a_n and b_m are unknown complex constants, and $\eta = kz + \omega t$ is a complex variation. The method has been successfully applied to the KDV and the Dodd-Bullough Mikhailov equation [15], KDV equation with variable coefficients [16], discrete sine-Gordon equation [17], and nonlinear Schrödinger equations [18]. Our approach, which also involves rational fraction between finite sum of exponentials functions (14), is clearly related to the exp-function method. However, a close examination of (19) and (14) points out several differences:

(1) First, in [15–18], the mathematical formula (19) has the status of an ansatz, an assumption (sic). On the contrary, our approach is strongly underpinned by the physical understanding of soliton as homoclinic or heteroclinic connections [11–13].

(2) More important: In (14), we end up with two rational fractions, one valid for the soliton left part, the other for the right part. For the NLS equation, it turns out that the satisfaction of the matching conditions forces the two rational fractions to be equal. But, the deep reason for that is the symmetry of the NLS soliton envelops with respect to the parity transformation. In the case of a localized solution which breaks the parity symmetry, the exp-function method is not available because it uses the same complex variation $\eta = kz + \omega t$ for the left and the right parts. On the contrary, we will show in the following that our method still leads to an analytical accurate approximation.

B. Modified Ginzburg-Landau equation

The GL equation is involved in Landau's description of the ferromagnetic phase transition. It is expressed as

$$\partial_t U = U - U^3 + \tfrac{1}{2}\partial_{zz} U, \quad (20)$$

where $U(t,z)$ is real. It is well known to possess topological solitons, either kink or antikink [$\pm\tanh(z)$]. Here, we consider the following modification:

$$\partial_t U = U - U^3 + \tfrac{1}{2}\partial_{zz} U - \theta + \theta^3, \quad (21)$$

where $\theta \in [-\frac{1}{\sqrt{3}}, +\frac{1}{\sqrt{3}}]$. In presence of a nonvanishing value of $H = \theta - \theta^3$, the $U \longrightarrow -U$ symmetry is broken and the topological solitons are expected to move with a constant velocity c. In the context of the ferromagnetic phase transition, the symmetry breaking term H stands for an additional external magnetic field. For small values of θ, one can perform standard perturbative computations and obtain

$$c = \tfrac{3}{2}\theta - \tfrac{69}{32}\theta^3 + O(\theta^5) \quad (22)$$

in very good agreement with numerical simulations. But, for large values of θ, the perturbative analysis is no more available. For example, for $\theta = 0.5$, Eq. (22) leads to $c \simeq 0.48$, but the velocity numerically measured is 0.75 ± 0.01. Also, to our

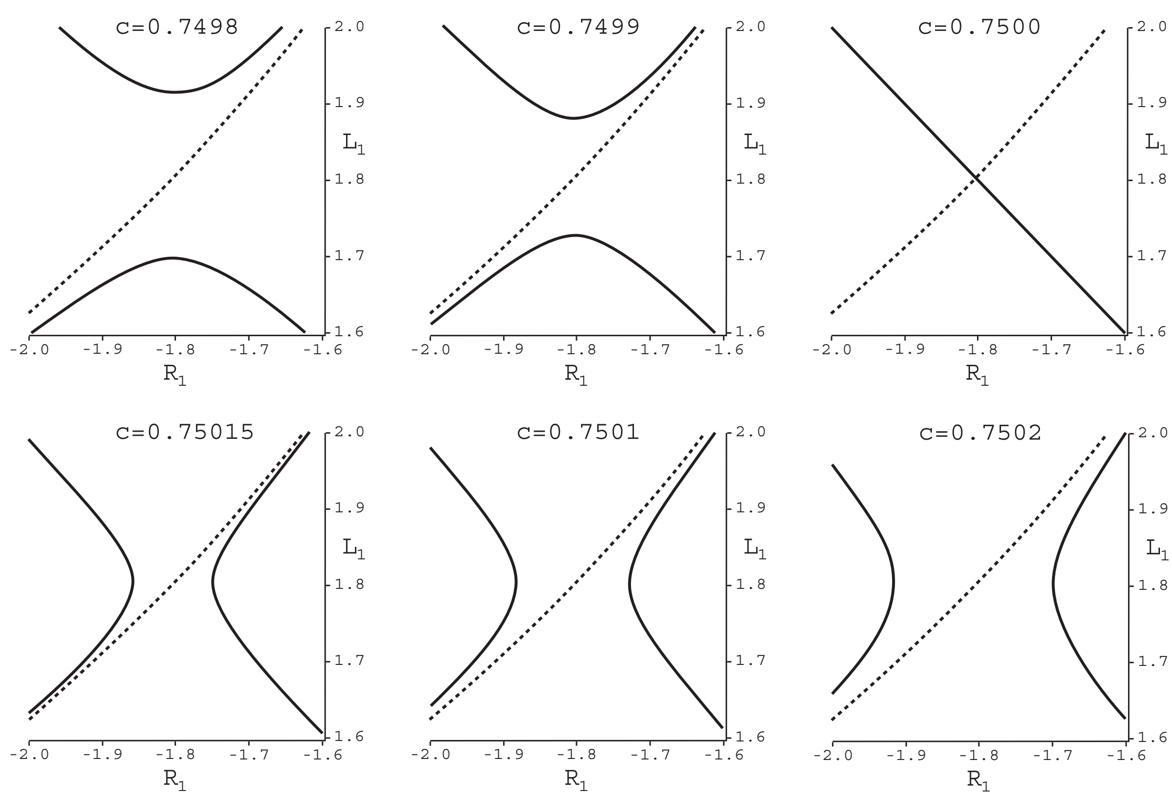

FIG. 1. Graphical resolution of Eq. (32) for $N = 3$. The dashed lines correspond to the points in the (R_1,L_1) plane where $P_c(R_1,L_1) = 0$. The continuous lines are associated with $Q_c(R_1,L_1) = 0$. The various plots correspond to different values of c between 0.7498 and 0.7502. An intersection point only exists for $c \simeq 0.7500$.

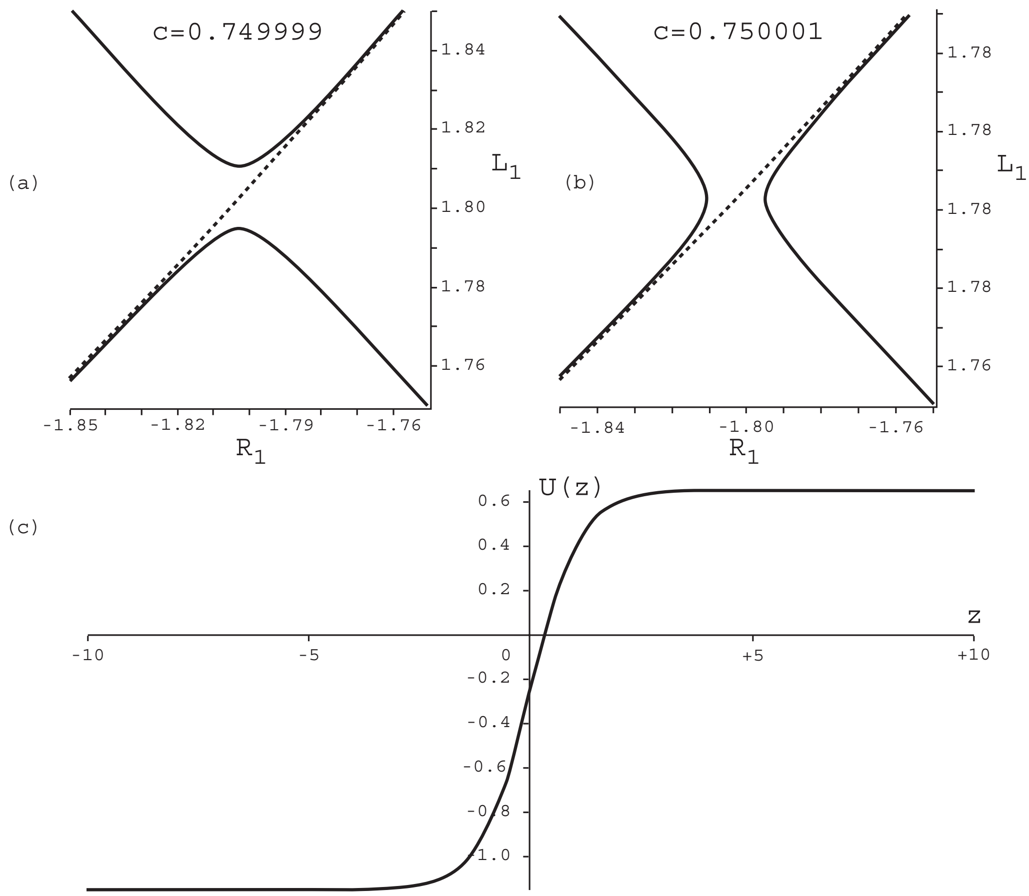

FIG. 2. (a) and (b) are the same as Fig. 1 but with $N = 9$ which allows a better resolution (scales have been changed). (c) is the plot of the two Padé approximants $U_L^p(z)$ and $U_R^p(z)$ versus z, for $c = 0.749\,999$ and $L_1 = -R_1 = 1.803$.

best knowledge, there is no analytical method to solve the problem in such a regime. It is therefore a decisive test for our method.

Stationary homogeneous solutions of Eq. (21) are

$$\begin{aligned} U_- &= -\frac{\theta}{2} - \sqrt{1 - \frac{3}{4}\theta^2}, \\ U_0 &= \theta, \\ U_+ &= -\frac{\theta}{2} + \sqrt{1 - \frac{3}{4}\theta^2}. \end{aligned} \tag{23}$$

Therefore, our problem is to find the single value of c for which Eq. (21) possesses a front solution $U(z - ct)$ which asymptotically connects U_- $[=U(-\infty)]$ to U_+ $[=U(+\infty)]$.

We will proceed in the same way as before. The front solution $U(\xi)$ ($\xi{=}z - ct$) satisfies

$$0 = U - U^3 + c\partial_\xi U + \tfrac{1}{2}\partial_{\xi\xi} U - \theta + \theta^3. \tag{24}$$

After linearization around the stationary solutions $U_s = U_\pm$, we get

$$U = U_s + u \Longrightarrow 0 = -(U_s - \theta)(2U_s + \theta)u + c\partial_\xi u + \tfrac{1}{2}\partial_{\xi\xi} u, \tag{25}$$

then

$$u = e^{kz} \Longrightarrow k_\pm(U_s) = -c \pm \sqrt{c^2 + 6U_s^2 - 2}. \tag{26}$$

Note that for $\theta \in [\frac{-1}{\sqrt{3}}, \frac{+1}{\sqrt{3}}]$, k_+ is always positive and k_- always negative. We introduce

$$b_L = k_+(U_-) \qquad \text{and} \qquad b_R = k_-(U_+). \tag{27}$$

Far from the core, on the left side, a solution of (24) is

$$U_L = U_- + L_1 e^{b_L\xi} + L_2 e^{2b_L\xi} + L_3 e^{3b_L\xi} + \dots L_N e^{Nb_L\xi}, \tag{28}$$

where N is the order of the approximation and

$$\begin{aligned} L_2 &= -\frac{3U_- L_1^2}{-2b_L^2 - 1 + 3U_-^2 - 2cb_L}, \\ L_3 &= \frac{2L_1^3\left(15U_-^2 + 2b_L^2 + 1 + 2cb_L\right)}{\left(-2b_L^2 - 1 + 3U_-^2 - 2cb_L\right)\left(-9b_L^2 - 2 + 6U_-^2 - 6cb_L\right)}, \\ &\dots . \end{aligned} \tag{29}$$

For the right side, we have

$$U_R = U_+ + R_1 e^{b_R\xi} + R_2 e^{2b_R\xi} + R_3 e^{3b_R\xi} + \dots R_N e^{Nb_R\xi} \tag{30}$$

with

$$R_2 = -\frac{3U_+R_1^2}{-2b_R^2 - 1 + 3U_+^2 - 2cb_R},$$

$$R_3 = \frac{2R_1^3\left(15U_+^2 + 2b_R^2 + 1 + 2cb_R\right)}{\left(-2b_R^2 - 1 + 3U_+^2 - 2cb_R\right)\left(-9b_R^2 - 2 + 6U_+^2 - 6cb_R\right)},$$
$$\ldots. \tag{31}$$

It is worth noting the fact that $k_+ \neq k_-$ [Eq. (26)] definitively rules out the exp-function method which only involves a single asymptotic complex behavior ($e^{\pm\eta}$).

We next compute the Padé approximants (U_R^p and U_L^p) and then obtain the two matching conditions

$$\begin{aligned} U_L^p(0) - U_R^p(0) &= P_c(R_1, L_1) = 0, \\ \partial_\xi U_L^p(0) - \partial_\xi U_R^p(0) &= Q_c(R_1, L_1) = 0, \end{aligned} \tag{32}$$

where P_c and Q_c are long and tiresome polynomials in R_1 and L_1 parametrized by c.

Figures 1 and 2 stand for the graphical resolution of Eq. (32). The order of the approximation N is, respectively, equal to 3 for Fig. 1 and 9 for Fig. 2. The noticeable topological change around $c \simeq 0.750\,00$ is associated with the velocity selection mechanism. Hence, we expected the selected velocity to satisfy $0.749\,999 < c < 0.750\,001$. Therefore, the algebraic computation leads to a selected velocity in very good agreement with the numerical observation, even for low value of N.

IV. DISCUSSION

In the system dynamics point of view, solitons are thought of as homoclinic or heteroclinic connections, which asymptotically connect a homogeneous stationary solution. Although analytic asymptotic expansions are easy to compute, there were no previous attempts in the literature to derive an analytical description from the matching conditions because the asymptotic expansions were recognized to be valid only far from the core.

Here, we have identified the reason why the asymptotic expansions fail to describe the soliton near its core. It deals more with the convergence radius than with the number of terms of the asymptotic expansions. Therefore, the technical solution is well known and is called the Padé approximant.

On one hand, the resulting method is straightforward, accessible, and is valid for a large class of equations. It provides a physical background to the newfound exp-function method, but turns out to be more general. In contrast with classical soliton methods, it can be used to describe the topological solutions of nonconservative partial differential equations. It provides analytical expressions with controllable precision, and is accurate enough to investigate some selection mechanism (as the front velocity in the MGL equation).

On the other hand, the algebraic computations, although basic, are so long that computer programs to do mathematics can fail because of a lack of memory. Also, once obtained, the matching equations take the form of nonlinear multivariate polynomials. Certainly, the complexity has decreased because we start with nonlinear differential equations and end up with nonlinear algebraic equations, but a final numerical investigation is always required. Lastly, multisolitons solutions are not described.

One of the most interesting natural extensions of this work deals with the addition of fourth order spatial derivatives in the NLS, SG, and MGL soliton equations. First, because usual solitons technics are mainly restricted to second order spatial derivatives (especially the exp-function method). Second, because in such a case, each asymptotic behavior generically involves two exponential decay rates and therefore the possibility of spatial oscillations [19]. Third, because surprising complex selection mechanisms have been qualitatively predicted [14] in this regime. From a technical point of view, Padé approximants with two variables are then expected to be involved. Works in this direction are in progress.

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APPENDIX

1. Korteweg–de-Vries equation

The KDV equation is expressed as

$$\partial_t u + 6u\partial_z u + \partial_{zzz} u = 0, \tag{A1}$$

where $u(t,z)$ is real. Looking for a solution of the form $u(t,z) = u(z - ct)$, we are left with

$$-c\partial_\xi u + 6u\partial_\xi u + \partial_{\xi\xi\xi} u = 0, \qquad \xi = z - ct, \tag{A2}$$

where c is real and assumed to be positive without loss of generality. Solutions of the linear problem are expressed as

$$u(\xi) = e^{k\xi} \Longrightarrow k_\pm = \pm\sqrt{c}. \tag{A3}$$

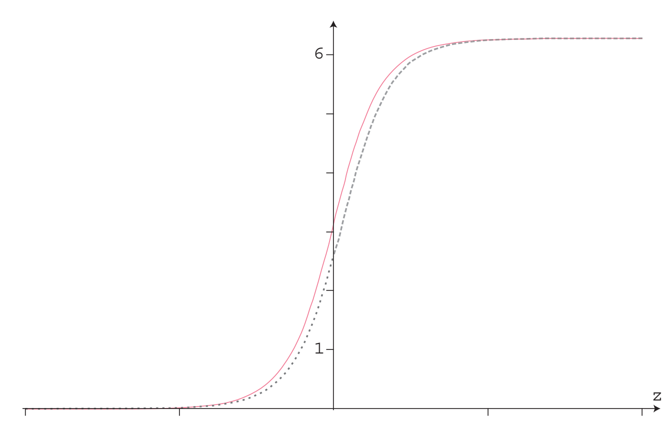

FIG. 3. (Color online) Sine-Gordon soliton. The continuous line stands for the exact solution. The dashed lines (one on the left, the other on the right) correspond to the [2,3] Padé approximants.

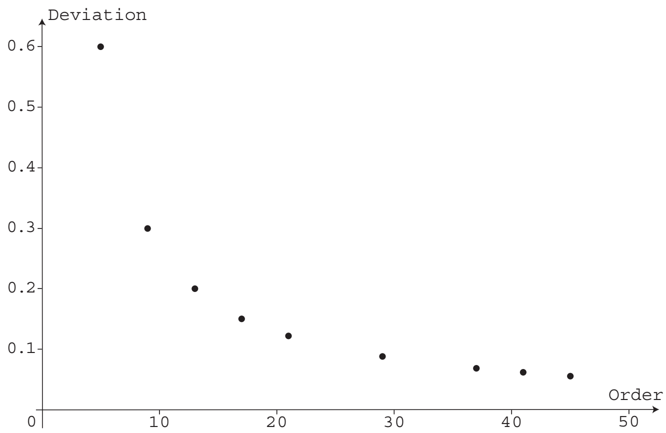

FIG. 4. Convergence of the method for the sine-Gordon equation. The plot displays the maximal deviation between the exact sine-Gordon soliton and the Padé approximants versus the order of the asymptotic expansion.

As previously, $b = \sqrt{c}$ is defined as the real part of k_+. The asymptotic left and right solutions are

$$\begin{aligned} U_L(\xi) &= L_1 e^{b\xi} + L_2(e^{b\xi})^2 + L_3(e^{b\xi})^3 + \dots, \\ L_2 &= -\frac{L_1^2}{c}, \quad L_3 = \frac{3}{4}\frac{L_1^3}{c^2}, \\ U_R(\xi) &= R_1 e^{-b\xi} + R_2(e^{-b\xi})^2 + R_3(e^{-b\xi})^3 + \dots, \\ R_2 &= -\frac{R_1^2}{c}, \quad R_3 = \frac{3}{4}\frac{R_1^3}{c^2}. \end{aligned} \tag{A4}$$

The corresponding [1,2] approximants are expressed as

$$\begin{aligned} U_L^p(\xi) &= \frac{L_1^3 e^{+b\xi}}{L_1^2 - L_2 L_1 e^{+b\xi} + \left(L_2^2 - L_1 L_3\right)(e^{+b\xi})^2} \\ &= \frac{4 L_1 c^2 e^{+b\xi}}{(2c + L_1 e^{+b\xi})^2}, \\ U_R^p(\xi) &= \frac{R_1^3 e^{-b\xi}}{R_1^2 - R_2 R_1 e^{-b\xi} + \left(R_2^2 - R_1 R_3\right)(e^{-b\xi})^2} \\ &= \frac{4 R_1 c^2 e^{-b\xi}}{(2c + R_1 e^{-b\xi})^2}. \end{aligned} \tag{A5}$$

Equation (A2) is a third order ordinary differential equation and therefore there are three matching conditions for only two variables. It turns out that with the constraints $R_1 = A = L_1$, the matching conditions for the left and right expansions (U_R^p and U_L^p) as well as for the third order derivatives ($\partial_{\xi\xi\xi} U_R^p$ and $\partial_{\xi\xi\xi} U_L^p$) are automatically satisfied. We are then left with the second order derivatives matching condition which can be reduced to

$$A = 2c \tag{A6}$$

and after substitution in (A5), we obtain

$$U_L^p(\xi) = \frac{c/2}{\cosh(\frac{b}{2}\xi)^2} = U_R^p(\xi), \tag{A7}$$

i.e., again the exact analytical soliton expression!

2. Sine-Gordon equation

The SG equation is expressed as

$$\partial_{tt} u - \partial_{zz} u + \sin(u) = 0, \tag{A8}$$

where $u(t,z)$ is real. Restricting ourselves to solution with the form $u(t,z) = u(z - ct)$, we are left with

$$(c^2 - 1)\partial_{\xi\xi} u + \sin(u) = 0, \qquad \xi = z - ct, \tag{A9}$$

$u = 0$ and $u = 2\pi$ are fixed points with the same linear stability

$$u = \begin{cases} 0 \\ \text{or} \\ 2\pi \end{cases} + \epsilon e^{kz} \Longrightarrow k^2 = \frac{1}{1 - c^2}. \tag{A10}$$

In what follows, we assume $c^2 < 1$ and define $b = \frac{1}{\sqrt{1-c^2}}$. The asymptotic solutions are

$$\begin{aligned} U_L(\xi) &= 0 + L_1 e^{+b\xi} + L_3(e^{+b\xi})^3 + L_5(e^{+b\xi})^5 + \dots, \\ L_3 &= -\frac{L_1^3}{48}, \qquad L_5 = \frac{L_1^5}{1280}, \\ U_R(\xi) &= 2\pi + R_1 e^{-b\xi} + R_3(e^{-b\xi})^3 + R_5(e^{-b\xi})^5 + \dots, \\ R_3 &= -\frac{R_1^3}{48}, \qquad R_5 = \frac{R_1^5}{1280}, \end{aligned} \tag{A11}$$

and the [2,3] Padé approximants are

$$\begin{aligned} U_L^p(\xi) &= \frac{48 L_1 e^{b\xi}}{48 + L_1^2 (e^{b\xi})^2}, \\ U_R^p(\xi) &= \frac{6R_1^2(-16 + 3\pi^2)(e^{-b\xi})^2 + 48 R_1 \pi (e^{-b\xi}) + 480\pi^2}{240\pi - 96 R_1 (e^{-b\xi}) + 9 R_1^2 \pi (e^{-b\xi})^2 - 2R_1^3 (e^{-b\xi})^3}. \end{aligned} \tag{A12}$$

The two matching conditions do not depend on c. Their numerical resolution with a Newton's method leads to

$$L_1 = 3.10309\dots, \qquad R_1 = -5.49222\dots. \tag{A13}$$

Figure 3 shows the comparison between the exact soliton solution and its Padé [2,3] approximants. The highest deviation is obtained near the core and corresponds to $\simeq 0.6$, i.e., about 10% of the soliton amplitude. Increasing the order of the Padé approximants leads to a decrease of the deviation as shown in Fig. 4.

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