Time and energy in team-based search

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When an object needs to be found in a random environment by a team of searchers, we obtain a formula for the total number of searchers needed if at least k of them must find the object by some large time S. We then compute the energy consumed by the N searchers if they are all stopped as soon as k are successful, and we show that the energy consumed decreases as N increases. We also consider the case in which the successful ones stop but the unsuccessful ones continue until a time-out or until they are destroyed by some other "natural" cause, and in this case we see that the energy consumed increases with N as one might expect. The transform-based analysis used assumes that the searchers' motion is described by diffusion processes, that the search space is infinite and homogeneous, that searchers can be destroyed or become permanently lost as they proceed, and that a time-out mechanism is used so that any searcher that exceeds this time-out and has not succeeded in its quest will be removed and replaced by a new searcher that behaves stochastically and independently of its predecessor.

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I. INTRODUCTION

There are many examples in science and engineering [1–9] that involve search in a random and imperfectly known medium, where the search space is much larger than the dimensions of the searcher and of the object being sought, so that the searcher moves at random in an infinite space based on imperfect knowledge (or bad advice) about which way to go. The searcher can also be destroyed or trapped in a location from which it cannot extract itself. Examples include the motion of a particle toward an oppositely charged site in a random field, proteins searching for a binding site on DNA [10], a search for the destination node in a very large network [11], or robots searching for concealed objects [12]. Multiple searchers [13] increase the chances of success if any one of them may be destroyed. In [14], the average travel time of a packet to a destination in an infinitely large multihop network is obtained using a *mixed* discrete and Brownian motion. Blocking and resending at specific distances from the destination during a search [15], and the analysis of time-outs to reduce the time needed to find the object, were studied for single [16] and multiple [17] diffusive searchers. Other work [18] addresses spatially nonhomogeneous environments, e.g., when search becomes difficult in the vicinity of the object due to attempts to impede the searcher.

A. The model

If Z_t is the non-negative *distance* of a searcher to the object being sought at time $t \geqslant 0$ given that the initial distance at time $t = 0$ is $Z_0 = D \geqslant 0$, the total search time for a single searcher is $T = \inf\{t \geqslant 0 | Z_t = 0\}$. We model $\{Z_t : t \geqslant 0\}$ as a homogeneous diffusion process [19,20] in which the mean change in the searcher's distance to the object being sought in a small time interval Δt is $b\Delta t$, while the variance of the distance traveled by the searcher over the same time interval is $c\Delta t$, where $b = \lim_{\Delta t\to 0} \frac{E[Z_{t+\Delta t}-Z_t|Z_t=z]}{\Delta t}$ and $c = \lim_{\Delta t\to 0} \frac{E[(Z_{t+\Delta t}-Z_t)^2-(E[Z_{t+\Delta t}-Z_t])^2|Z_t=z]}{\Delta t}$. When $b < 0$, on average the searcher gets closer over time to the object being sought. If $b \geqslant 0$, the searcher receives wrong directions or is misled. Also, we assume that the searcher may get lost or be destroyed with probability $\lambda\Delta t$ in a small time interval $[t,t+\Delta t)$, where $\lambda \geqslant 0$ is the destruction rate. A time-out mechanism restarts the search from the initial location if the object is not found by τ time units after the search starts, or after its preceding time-out. After the time-out, an additional delay with an exponentially distributed random variable with parameter μ is incurred before the search restarts. The probability that a single searcher has reached the object being sought by time t is denoted by $G(t|D) \equiv \Pr[T \leqslant t]$ and its probability density function (pdf) is $g(t|D)$. Since every expression is conditioned on the initial distance D, in the sequel we will simply write $G(t|D) = G(t)$ and $g(t|D) = g(t)$. In many applications, both the time and energy needed for a successful search is of interest (predators rely on their stored energy to find their prey, robots run on batteries, and the nodes that forward packets in wireless networks are often battery-operated).

If the searcher consumes energy only while it is *moving* at one energy unit per unit time, this is equivalent to stating that the energy consumed by a searcher J is the same as the time it spends in motion, to the exclusion of epochs spent waiting for a time-out to relaunch it. In [13], N searchers are sent out from the same initial location in a quest for an object located at distance D, each moving independently of the others according to the diffusion process described above, and the resulting mixed partial-ordinary differential equations for a Lévy flight process are solved to compute the average search time.

B. Main results

In this paper, we focus on the case in which at least k out of N searchers must be successful, and we obtain approximate and asymptotic estimates for the search times and exact expressions for the energy consumption. Indeed, in

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hunting or foraging, search may take place as a group effort [1], and k out of N members of a team must be successful. In some communication networks [21], packets are encoded so that the information transmitted is correctly received when at least k out of N transmitted packets arrive at the destination. Thus in this paper we study *both the time and the energy* needed to find an object by at least k searchers out of the N, when searchers may be destroyed, and a "time-out" limits the lifetime of an unsuccessful searcher. The energy consumed is computed (i) when the search is stopped as soon as the first k searchers find the object, and (ii) when the remaining searchers (other than the first k successful ones) continue their search until successful completion or until they are destroyed or stopped by their time-out. The latter case is relevant when the source cannot communicate with searchers or when successful searchers cannot communicate with their peers.

Our main results include a formula for the *total number of searchers that are needed* if at least k of them must find the object by time S, which is expressed as $N_{S,k} \cong \frac{k}{G(S)}$ (cf. Sec. II C) when S is large. In Sec. II A, we show that for large values of time, $G(t)$ is exponentially distributed with parameter $\beta_T = [(\frac{1}{r} + \frac{1}{\mu})e^{[b+\sqrt{b^2+2c(\lambda+r)}]D/c}]^{-1}$.

Then in Sec. III we obtain the energy consumed by the N searchers if they are all stopped as soon as k are successful: we see that the energy consumed decreases as N increases. However, when the successful searchers stop but the unsuccessful ones continue until a time-out or until they are destroyed by some other "natural" cause, we observe that the energy consumed increases with N, as shown in Fig. 3.

II. OBTAINING $G(t)$

The probability that k out of N independent searchers will be successful by time t is obviously

$$G_{k,N}(t) = \binom{N}{k} G(t)^k [1 - G(t)]^{N-k}. \quad (1)$$

We know that the time until return to the origin of a pure diffusion process starting at distance D is [22]

$$g_0(t) = \frac{D}{\sqrt{2\pi c t^3}} e^{-\frac{(D+bt)^2}{2ct}}, \quad (2)$$

where the subscript 0 indicates that the quantity refers to a pure diffusion. The PDF of the searcher's distance z to the destination at time t is

$$f_0(z,t) = \frac{e^{-\frac{b^2 t}{2c}} e^{-\frac{b}{c}(D-z)}}{\sqrt{2\pi c t}} \left[e^{-\frac{(z-D)^2}{2ct}} - e^{-\frac{(z+D)^2}{2ct}}\right] \quad (3)$$

and its cumulative distribution function (CDF) [23] is

$$G_0(t) = \frac{1}{2}\left[\operatorname{erfc}\left(\frac{D+bt}{\sqrt{2ct}}\right) + e^{-2bD/c} \operatorname{erfc}\left(\frac{D-bt}{\sqrt{2ct}}\right)\right], \quad (4)$$

where $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy$ and $b < 0$ is necessary for $E[T_0] = -D/b$ to be finite, and if $b > 0$ there is a nonzero probability that the searcher will never reach the destination:

$$G_0(\infty) = \begin{cases} 1, & b \leqslant 0, \\ e^{-2bD/c}, & b > 0. \end{cases}$$

If $b = 0$, the searcher will almost surely reach the destination, but in a time which is infinite on average. Writing the Laplace transform (LT) of any $\alpha(t)$ as $\bar{\alpha}(s) = \int_0^\infty \alpha(t) e^{-st} dt$, the above quantities yield

$$\bar{g}_0(s) = e^{-(b+\sqrt{b^2+2cs})D/c},$$
$$\bar{f}_0(z,s) = \frac{e^{b(z-D)/c}}{\sqrt{b^2+2cs}}\left[e^{-\frac{\sqrt{b^2+2cs}}{c}|z-D|} - e^{-\frac{\sqrt{b^2+2cs}}{c}(z+D)}\right]. \quad (5)$$

In the model with loss and time-out, let X and Y be the mutually independent random variables representing the time to the next loss and the time to the next time-out, respectively, which are exponentially distributed with parameters λ and r. Then $\gamma_\iota(t)$, the PDF of the duration of a search time until its first interruption, is $\gamma_\iota(t)dt = \Pr[t \leqslant \min(X,Y) \leqslant t + dt, T_0 > t]$ since T_0 is the total search time if there is no interruption and its PDF is given in (2). Therefore,

$$\gamma_\iota(t) = (\lambda + r)e^{-(\lambda+r)t}[1 - G_0(t)],$$
$$\bar{\gamma}_\iota(s) = \frac{\lambda + r}{s + \lambda + r}[1 - \bar{g}_0(s + \lambda + r)]. \quad (6)$$

Search is interrupted randomly several times in this manner, and after each interruption it starts again at the origin after a further delay. The last and hence successful attempt at reaching the destination has a duration whose PDF $\gamma_d(t)dt = \Pr[t \leqslant T_0 \leqslant t + dt, \min(X,Y) > t]$ or

$$\gamma_d(t) = g_0(t)e^{-(\lambda+r)t}, \quad \bar{\gamma}_d(s) = \bar{g}_0(s + \lambda + r). \quad (7)$$

If the searcher is successful in locating the object being sought in its first attempt, then the search time and energy consumption are equivalent. On the other hand, if the search is interrupted at least once, then T will exceed J by the amount of time spent in the wait-for-restart states. Therefore, the joint density of T and J can be obtained by accounting for the possibilities of locating the object being sought in $1,2,\ldots$ attempts while including the time spent in the wait-for-restart states in T but not in J. Let $\phi(x,t)$ be the joint probability density of search time T and energy consumption J: $\phi(x,t)dx\,dt = \Pr[x \leqslant J \leqslant x + dx, t \leqslant T \leqslant t + dt]$. Since each attempt is independent of its predecessors, we have for $t \geqslant x$,

$$\phi(x,t) = \gamma_d(t)\delta(t - x) + \int_0^x \gamma_\iota(y)\psi(t - x)\gamma_d(x - y)dy + \cdots$$

and $\phi(x,t) = 0$ for $t < x$. $\psi(t)$ is the PDF of the time interval between the blocking or time-out of a search and the beginning of a new one (during which no energy is consumed):

$$\psi(t) = \frac{r}{\lambda + r}\mu e^{-\mu t} + \frac{\lambda}{\lambda + r}\int_0^t r e^{-ry}\mu e^{-\mu(t-y)}dy,$$
$$\bar{\psi}(s) = \frac{s + \lambda + r}{\lambda + r}\frac{\mu r}{(s + \mu)(s + r)}. \quad (8)$$

Evaluating the LT of $\phi(x,t)$ with respect to the time variable t yields

$$\bar{\phi}(x,s) = \gamma_d(x)e^{-sx} + \bar{\psi}(s)e^{-sx}\int_0^x \gamma_\iota(y)\gamma_d(x - y)dy + \cdots,$$

and taking the LT with respect to the energy variable x and summing the resulting infinite geometric series, we obtain the two-dimensional LT with complex variables ξ and s,

$$\hat{\phi}(\xi,s) = \frac{\bar{\gamma}_d(s+\xi)}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s+\xi)}. \tag{9}$$

The LT of the PDF of the total search time is

$$\begin{aligned}\bar{g}(s) = \hat{\phi}(0,s) &= \frac{\bar{\gamma}_d(s)}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s)}\\ &= \frac{(s+\mu)(s+r)}{s(s+\mu+r)e^{[b+\sqrt{b^2+2c(s+\lambda+r)}]D/c}+\mu r}.\end{aligned} \tag{10}$$

In the special case in which $b=0$, $\lambda=0$, and $\mu\to\infty$, $\bar{g}(s)$ reduces to the expression obtained in [16].

A. Approximate inversion of the transform

Inversion of the LTs for small and large values of the real variable is performed by taking the limit of the corresponding Laplace variable as it tends to ∞ and 0:

$$\bar{g}(s) \sim \begin{cases} \frac{(s+\mu)(s+r)}{s(s+\mu+r)e^{[b+\sqrt{b^2+2c(s+\lambda+r)}]D/c}}, & s \text{ large},\\ \frac{\mu r}{s(\mu+r)e^{[b+\sqrt{b^2+2c(\lambda+r)}]D/c}+\mu r}, & s \text{ small}.\end{cases}$$

As a consequence, we have

$$g(t) \sim \begin{cases} \gamma_d(t) + \frac{\mu r}{\mu+r}\left[\Gamma_d(t) - \int_0^t \gamma_d(t-\tau)e^{-(\mu+r)\tau}d\tau\right], & t \text{ small},\\ \beta_T e^{-\beta_T t}, & t \text{ large},\end{cases} \tag{11}$$

where $\Gamma_d(t) = \int_0^t \gamma_d(\tau)d\tau$ and

$$\beta_T^{-1} = \left[\frac{1}{r}+\frac{1}{\mu}\right]e^{[b+\sqrt{b^2+2c(\lambda+r)}]D/c}. \tag{12}$$

Similarly for the PDF of the energy consumption $\bar{h}(\xi) \equiv \hat{\phi}(\xi,0)$, we have

$$\bar{h}(\xi) \sim \begin{cases} \frac{\xi+\lambda+r}{\xi e^{[b+\sqrt{b^2+2c(\xi+\lambda+r)}]D/c}}, & \xi \text{ large},\\ \frac{\lambda+r}{\xi e^{[b+\sqrt{b^2+2c(\lambda+r)}]D/c}+\lambda+r}, & \xi \text{ small},\end{cases}$$

so that

$$h(x) \sim \begin{cases} \gamma_d(x) + (\lambda+r)\Gamma_d(x), & x \text{ small},\\ \beta_J e^{-\beta_J x}, & x \text{ large},\end{cases} \tag{13}$$

where

$$\beta_J^{-1} = \frac{e^{[b+\sqrt{b^2+2c(\lambda+r)}]D/c}}{\lambda+r}.$$

B. Numerical inversion

Since it may not be possible to analytically invert $\bar{g}(s)$ and $\bar{h}(\xi)$ for all values of t and x, we perform numerical inversion using MATLAB [24] with the algorithm proposed in [25]. Figure 1 shows that the pdf of the search time and energy consumption using asymptotics and the numerical inversion of [25] agree well for a wide range of delay and energy values.

C. Searchers needed for *k* successes in time *S*

Let T_i, $i=1,\dots,N$, be independent random variables, each of them with a probability distribution function $G(t)$; they represent the time it takes for each of the N searchers to find the object. Let $T_{1,N}\leqslant T_{2,N}\leqslant\cdots\leqslant T_{N,N}$ be the variables T_i rearranged in ascending order, i.e., the corresponding order statistics, and define $G^{-1}(p)=\inf\{t: G(t)\geqslant p\}$, $0<p<1$, the *quantile function* of the distribution of the search time for a single searcher. When N is large, it is known that $T_{\lceil pN\rceil,N}$, the pth sample quantile, is asymptotically normally distributed [26]:

$$T_{\lceil pN\rceil,N} \sim \mathcal{N}\left(G^{-1}(p),\frac{p(1-p)}{N\{g[G^{-1}(p)]\}^2}\right). \tag{14}$$

Thus for large N the distribution of the time for k out of N searchers to be successful tends to a constant equal to the $p\approx k/N$th quantile of $G(t)$. As a consequence, the number of searchers $N_{S,k}$ required to find the object in time S when N is large is given approximately by

$$N_{S,k} \cong \left\lceil \frac{k}{G(S)}\right\rceil. \tag{15}$$

Since convergence to the normal distribution (14) is fast, the expression (15) provides a good approximation even for relatively small $N_{S,k}$. The good agreement between the asymptotic approximation of (15) and $G_{N,k}(S)$ from (1), and the numerical inversion of [25] discussed in Sec. II, are illustrated in Fig. 2.

III. ENERGY EXPENDITURE

To derive the energy required for k out of N independent searchers to locate the object, we need to evaluate the time-dependent pdf for the energy expended by a searcher. Let the random variables S_t and J_t represent the state of a searcher at time $t\geqslant 0$ and its energy consumption up to t, respectively, and define the joint pdf $h_p(x,t)dx = \Pr[x\leqslant J_t\leqslant x+dx, S_t = p]$, $p\in\{d,\iota,a\}$, where we have the following cases:

Case d: The searcher reached the *destination* at some time $\tau\leqslant t$ so that

$$h_d(x,t) = \int_x^t \phi(x,\tau)d\tau, \quad \hat{h}_d(\xi,s) = \frac{\hat{\phi}(\xi,s)}{s}. \tag{16}$$

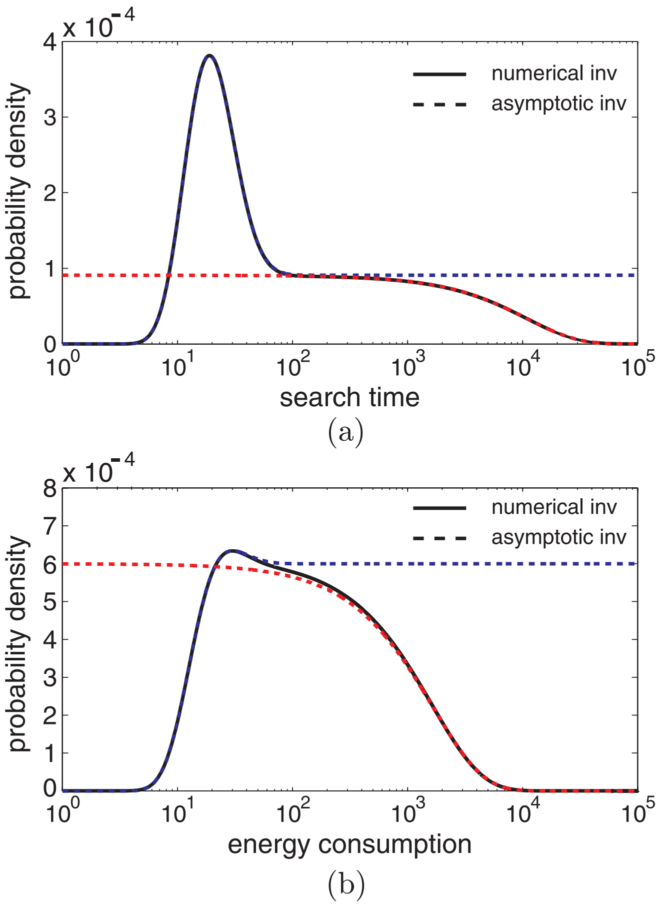

FIG. 1. (Color online) The figure shows the pdf of search time $g(t)$ (above) and energy consumption $h(t)$ (below) when the searcher loss rate is $\lambda = 0.05$, $b = 0.1$, and $c = 1$, average time-out $1/r = 100$, the average time that separates a time-out from the instant when a new searcher is sent out is $1/\mu = 10$, and the initial distance of the object from the searchers is $D = 10$. For this example that has a small value of b and hence relatively high uncertainty in the search direction, the pdf of the search time has a long tail, which is apparent from the logarithmic scale on the horizontal axis.

Case ι: The searcher is *idle* at time t due to a blocking or time-out that has not yet ended:

$$h_\iota(x,t) = \gamma_\iota(x)[1-\Psi(t-x)] + \int_0^x \int_0^{t-x} \gamma_\iota(y_1)\psi(y_2)\gamma_\iota \times (x-y_1)[1-\Psi(t-x-y_2)]dy_2 dy_1 + \cdots,$$

where $\Psi(t) = \int_0^t \psi(\tau)d\tau$. Here the ith term denotes the case in which the time instant t occurs after the search process is suspended i times but before restarting the $(i+1)$th search attempt. Taking the double LT of the above equation yields:

$$\hat{h}_\iota(\xi,s) = \frac{[1-\bar{\psi}(s)]\bar{\gamma}_\iota(s+\xi)}{s[1-\bar{\psi}(s)\bar{\gamma}_\iota(s+\xi)]} = \frac{1}{s}\left[1-\frac{1-\bar{\gamma}_\iota(s+\xi)}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s+\xi)}\right]. \tag{17}$$

Case a: The searcher is *active* (i.e., moving through the search space) at time t:

$$h_a(x,t) = e^{-(\lambda+r)t}[1-G_0(t)]\delta(x-t) + \int_0^x \gamma_\iota(y) \times \psi(t-x)e^{-(\lambda+r)(x-y)}[1-G_0(x-y)]dy + \cdots.$$

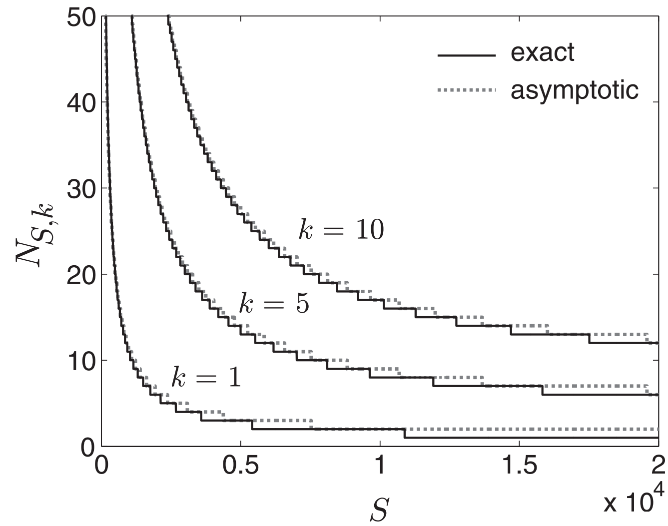

FIG. 2. Comparison of the asymptotic approximation with exact analysis for the total number of searchers $N_{S,k}$ that are required, so that k of them find the object within time S, for different values of k. The parameters are the same as in Fig. 1.

In the first term, no time-out or blocking has occurred up to t and consequently the total energy consumption is equal to t. The ith term corresponds to the case in which at time t the search is ongoing after it was restarted $i-1$ times so that the pdf of the energy utilization up to t is given by the convolution of the pdf of $i-1$ interrupted search periods (each followed by an idle period in which energy is not consumed) and a single search period which does not end before the time instant t. We then end up with

$$\hat{h}_a(\xi,s) = \frac{1-\bar{g}_0(s+\xi+\lambda+r)}{[s+\xi+\lambda+r][1-\bar{\psi}(s)\bar{\gamma}_\iota(s+\xi)]} = \frac{1}{\lambda+r}\frac{\bar{\gamma}_\iota(s+\xi)}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s+\xi)}. \tag{18}$$

Note that $\sum_{p=\{d,\iota,a\}} h_p(x,t) \equiv h(x,t)$ is the pdf of the energy consumed by the searcher up to t:

$$\hat{h}(\xi,s) = \frac{1}{s}\left[1-\frac{\xi}{\lambda+r}\frac{\bar{\gamma}_\iota(s+\xi)}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s+\xi)}\right] \tag{19}$$

and $\lim_{t\to\infty} h(x,t) = h(x)$ as expected.

When search is suspended immediately after the object being sought is found by k searchers, the total energy consumption can be obtained directly using our preceding analysis:

Result 1. Let $J_{k,N}^{-}$ be the total energy consumption up to the time at which a k-subset of N searchers finds the object being sought; the LT of its pdf is given by

$$\bar{h}_{k,N}^{-}(\xi) = \frac{N!}{(k-1)!(N-k)!}\int_0^\infty \bar{h}_d(\xi,t)^{k-1}\bar{\phi}(\xi,t) \times [\bar{h}_\iota(\xi,t)+\bar{h}_a(\xi,t)]^{N-k}dt. \tag{20}$$

Proof. For the total consumption to be equal to x with a search time t, it is necessary that exactly $k-1$, 1, and $N-k$ searchers locate the object being sought in the intervals $[0,t]$, $[t,t+dt]$, and $[t+dt,\infty]$, respectively, and that the energy expended by each individual searcher is at most t while their sum is x. The probabilities that a search succeeds in the three respective intervals while consuming w units of energy up to t are $h_d(w,t)dw$, $\phi(w,t)dw\,dt$, and $[h_\iota(w,t)+h_a(w,t)]dw$; the result then follows by accounting for all possible combinations, convolving with respect to the energy variable (which is equivalent to multiplication in the ξ domain), and integrating over all possible values of t.

On the other hand, if all active but unsuccessful searchers continue searching after the first k successful ones complete their search, we need to know the energy expended by an active searcher up to t and its distance to the object being sought so as to compute the additional energy consumed before it actually stops moving. Define

$$f(z,x,t)dz\,dx = \Pr[x \leqslant J_t \leqslant x+dx, z \leqslant Z_t \leqslant z+dz],$$

which can be derived using $f_0(z,t)$ from (3) as

$$\begin{aligned} f(z,x,t) = e^{-(\lambda+r)t} f_0(z,t)\delta(x-t) + \int_0^x \gamma_\iota(\tau)\psi(t-x) \\ \times f_0(z,x-\tau)e^{-(\lambda+r)(x-\tau)}d\tau \\ + \cdots, \quad x \leqslant t, z > 0, \end{aligned}$$

where the first term is the probability that the searcher reaches distance z in time t without being interrupted so that $x = t$; the second term is the probability that the search is stopped at some time $\tau \in [0,x]$, it is restarted after $t-x$ time units, and distance z is reached during the next search attempt in a time interval $x-\tau$ in which no interruption occurs. More generally, the ith term represents the case in which the time instant t lies in the ith attempt to locate the object, when the searcher is at distance z and has consumed x units of energy. The two-dimensional LT of $f(z,x,t)$ is

$$\hat{f}(z,\xi,s) = \frac{\bar{f}_0(z,s+\xi+\lambda+r)}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s+\xi)}. \tag{21}$$

Note that $f(z,x,t)$ satisfies the following equalities:

$$\int_0^t f(z,x,t)dx = f(z,t), \qquad \int_0^\infty f(z,x,t)dz = h_a(x,t)$$

so that

$$\bar{f}(z,s) = \hat{f}(z,0,s) = \frac{\bar{f}_0(z,s+\lambda+r)}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s)}. \tag{22}$$

Result 2. If there is no mechanism to immediately stop active searchers after the completion of the search, then the pdf of the total energy consumed $J^+_{k,N}$ is given by

$$\begin{aligned} \bar{h}^+_{k,N}(\xi) = \frac{N!}{(k-1)!(N-k)!}\int_0^\infty \bar{h}_d(\xi,t)^{k-1}\bar{\phi}(\xi,t) \\ \times [\bar{h}_\iota(\xi,t)+\bar{h}_c(\xi,t)]^{N-k}dt, \end{aligned} \tag{23}$$

where $h_c(\xi,t)$ is the pdf for the total energy expended by an unsuccessful active searcher until it stops moving after the search ends at t:

$$\hat{h}_c(\xi,s) = \frac{1}{s(\xi+\lambda+r)}\frac{s\bar{\gamma}_\iota(s+\xi)+\xi[\bar{\gamma}_d(\xi)-\bar{\gamma}_d(s+\xi)]}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s+\xi)}.$$

To show this result, notice that the pdf $h^+_{k,N}$ is evaluated in the same manner as $h^-_{k,N}$ except that we use h_c instead of h_a to take account of the additional energy consumed by active searchers upon the completion of the search. More precisely, $h_c(x,t)$ takes the following form:

$$h_c(x,t) = \int_0^\infty\int_0^x f(z,x-u,t)[\gamma_\iota(u|z)+\gamma_d(u|z)]du\,dz, \tag{24}$$

where $\gamma_\iota(.|z)$ and $\gamma_d(.|z)$ are computed as in (6) and (7), respectively, but with the initial distance being z instead of D. Hence, if the object being sought is found at some time t while a searcher is at distance $z>0$ and has consumed $x-u$ units of energy, then the searcher will continue to move and consume additional u units of energy with probability $\gamma_\iota(u|z)du$ if it is interrupted before reaching the destination or $\gamma_d(u|z)du$ otherwise. Finally, $h_c(y,t)$ is obtained by integrating over all possible values of the distance $z>0$ and energy $u\in[0,x]$ at the time instant t, and its two-dimensional LT follows directly.

The total average energy consumption of a k-out-of-N search is

$$\begin{aligned} E[J^\pm_{k,N}] &= -\lim_{\xi\to 0}\frac{d\bar{h}^\pm_{k,N}(\xi)}{d\xi} = -\lim_{\xi\to 0}\frac{N!}{(k-1)!(N-k)!} \\ &\times \int_0^\infty dt\Big[G(t)^{k-1}[1-G(t)]^{N-k}\frac{\partial\bar{\phi}(\xi,t)}{\partial\xi} \\ &+(k-1)G(t)^{k-2}g(t)[1-G(t)]^{N-k}\frac{\partial\bar{h}_d(\xi,t)}{\partial\xi} \\ &+(N-k)G(t)^{k-1}g(t)[1-G(t)]^{N-k-1} \\ &\times\left(\frac{\partial\bar{h}_\iota(\xi,t)}{\partial\xi}+\frac{\partial\bar{h}_*(\xi,t)}{\partial\xi}\right)\Big], \end{aligned}$$

where h_* is substituted by either h_a or h_c, depending on whether active searchers stop or continue after the object being sought is found. The LTs of $\frac{\partial\bar{\phi}(\xi,t)}{\partial\xi}$, $\frac{\partial\bar{h}_d(\xi,t)}{\partial\xi}$, and $\frac{\partial}{\partial\xi}[\bar{h}_\iota(\xi,t)+\bar{h}_*(\xi,t)]$ are, respectively,

$$\begin{aligned} \hat{\phi}'(0,s) &= \frac{-\bar{g}(s)}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s)}\frac{1}{(s+\mu)(s+r)} \\ &\quad\times\left[\mu r\hat{h}_a(0,s)+\frac{s(s+\mu+r)D}{\sqrt{b^2+2c(s+\lambda+r)}}\right], \\ \hat{h}'_d(0,s) &= \frac{\hat{\phi}'(0,s)}{s}, \\ \hat{h}'_\iota(0,s)+\hat{h}'_a(0,s) &= -\frac{\hat{h}_a(0,s)}{s}-\hat{h}'_d(0,s), \\ \hat{h}'_\iota(0,s)+\hat{h}'_c(0,s) &= -\frac{\hat{h}_a(0,s)}{s}-\hat{h}'_d(0,s)+\frac{1}{\lambda+r} \\ &\quad\times\left[\frac{\bar{g}_0(\lambda+r)}{1-\bar{\psi}(s)\bar{\gamma}_\iota(s)}-\bar{g}(s)-\hat{h}_a(0,s)\right]. \end{aligned}$$

Hence only one-dimensional LT numerical inversion operations need to be performed in order to compute $E[J^-_{k,N}]$ and $E[J^+_{k,N}]$.

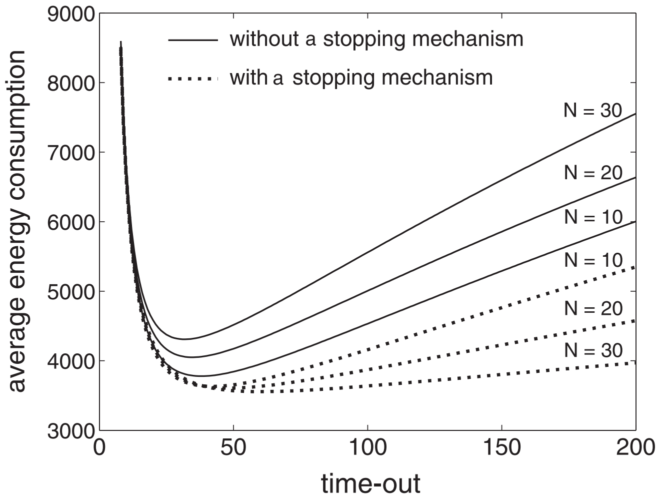

FIG. 3. Average energy consumption *with and without a stopping mechanism* vs time-out $1/r$ for $k=5$ and various values of N, with $b=0.15$, $c=1.25$, $\lambda=0.001$, $\mu=0.1$, and $D=10$. When N increases, the *energy consumed decreases with the stopping mechanism*, and the opposite is true without the stopping mechanism.

In Fig. 3, we show $E[J^-_{k,N}]$ and $E[J^+_{k,N}]$ for $k=5$ versus the time-out $1/r$ for different values of N. The numerical results indicate that the minimum amount of energy consumed until the object is found (i.e., $E[J^-_{k,N}]$) does not vary much with the number of searchers N. However, in the absence of a stopping mechanism, the minimum energy consumed *increases* with N and the time-out value that minimizes the energy expended is smaller. The intuitive but interesting observation is that when N increases, the *energy consumed increases if there is no stopping mechanism*, while the opposite occurs *with* the stopping mechanism.

IV. FUTURE WORK

There are several directions in which we can extend this work. We have assumed that the N diffusive searchers move independently of each other; but there are cases that involve collaborative behavior or use of memory, e.g., a searcher which has exhaustively searched a particular area may leave "negative hints" that would encourage others to move elsewhere. It would also be interesting to generalize the analysis to nonhomogeneous environments and study attacks by a swarm of searchers. Furthermore, approaches using the "multiple class" artefact [27,28] could be used to study how effective search teams could be composed from diverse individuals, and techniques for allocating the work to different searchers [29] could also be considered.

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