Kosterlitz-Thouless phase transition and re-entrance in an anisotropic three-state Potts model on the generalized kagome lattice

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The unusual re-entrant phenomenon is observed in the anisotropic three-state Potts model on a generalized kagome lattice. By employing the linearized tensor renormalization group method, we find that the re-entrance can appear in the region not only under a partially ordered phase that is commonly known but also a phase without a local order parameter, which is found to fall into the universality class of Kosterlitz-Thouless (KT). The region of the re-entrance depends strongly on the ratios of the next-nearest-neighbor couplings $\alpha = J_2/|J_1|$ and $\beta = J_3/|J_1|$. The phase diagrams in the plane of temperature versus β for different α are obtained. Through massive calculations, it is also disclosed that the quasientanglement entropy can be applied to accurately determine the KT transition temperature.

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I. INTRODUCTION

Two-dimensional (2D) frustrated classical lattices possess many interesting physical properties (see, e.g., Refs. [1–8]. One of them is the so-called re-entrant phenomenon [1,3,6,9], which is defined as the occurrence of a disordered phase (usually a paramagnetic phase) in the region below an ordered or a partial ordered phase on the temperature scale. This disordered phase is called the re-entrant phase that has been detected experimentally in spin glasses [10] and studied theoretically in several exactly solvable 2D frustrated Ising systems on lattices such as the centered square lattice [1], generalized kagome lattice [3], centered honeycomb lattice [2], and other three-dimensional lattices [5]. Up to now, the reason for the occurrence of the re-entrant phase is still under debate.

Some conjectures [1,3,5] have been proposed to understand the re-entrant phenomenon, in which the essential point is that the re-entrant phase is probably caused by frustrations and, in the ground state, there should be at least one partially ordered phase adjacent to an ordered phase or another partially ordered phase between which the re-entrant phase can appear at finite temperature. Above the re-entrant phase, there is usually a partially ordered phase whose disordered sublattices supplement the entropy that is lost due to the formation of the ordered sublattice. In addition, other factors such as the coordination numbers [5] of the sites on the disordered sublattice and the freedom of the on-site spin [9] (e.g., the value of q in the Potts model) may also have effects on the re-entrant phenomenon.

Alternatively, to explore the incentives of the re-entrant phenomenon, the q-state Potts model [11] can provide some clues. The Potts model is the generalization of the Ising model by extending the on-site freedom from $q = 2$ to $q > 2$. The q-state Potts model does not have exact solutions. For some lattices such as the piled-up domino model (frustrated Villain lattice) [9], the Potts model possesses a re-entrant phase that could maintain in the region under one partially ordered phase when $1 \leqslant q \leqslant 4$ (except for $q = 2$), which indicates that the occurrence of re-entrant phenomenon is closely related to the value of q.

In this work, we shall focus on the anisotropic three-state Potts model on a generalized kagome lattice, as shown in Fig. 1. This model contains three different couplings: J_1 is the diagonal coupling and is presumed to be ferromagnetic (F), J_2 and J_3 are couplings along the vertical and horizontal directions, respectively, both of which can be either antiferromagnetic (AF) or F. For the Ising case ($q = 2$), there exist two partial disordered configurations A and B in the ground states, as shown in Fig. 1. Configuration A corresponds to the case of $J_3 < 0$ (F) and $J_2 > 0$ (AF), and B indicates the case of J_2, $J_3>0$. The spins on the central site in both cases are in the free states, therefore the order is defined as a partial order or a partial disorder. By the exact solution the re-entrant phase is found between the F phase and the partially ordered phase with configuration A [3], which is induced by frustrations. It is interesting to ask whether there is any re-entrant phenomenon in the present anisotropic three-state Potts model on the generalized kagome lattice, in which J_2 is set to be negative and J_3 positive with $J_1 = -1$ (F). Such a mixed (with AF and F interactions) Potts model cannot be solved exactly. We shall perform numerical simulations to obtain the specific heat, susceptibility, correlation length, and quasientanglement entropy of the system to determine the phase diagram precisely. It is discovered that although the thermal fluctuation depresses any partial order with configuration A and B in the ground state, our numerical results strongly support that there still exists a re-entrance but with the Kosterlitz-Thouless (KT) type phase transition at certain values of $J_2/|J_1|$ and $J_3/|J_1|$. In addition, it is found that the re-entrance disappears when $q > 3$.

This paper is organized as follows. In Sec. II, the model Hamiltonian and the partition function are defined, and the tensor network representation of this model is given. In Sec. III, the specific heat, susceptibility, and the quasientanglement entropy are calculated, and the KT-type phase transition is discussed. In Sec. IV, by examining the singularities of the thermodynamic quantities, the phase diagrams in $\beta - T$ plane

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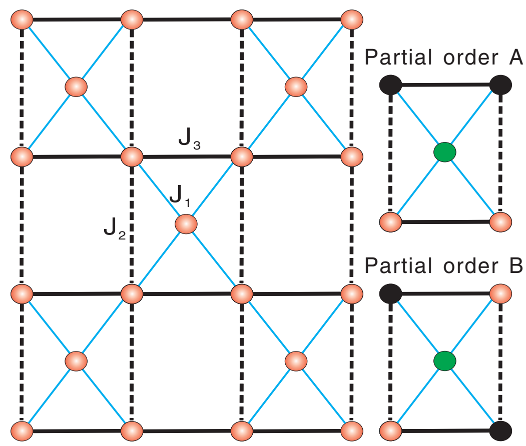

FIG. 1. (Color online) The geometric structure (left) and partial order configurations (right) of the generalized kagome lattice. The thin blue line represents ferromagnetic coupling J_1, the thick solid line (J_3) and dashed black line (J_2) could be either ferromagnetic or antiferromagnetic coupling. The black and red dots in partial order A and B configurations indicate different directions of the on-site spin and the centered green site is a free spin.

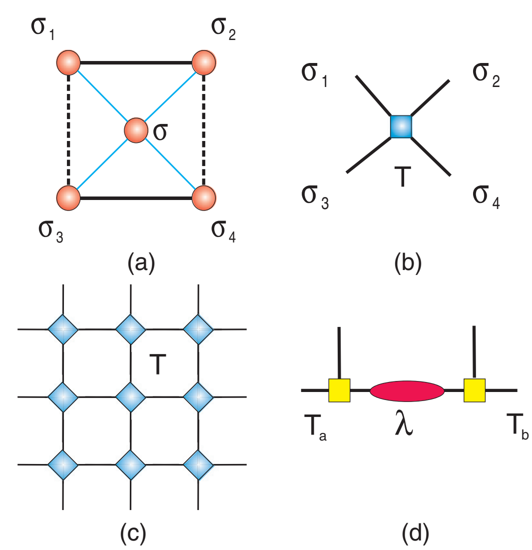

FIG. 2. (Color online) Tensor network representation in LTRG calculations of this present system. (a) the cell lattice for tensor construction; (b) the schematic representation of tensor **T**; (c) the tensor network of the partition function; and (d) the matrix product state [12] obtained by contracting the tensor network in (c).

are presented for $\alpha = -1$ ($\alpha < 0$), $0 < \alpha < 1$, and $\alpha > 1$, respectively, and the re-entrant phenomena are observed. Finally, a summary is given.

II. MODEL AND PARTITION FUNCTION

The Hamiltonian of the anisotropic q-state Potts model on the generalized kagome lattice reads

$$H = J_1 \sum_{\langle i,j\rangle_\star} \delta_{\sigma_i\sigma_j} + J_2 \sum_{\langle i,j\rangle_\perp} \delta_{\sigma_i\sigma_j} + J_3 \sum_{\langle i,j\rangle_\parallel} \delta_{\sigma_i\sigma_j}, \qquad (1)$$

where J_i ($i = 1,2,3$) are coupling constants, $\delta_{\sigma_i\sigma_j}$ is the Kronecker symbol with $\sigma_i = 1,2,\ldots,q$ and $\langle\rangle_{\star,\perp,\parallel}$ represents the nearest-neighbor pairs along the diagonal, vertical, and parallel directions as shown in Fig. 1. In the following we shall focus on $q = 3$. For clarity, we define $\alpha = J_2/|J_1|$, and $\beta = J_3/|J_1|$.

By means of the Trotter-Suzuki decomposition the partition function of the system can be written as

$$Z = \mathrm{Tr}\, e^{-H/T} = \mathrm{Tr}\prod e^{-\epsilon h}, \qquad (2)$$

$$h = J_1\big(\delta_{\sigma\sigma_1} + \delta_{\sigma\sigma_2} + \delta_{\sigma\sigma_3} + \delta_{\sigma\sigma_4}\big) + J_2\big(\delta_{\sigma_1\sigma_3} + \delta_{\sigma_2\sigma_4}\big) + J_3\big(\delta_{\sigma_1\sigma_2} + \delta_{\sigma_3\sigma_4}\big), \qquad (3)$$

where T is temperature, the Boltzmann's constant k_B is taken as unity, and ϵ is the Trotter slice. The relative positions of σ_i ($i = 1,2,3,4$) and σ are illustrated in Fig. 2(a).

To study this present anisotropic three-state Potts model, we employ the linearized tensor renormalization group (LTRG) method [12], a recently developed numerical algorithm for calculating the thermodynamic properties of the low-dimensional quantum lattice systems with high accuracy and efficiency, which has been successfully applied to a few one-dimensional quantum spin lattice models [14,15]. In the tensor network representation of the partition function (Fig. 2), $e^{-\epsilon h}$ can be seen as a fourth-order tensor **T** as shown in Fig. 2(b), and the partition function Z can be represented as a tensor network in Fig. 2(c). During the LTRG calculations, we keep the bond dimension cutoff D_c of the tensor network at least 60 and the truncation error is less than 10^{-7}.

III. THERMODYNAMIC QUANTITIES AND KOSTERLITZ-THOULESS PHASE TRANSITION

The Kosterlitz-Thouless phase transition in the mixed Potts model [16,17] is characterized by the divergence of the correlation length ξ near the critical temperature $T_{c,\mathrm{KT}}$ of the form

$$\xi \sim e^{\frac{\mathrm{const}}{\sqrt{T - T_{c,\mathrm{KT}}}}}, \qquad (4)$$

while the specific heat shows no divergence and has a broad peak near $T_{c,\mathrm{KT}}$. Below $T_{c,\mathrm{KT}}$, there is a phase (coined as the floating phase) with an algebraically decaying correlation function. Such kind of phase was explained as the resemble of vortices that are closely bound in pairs in 2D XY models [17]. In the present three-state Potts model, we also found such a phase and in the following we will follow Ref. [16] to call it the floating phase.

Now let us first explore the thermodynamic properties of the generalized kagome Potts model with parameters $\alpha = J_2/|J_1| = -1$ and $\beta = J_3/|J_1| > 0$. The temperature dependence of the specific heat C for different β is obtained, as shown in Fig. 3. It is seen that, for $\beta < 1$, the specific heat has sharp peaks at critical temperatures [Fig. 3(a)], which indicate that the second-order phase transitions may take place in this case. It can be easily understood that when $\beta < 1$ the ferromagnetic couplings J_2 and J_1 are dominant, and for a given $\beta < 1$ the system may undergo a phase transition from ferromagnetic phase to paramagnetic phase, which will also be confirmed in calculations of magnetization (see Fig. 4). It is observed that the critical temperature T_c decreases with increasing β when $\beta < 1$. For $\beta > 1$, the specific heat

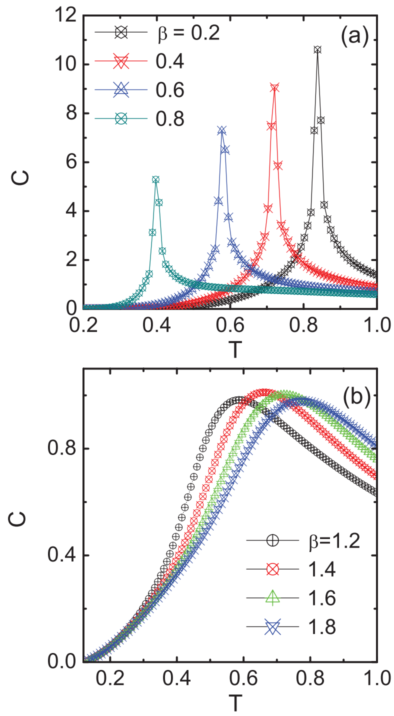

FIG. 3. (Color online) Temperature dependence of the specific heat of the anisotropic three-state Potts model on the generalized kagome lattice with $\alpha = -1$. (a) $\beta < 1$, and (b) $\beta > 1$.

shows broad peaks at low temperatures, and no divergence is found, displaying that no thermodynamic phase transitions can happen in this case. However, this does not imply that the topological phase transition such as the KT phase transition is unlikely. In fact, this is the case, as seen below.

To understand further the thermodynamic behavior of this model, we come to look at the magnetization per site defined as

$$m = \frac{1}{N}\sum_i \langle \sigma_i \rangle, \tag{5}$$

where N is the number of lattice sites, and $\langle ... \rangle$ denotes the thermal average. Note that in this definition m describes the mean value of magnetization on each site whose value equals

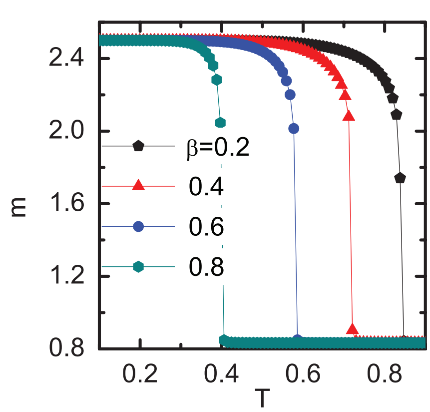

FIG. 4. (Color online) Temperature dependence of the magnetization per site of the anisotropic three-state Potts model on the generalized kagome lattice with $\alpha = -1$ and $\beta < 1$.

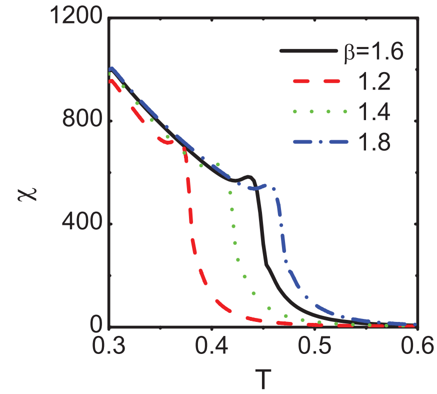

FIG. 5. (Color online) Temperature dependence of the susceptibility of the anisotropic three-state Potts model on the generalized kagome lattice with $\alpha = -1$ and $\beta > 1$.

$\frac{5}{6}$ when the lattice is in disordered state. Figure 4 presents the temperature dependence of magnetization per site at $\alpha = -1$ for different $\beta < 1$. The sharp changes of m at critical temperatures can be seen, showing that for a given $\beta < 1$ the system indeed has a thermodynamic order-disorder phase transition with increasing temperature, being consistent with the results of the specific heat [Fig. 3(a)]. For $\beta > 1$, $m = 0$ at finite temperature, showing that in this case no long-range order appears. To examine if there is a topological phase transition in the case of $\beta > 1$, we calculated the susceptibility χ of the system defined by

$$\chi = \frac{\partial m}{\partial h}, \tag{6}$$

where h is the uniform magnetic field. The results are given in Fig. 5, where, for different values of $\beta > 1$, there appear singularities (kinks) at certain temperatures, implying that there might be the occurrence of a kind of nonthermodynamic phase transition.

To confirm that such topological phase transition is of the KT type, we investigate whether the correlation length ξ has the form of Eq. (4). Following the lines introduced in Ref. [18], we obtained the results of correlation length versus temperature, as shown in Fig. 6(a). Apparently, when $\beta > 1$ near the transition temperature $T_{c,\mathrm{KT}}$, the correlation length ξ shows the same behavior as that given in Eq. (4), where a slight deviation could be modified by increasing the bond dimension D_c. This result supports that the phase transition for $\beta > 1$ is indeed of the KT type. For $\beta < 1$, the correlation length ξ reveals a distinct behavior from the case of $\beta > 1$, as shown in the inset of Fig. 6(a), being consistent with the previous observation that the occurrence of phase transition in the case of $\beta < 1$ is thermodynamic. Besides, by fitting the curve, we obtained the critical exponent $\nu = 0.820$ for $\beta < 1$ in this model, which is close to the conjectured value $\frac{5}{6}$ [11].

In addition, making use of the diagonal elements of the diagonal λ matrix in the matrix product state (MPS), shown in Fig. 2(d), which are obtained by the contraction of the tensor network in Fig. 2(c), we can define the quasientanglement entropy following the definition of von Neumann entropy [13] in one-dimensional quantum lattice systems

$$S_Q = -\Sigma_i \lambda_i \log \lambda_i . \tag{7}$$

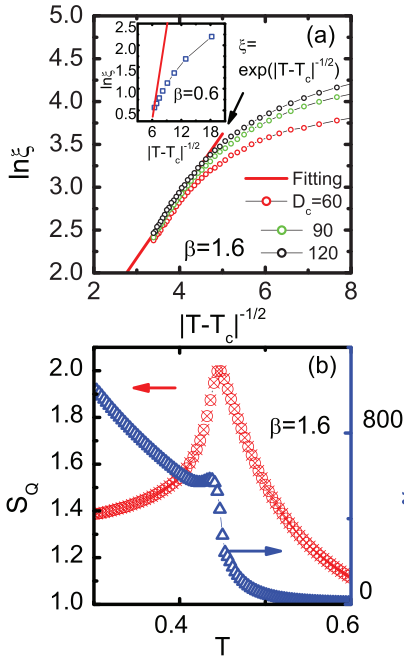

FIG. 6. (Color online) (a) Correlation length of the anisotropic three-state Potts model on the generalized kagome lattice with $\alpha = -1$ and $\beta > 1$, and the inset is for $\beta < 1$. (b) Quasientanglement entropy and susceptibility of this model with $\beta = 1.6$.

Actually, the MPS obtained by contracting the tensor network is equivalent to the ground state of its corresponding quantum 1D model [19]. The singular point of the quasientanglement entropy, which corresponds to the phase transition in quantum 1D models can be applied to determine T_c or $T_{c,\mathrm{KT}}$ for the 2D classical lattice. The temperature dependence of the quasientanglement entropy S_Q at $\beta = 1.6$ is calculated, as presented in Fig. 6(b), where a sharp peak of S_Q is seen at the temperature that is the same as the temperature at which the susceptibility exhibits a singularity. It appears that the quasientanglement entropy defined by Eq. (7) can be employed to detect the KT phase transition.

IV. RE-ENTRANT PHENOMENA AND PHASE DIAGRAMS

The KT phase transition implies the quenching of the partial order below $T_{c,\mathrm{KT}}$ that further supports the disordered ground state when $\beta > 1$ owing to the large degeneracies in the ground state and the strong fluctuations in the Potts model. Under $T_{c,\mathrm{KT}}$, the floating phase is very sensitive to external perturbations. Figure 7(a) gives the induced m by a tiny field ($\sim 10^{-3}$) for $\beta = 1.6$, where one may see that m increases fast with increasing the small field, showing the character of the floating phase. The inset of Fig. 7(a) demonstrates that m decreases with temperature in a small field, and at a certain temperature, it has a rapid drop to the value of $5/6$, indicating again that the system is in the floating state.

From the above discussions, we come to conclude that the anisotropic three-state Potts model on the generalized kagome lattice for $\beta > 1$ has the topological phase transition of the KT type. An interesting question then arises: can a re-entrant phase exist under a phase without a partial order through the KT transition? Let us answer this question below.

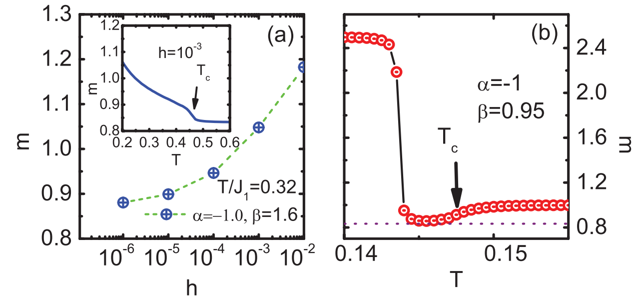

FIG. 7. (Color online) (a) The magnetization versus small magnetic field at temperature $T/|J_1| = 0.32$. The inset is the magnetization versus temperature in a small field $h = 10^{-3}$. Here $\beta = 1.6$. (b) Temperature dependence of m in the re-entrance region.

To search for the re-entrant region, we would collect the information from the specific heat, susceptibility, and the quasientanglement entropy to determine the phase diagrams. We first begin with $\alpha = -1$ and take β as the variable. In calculations we determined the critical temperature of ferromagnetic phase transition by means of the peak of the specific heat and the KT phase transition temperature according to the peak of the quasientanglement entropy. By performing a large-scale calculations, we found that the parameter range of β where the re-entrant phenomenon occurs is in between $[0.94, 1]$, and in this small region the system is most frustrated. Figure 7(b) gives the temperature dependence of magnetization for $\alpha = -1$ and $\beta = 0.95$, and it can be seen that with increasing temperature the magnetization decreases sharply to the value of $5/6$, and after experiencing a flat change, then increases remarkably, suggesting that there should be two phase transitions in this case, namely, one from F state to paramagnetic (P) state, and another from P state to floating state. Such transitions can also be clearly seen from the susceptibility, as shown in Fig. 8, where the temperature dependence of the susceptibility is presented for $\beta = 0.952$ and $\alpha = -1$. There are two peaks: the first peak at $T_{c,f} \approx 0.136$ corresponds to the phase transition from F to P phase; and the second peak at the critical temperature $T_{c,xi} \approx 0.140$ indicates the phase transition from P to X phase. The inset of Fig. 8 gives the temperature dependence of the quasi-entanglement entropy, where three peaks are observed: the two peaks at the critical temperatures 0.136 and 0.140 are too close to be

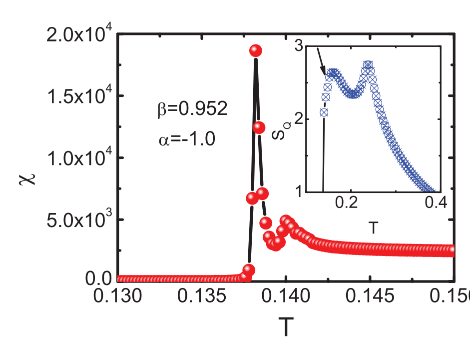

FIG. 8. (Color online) Temperature dependence of the susceptibility of the anisotropic three-state Potts model on the generalized kagome lattice in the re-entrance region. The inset shows the corresponding quasientanglement entropy.

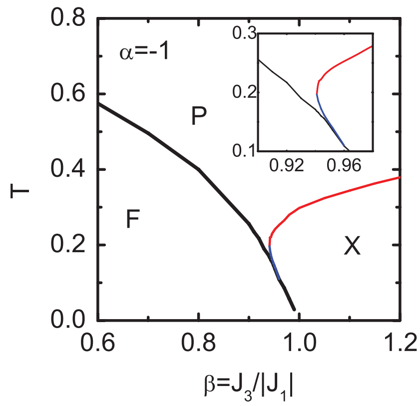

FIG. 9. (Color online) The phase diagram of the anisotropic three-state Potts model on the generalized kagome lattice with $\alpha = -1$. The inset indicates the re-entrant region. F: ferromagnetic phase; P: paramagnetic phase; X: Floating phase.

visibly separated, which just correspond to the two peaks of susceptibility, while the third peak occurs at $T_{c,xf} \approx 0.242$, indicating the phase transition from X phase to P phase. Therefore, for $\beta = 0.952$, with increasing temperature, the system undergoes phase transitions from P to X to P phase, which is nothing but the re-entrant phenomenon.

The whole phase diagram for $\alpha = -1$ is depicted in Fig. 9, where three phases including F, P, and X phases are identified. One may note that there exists a small region $\beta \in [0.94, 1]$ where the re-entrance occurs, as shown in the inset of Fig. 9. It can be observed that in this region and with increasing temperature, the system goes first into the P phase from the F phase, and then enters into the X phase, and finally goes into again the P phase. In addition, some further computations reveal that the region of β for which the re-entrant phenomenon can appear is proportional to $|\alpha|$ ($\alpha < 0$) and shrinks to zero at $\alpha = 0$ that is just the typical kagome lattice. It is also worth mentioning that the phase diagram in Fig. 9 is similar to that of the generalized kagome Ising model [3]. Since in the three-state Potts model the thermal fluctuations are stronger than in the Ising model, the phase boundaries in Fig. 9 stand lower than in the diagram of Ref. [3]. In the latter diagram the region of the partially ordered state A is replaced by the floating phase labeled by X.

When $0 < \alpha < 1$, there is also a re-entrant property, as shown in Fig. 10. In the phase diagram with $\alpha = 0.25$, there are four phases, including F, P, X, and Y phases, where Y is also a floating phase appearing in a very small region when $\beta > 0.8$. There exists a re-entrant region between the F and X phases when $\beta < -24.94$. This region, determined by the same procedure as above, is the same as the generalized kagome Ising model [3,5], but the critical temperature $T_{c,f}$ is lower than that in the Ising case due to strong thermal fluctuations. It is found that the area of X phase increases with the value of α in the cost of the decreased scope of the F phase. It is clear that in the inset of Fig. 10 the ferromagnetic phase transition temperature $T_{c,f}$ goes to zero when α increases close to 1. The finite temperature phase transition from X to P phase is also of the KT type, and the critical temperature can be obtained through the calculation of the quasientanglement entropy or susceptibility.

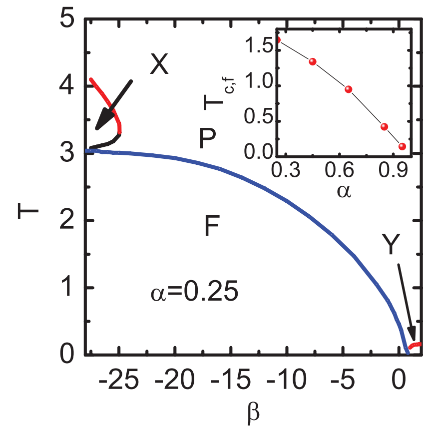

FIG. 10. (Color online) Phase diagram of the anisotropic three-state Potts model on the generalized kagome lattice with $0 < \alpha < 1$. F: Ferromagnetic; P: Paramagnetic; X and Y: floating phases. The inset is α dependence of the ferromagnetic phase transition critical temperature $T_{c,f}$.

When $\alpha > 1$ the F phase disappears, and there will be only two floating phases X and Y in the phase diagram, as shown in Fig. 11. In this case, no re-entrant phenomenon occurs. In addition, the partially ordered state with configuration B in the Ising model (Ref. [3]) when α, $\beta > 0$ is vanished due to highly thermal fluctuations in the three-state Potts model as in Fig. 10. Besides, we found through a number of calculations that when $q > 3$, the q-state Potts model on the generalized kagome lattice exhibits no re-entrant property.

V. SUMMARY

To summarize, we have studied the anisotropic three-state Potts model on the generalized kagome lattice by utilizing the LTRG method. The phase diagrams in the plane of temperature versus β for the cases of $\alpha < 0$, $0 < \alpha < 1$, and $\alpha > 1$ are obtained. Different phases including ferromagnetic, paramagnetic, and floating phases are identified. The phase boundaries are determined by observing the singularities in the specific heat, magnetization, susceptibility, and quasientanglement entropy. For the two cases of $\alpha < 0$ and $0 < \alpha < 1$,

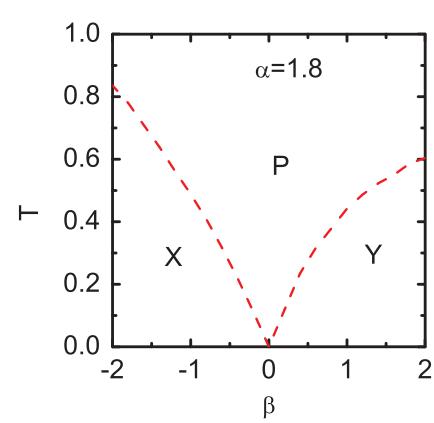

FIG. 11. (Color online) Phase diagram of the anisotropic three-state Potts model on the generalized kagome lattice with $\alpha > 1$. P: Paramagnetic; X and Y: floating phases.

the re-entrant phenomena are observed in small regions, respectively. By studying the behaviors of the correlation length, we found that the phase transition for $\beta > 1$ is of the KT type, in sharp contrast to the case of $\beta < 1$ where the correlation length has a quite different behavior from that of $\beta > 1$.

By comparing with the generalized kagome Ising model, where the partially ordered state phases exist, the present three-state Potts model possesses floating phases X and Y owing to strong thermal fluctuations. Such floating phases do not have any local order parameters. Through the present study, we may remark that the appearance of the re-entrant phenomena in 2D classical lattices is mainly caused by frustrations as well as the geometrical structure of the lattices, which has also close relation with the freedom q of the local spin. Importantly, the existence of the partially ordered state in the ground state may not be the necessary condition for the re-entrant phenomena.

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[1] V. Vaks, A. Larkin, and Y. Ovshinnikov, Sov. Phys. JEPT **22**, 820 (1996).

[2] H. T. Diep, M. Debauche, and H. Giacomini, Phys. Rev. B **43**, 8759 (1991).

[3] M. Debauche, H. T. Diep, P. Azaria, and H. Giacomini, Phys. Rev. B **44**, 2369 (1991); P. Azaria, H. T. Diep, and H. Giacomini, Phys. Rev. Lett. **59**, 1629 (1987).

[4] Wei Li, Shou-Shu Gong, Yang Zhao, Shi-Ju Ran, Song Gao, and Gang Su, Phys. Rev. B **82**, 134434 (2010).

[5] H. T. Diep, *Frustrated Spin Systems* (World Scientific, Singapore, 2004).

[6] E. H. Boubcheur, R. Quartu, H. T. Diep, and O. Nagai, Phys. Rev. B **58**, 400 (1998).

[7] C. Santamaria and H. T. Diep, J. Appl. Phys. **81**, 5276 (1997).

[8] Q. N. Chen, M. P. Qin, J. Chen, Z. C. Wei, H. H. Zhao, B. Normand, and T. Xiang, Phys. Rev. Lett. **107**, 165701 (2011).

[9] D. P. Foster and C. Gérard, Phys. Rev. B **70**, 014411 (2004).

[10] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).

[11] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).

[12] Wei Li, Shi-Ju Ran, Shou-Shu Gong, Yang Zhao, Bin Xi, Fei Ye, and Gang Su, Phys. Rev. Lett. **106**, 127202 (2011).

[13] Luigi Amico, Rosario Fazio, Andreas Osterloh, and Vlatko Vedral, Rev. Mod. Phys. **77**, 513 (2005).

[14] Xin Yan, Wei Li, Yang Zhao, Shi-Ju Ran, and Gang Su, Phys. Rev. B **85**, 134425 (2012).

[15] Guang-Hua Liu, Wei Li, Wen-Long You, G. S. Tian and Gang Su, Phys. Rev. B **85**, 184422 (2012).

[16] Hans. J. Herrmannt and Héctor. O. Mártin, J. Phys. A: Math. Gen. **17**, 657 (1984); D. P. Foster and C. Gérard, *ibid.* **35**, L75 (2002).

[17] J. Kosterlitz and D. Thouless, J. Phys. C **6**, 1181 (1973).

[18] R. Orús and G. Vidal, Phys. Rev. B **78**, 155117 (2008).

[19] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, 2010).