Statistical treatment of the electromagnetic radiation-reaction problem: Evaluation of the relativistic Boltzmann-Shannon entropy

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The Vlasov-Maxwell statistical treatment of relativistic charged particles subject to electromagnetic (EM) radiation reaction (RR) represents an unsolved conceptual challenge. In fact, as shown here, the customary point-particle treatment based on the Landau-Lifschitz (LL) equation leads to a generally nonconstant Boltzmann-Shannon (BS) entropy even in the absence of binary collisions. This conclusion appears to be in contradiction with the intrinsic microscopic reversibility of the underlying physical system. In this paper the issue is addressed in the framework of a Hamiltonian treatment for extended charged particles in the presence of EM RR. It is shown that such a behavior actually has no physical ground, being a consequence of the asymptotic approximations involved in the construction of the LL equation. In particular, it is proved that the Hamiltonian structure of the underlying particle dynamics actually restores the conservation of the BS entropy. The connection between the two approaches is analyzed. As a result, it is pointed out that the fulfillment of the entropy law can still be warranted even in the framework of an asymptotic theory by introducing a suitable Hamiltonian approximation for the EM RR equation.

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I. INTRODUCTION

A basic difficulty with the statistical treatment of classical N-body systems based on the Vlasov-Maxwell statistical description lies in the very formulation of relativistic charged particle dynamics in the presence of the electromagnetic radiation reaction (EM RR). In fact, customary treatments are based on intrinsically asymptotic, nonvariational, and non-Hamiltonian approaches, which are exemplified by the Lorentz-Abram-Dirac (LAD) [1–3] or Landau-Lifschitz (LL) [4] equations. Fundamental problems arise when attempting to formulate classical statistical mechanics (CSM) based on these equations. Indeed, the treatment of the relativistic CSM for systems of this type involves a number of important issues, which concern the following:

(1) The evidence of the microscopic reversibility for single-particle dynamics in the presence of EM RR.

(2) The proper definition of a phase space: The problem is relevant for the LAD equation, which is a third-order ordinary differential equation (ODE). In fact, although the construction of kinetic theory is still formally possible [5,6], it involves the adoption of higher-dimensional phase space (e.g., 12-dimensional in the case of the Vlasov equation). As a consequence, the corresponding fluid statistical description remains inhibited because of possibly unphysical mixed acceleration and velocity moments. Therefore, the LAD equation must be discarded *a priori* for this reason.

(3) The lack of flow-preserving measures even in the absence of binary collisions, i.e., when adopting the Vlasov-Maxwell statistical description: This feature is relevant for the LL equation and is due to its non-Hamiltonian character. This implies that the corresponding relativistic Vlasov equation has a nonconservative 4-force.

(4) As shown in Ref. [7], this leads necessarily to a nonvanishing thermodynamic entropy 4-flow.

(5) The very notion of Boltzmann-Shannon (BS) statistical entropy for the kinetic probability density, to be distinguished from the definition of thermodynamic entropy [8,9] and extending the corresponding definition holding for the nonrelativistic theory: In particular, the BS entropy should be identified with a suitable 4-scalar prescribed in the framework of a relativistic treatment of CSM and determine a measure of the ignorance on the classical N-body system.

(6) The behavior of the BS entropy in the case of a kinetic theory based on the asymptotic LL equation.

(7) Finally, the possibility of assuring the exact validity of a constant H-theorem for the BS entropy, to be achieved in the framework of a nonasymptotic theory and in analogy with nonrelativistic systems (i.e., when EM RR can be neglected): If true, this conclusion would be consistent with the microscopic reversibility of the underlying physical system and would be important in order to warrant also the macroscopic reversibility of the N-body system in the absence of mutual particle collisions.

The interesting question is whether these problems can be solved in the framework of an axiomatic theory, capable of restoring at the same time the variational and Hamiltonian characters of relativistic particle dynamics [10]. For definiteness, we shall consider here the case of a flat Minkowski space-time. In this regard, an important contribution has been achieved in a series of recent papers, where a variational formulation for relativistic particle dynamics in the presence of EM RR has been obtained (see Refs. [11–14]). We stress the following in these works: (a) The dynamics of classical finite-size charged particles has been investigated, a feature which allows one to avoid the intrinsic divergences of the EM self-field characteristic of previous point-particle treatments; and (b) on the same grounds, renormalization models (such as the so-called relativistic massive Lorentz electrodynamics [15]) involving heuristic assumptions related to the concept of renormalized mass, charge, and spin as well as their assumed behavior during particle dynamics have been ruled out. Such a type of model, in fact, is based on the adoption of a purely local dynamical description. Indeed, EM RR is an intrinsic nonlocal phenomenon arising specifically due to the finite size of classical particles (see related extended discussions in Refs. [11–14]).

For this purpose, in particular, in the formulation of Refs. [11–14], particles are treated as being quasirigid, spherically symmetric, and with total rest mass m_o and total charge q, whose distributions have the same support on a spherical surface having an invariant radius $\sigma > 0$ (see Ref. [11]). Under these assumptions, it was proved that particle dynamics is variational and can be parametrized with respect to a suitably well-defined point (to be referred to as the center of symmetry) having position and velocity 4-vectors $r^\mu \equiv (ct,\mathbf{r})$ and $u^\mu \equiv \frac{dr^\mu}{ds}$ and proper time s. As a consequence of the symmetry properties of the corresponding Hamilton variational functional, the variational RR equation was proved to admit also both Lagrangian and Hamiltonian formulations in standard form. In particular, the problem was shown to admit an exact Hamiltonian structure in terms of the set $\{\mathbf{y},H_{\text{eff}}\}$, whereby the particle canonical state $\mathbf{y}$ obeys the Hamilton equations in terms of a suitable nonlocal effective Hamiltonian function H_{eff}:

$$\frac{d\mathbf{y}}{ds} = [\mathbf{y},H_{\text{eff}}]. \tag{1}$$

Here the notation is as follows. First, H_{eff} is defined as

$$H_{\text{eff}}(r,P,[r]) \equiv \frac{1}{2m_oc}\left(P_\mu - \frac{q}{c}A_{(\text{eff})\mu}\right)\left(P^\mu - \frac{q}{c}A^\mu_{(\text{eff})}\right), \tag{2}$$

where r and $[r]$ denote, respectively, local and nonlocal spatial dependences in terms of the particle position 4-vector r^μ. Furthermore, $A^\mu_{(\text{eff})}$ is the nonlocal effective EM 4-potential

$$A_{(\text{eff})\mu}(r,[r]) \equiv \overline{A}^{(\text{ext})}_\mu(r) + 2\overline{A}^{(\text{self})}_\mu(r,[r]), \tag{3}$$

with $\overline{A}^{(\text{ext})}_\mu(r)$ and $\overline{A}^{(\text{self})}_\mu(r,[r])$ being, respectively, suitable particle surface averages of the external and self-EM 4-potentials. For a spatially nonrotating extended particle, the latter is defined as

$$\overline{A}^{(\text{self})}_\mu(r,[r]) \equiv 2q\int_1^2 dr'_\mu \delta(\widetilde{R}^\alpha\widetilde{R}_\alpha - \sigma^2), \tag{4}$$

where here and in the rest of the paper the lower and upper extrema of integration "1" and "2" are identified with $\lim_{s\to\pm\infty} r^\mu(s)$ according to the prescription following the discussion reported in Ref. [14]. In addition, $\widetilde{R}^\alpha \equiv r^\alpha(s) - r^\alpha(s')$ represents the displacement bivector between actual and retarded particle positions, $r^\alpha(s)$ and $r^\alpha(s')$, respectively. Both are determined along the particle space-time trajectory and are evaluated at the proper times s and $s' < s$, which are related by the delay-time equation

$$\widetilde{R}^\alpha\widetilde{R}_\alpha - \sigma^2 = 0. \tag{5}$$

Second, the canonical state is identified with $\mathbf{y} = (r^\mu, P_\mu)$, where

$$P_\mu = m_oc\frac{dr_\mu(s)}{ds} + \frac{q}{c}\left[\overline{A}^{(\text{ext})}_\mu + 2\overline{A}^{(\text{self})}_\mu\right] \tag{6}$$

is the conjugate canonical momentum. Therefore, $\mathbf{y}$ spans the eight-dimensional phase-space $\Gamma \equiv \Gamma_r \times \Gamma_u$, where Γ_r and Γ_u are, respectively, the Minkowski M^4-configuration space and the four-dimensional velocity space, both with metric $\eta_{\mu\nu} \equiv \text{diag}(1,-1,-1,-1)$. Finally, $[\eta,\xi] \equiv [\eta,\xi]_{(\mathbf{x})}$ denotes the local Poisson brackets defined in terms of the canonical state $\mathbf{y}$ as

$$[\eta,\xi] = \left(\frac{\partial\eta}{\partial\mathbf{y}}\right)^T \cdot \underline{\underline{\mathbf{J}}} \cdot \left(\frac{\partial\xi}{\partial\mathbf{y}}\right), \tag{7}$$

with all components of $\mathbf{y}$ to be considered independent (i.e., $\mathbf{y}$ as unconstrained). Furthermore, $\underline{\underline{\mathbf{J}}}$ is the canonical Poisson matrix [16], while $\eta(\mathbf{y})$ and $\xi(\mathbf{y})$ denote two arbitrary smooth phase functions. As shown in Ref. [13], such a type of Hamiltonian theory can be extended also to the treatment of finite-size N-body systems subject to nonlocal EM interactions.

Based on these results, in this paper we intend to address the issues indicated above in the framework of an axiomatic formulation of kinetic theory and investigate the possible validity of H-theorems which hold for the BS entropy associated with the probability density function both for the exact Hamiltonian theory and the related asymptotic approximations. The scheme of the paper is as follows. In Sec. II the symmetry properties of the exact RR equation with respect to time-charge-parity (TCP) and time-reversal transformations are addressed, allowing for the proof of microscopic reversibility for the variational formulation of single-particle dynamics in the presence of EM RR. In Sec. III the relevant asymptotic approximations of the exact RR equation are recalled and their physical meaning is discussed. These are shown to satisfy as well the TCP and time-reversal invariance properties of the exact RR equation, thus proving that microscopic reversibility is preserved also by these approximate solutions. In Sec. IV the concept of statistical BS entropy is extended to relativistic CSM and the corresponding entropy production rate is determined for generally nonconservative systems. In Sec. V the BS entropy associated with the probability density for the Vlasov equation is explicitly calculated for the LL and the exact Hamiltonian RR equations. In Sec. VI the connection between the entropy laws holding in the two cases is established and the corresponding physical interpretation is pointed out. In particular, it is shown that the validity of the BS entropy conservation law can be restored even in the framework of an asymptotic treatment of the RR theory by adopting a suitable Hamiltonian approximation for particle dynamics. Finally, concluding remarks are given in Sec. VII.

II. TCP AND TIME-REVERSAL INVARIANCE OF THE EXACT RR EQUATION

In this section we preliminarily analyze the symmetry properties of the exact RR equation for finite-size classical charges obtained in Refs. [11–14] with respect to discrete transformations represented by TCP and time-reversal invariance properties. This analysis is necessary in order to prove the microscopic reversibility of the physical system when EM RR is taken into account.

The symmetry properties of the RR equations (1) can be conveniently deduced from the corresponding variational principle. As shown in Refs. [11–14] the nonlocal Hamilton action functional $S(r,u,[r])$ for classical finite-size charged particles can be represented as

$$S(r,u,[r]) = S_M(r,u) + S_C^{(\mathrm{ext})}(r) + S_C^{(\mathrm{self})}(r,[r]), \tag{8}$$

where S_M, $S_C^{(\mathrm{ext})}$, and $S_C^{(\mathrm{self})}$ are, respectively, the inertial mass and the EM coupling with the external and the self-fields. As said before, here r and u stand for local dependencies with respect to the 4-vector position r^μ and the 4-velocity u^μ, while $[r]$ stands for nonlocal dependencies on r^μ. In particular, the contributions in Eq. (8) are given by

$$S_M(r,u) = \int_{-\infty}^{+\infty} ds\left[m_o c u_\mu \frac{dr^\mu}{ds} - \frac{1}{2} m_o c u_\mu(s) u^\mu(s)\right], \tag{9}$$

$$S_C^{(\mathrm{ext})}(r) = \frac{q}{c}\int_1^2 dr^\mu \overline{A}_\mu^{(\mathrm{ext})}, \tag{10}$$

$$S_C^{(\mathrm{self})}(r,[r]) = \frac{q}{c}\int_1^2 dr^\mu \overline{A}_\mu^{(\mathrm{self})}, \tag{11}$$

where $\overline{A}_\mu^{(\mathrm{self})}$ is defined by Eq. (4) and carries nonlocal spatial dependencies. As proved in Ref. [11], the variational calculation provides a second-order delay-type ODE, which is equivalent to Eq. (1) and is given by

$$m_o c \frac{du_\mu(s)}{ds} = \frac{q}{c}\overline{F}_{\mu\nu}^{(\mathrm{ext})}(r) u^\nu(s) + \frac{q}{c}\overline{F}_{\mu k}^{(\mathrm{self})}(r,[r])\, u^k(s) \tag{12}$$

Here $\overline{F}_{\mu\nu}^{(\mathrm{ext})} \equiv \partial_\mu \overline{A}_\nu^{(\mathrm{ext})} - \partial_\nu \overline{A}_\mu^{(\mathrm{ext})}$ denotes the surface-averaged Faraday tensor associated with the external EM field, while $\overline{F}_{\mu k}^{(\mathrm{self})}(r,[r])$ is given by

$$\overline{F}_{\mu k}^{(\mathrm{self})}(r,[r]) = -2q\int_1^2 dr'_\mu \frac{\partial}{\partial r^k}\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2) + 2q\int_1^2 dr'_k \frac{\partial}{\partial r^\mu}\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2). \tag{13}$$

Hence, the 4-vectors,

$$G_\mu^{(\mathrm{ext})} \equiv \frac{q}{c}\overline{F}_{\mu\nu}^{(\mathrm{ext})}(r) u^\nu(s), \tag{14}$$

$$G_\mu^{(\mathrm{self})} \equiv \frac{q}{c}\overline{F}_{\mu k}^{(\mathrm{self})}(r,[r]) u^k(s), \tag{15}$$

denote, respectively, the external EM force and the self-EM RR force.

Let us now consider the TCP-transformation law, which is defined as follows:

$$\begin{cases} T : t \to -t, \\ C : q \to -q, \\ P : \mathbf{r} \to -\mathbf{r}. \end{cases} \tag{16}$$

This will be intended as acting both on the particle dynamical variables as well as on the external EM potential and the corresponding sources. We notice that the TCP transformation defined by Eq. (16) is effectively equivalent to the $C\Pi$ transformation defined as

$$\begin{cases} C : q \to -q, \\ \Pi : r^\mu \to -r^\mu. \end{cases} \tag{17}$$

Therefore, in order to prove the TCP invariance of the functional $S(r,u,[r])$, it is sufficient to prove its $C\Pi$ invariance. In particular, it is immediate to verify that each term entering the functional (8) is left unchanged under the $C\Pi$ transformation, which must apply the same to both local and nonlocal dependencies. We further notice that in the case of $S_C^{(\mathrm{ext})}(r)$ the 4-vector $\overline{A}^{(\mathrm{ext})\mu}$ is left unchanged because Maxwell's equations are TCP invariant. Since the RR equation (12) is variational, it exhibits the same symmetry properties of the variational functional $S(r,u,[r])$ and therefore it is also TCP invariant according to Eq. (16).

A comment is in order regarding the physical relevance of this conclusion. In fact, in classical mechanics the customary viewpoint is that TCP invariance is a direct consequence of Lorentz symmetry, to be intended as a local property. However, when dealing with nonlocal interactions, as in the case of the EM RR phenomenon, in principle there is no reason to exclude *a priori* possible TCP-invariance violations, while still demanding full Lorentz covariance of the theory. TCP-symmetry-breaking effects might arise due to nonlocal interactions, even in the framework of a covariant theory (as the one developed in Refs. [11–14]). In fact, Lorentz covariance, both at classical and quantum levels, is usually achieved imposing suitable commutation rules with the local generators of the Poincarè algebra [Dirac generator formalism (DGF)]. A crucial point is, however, that, as proved in Ref. [13], DGF becomes incorrect in the case of the nonlocal EM RR interaction. Thus, for example, the so-called Lorentz-symmetry condition, i.e., the commutation rule with the canonical momentum, is not sufficient to warrant the full Lorentz covariance of the theory. As a result, a proper modification of Poincarè algebra is required to deal with nonlocal interactions. This is realized by means of the nonlocal generator formalism developed in Ref. [13]. Therefore, the rigorous result obtained here concerning the TCP invariance of the RR equation is nontrivial.

Let us next investigate the property of time-reversal invariance. In fact, an important issue is whether the exact RR equation is also time-reversal symmetric, namely, it is invariant with respect to the time-reversal transformation

$$t \to Tt = -t, \tag{18}$$

$$\mathbf{r} \to T\mathbf{r} = \mathbf{r}, \tag{19}$$

$$s \to Ts = s, \tag{20}$$

where T is here referred to as the time-reversal operator. For the world line of a particle parametrized in terms of t one has $\mathbf{r}(t) \to \mathbf{r}(-t)$, so that

$$Tr^\mu(t) = -r_\mu(-t), \tag{21}$$

$$T\frac{dr^\mu(t)}{ds} = -\frac{dr_\mu(-t)}{ds}. \tag{22}$$

These transformations imply that the initial conditions for the state $x \equiv [r^\mu(t), u^\mu(t)]$ and $Tx \equiv [Tr^\mu(t), Tu^\mu(t)]$ must be defined consistently, namely, in such a way as to satisfy an existence and uniqueness theorem (see Ref. [11]). From the mathematical point of view, this means that the initial conditions must define a well-posed problem in both cases. As a further consequence, when T acts on the retarded proper time s', and s' is parametrized in terms of the retarded coordinate time t', by definition it is left unchanged, i.e.,

$$Ts'(t') = s'(t'). \tag{23}$$

Under these premises, one can prove that the RR equation (12) is invariant with respect to the time-reversal transformation (18)–(20). The result follows again by inspecting the variational functional $S(r,u,[r])$. In fact, first the inertial contributions in S_M are all invariant thanks to properties (21) and (22). The coupling term with the external EM field $S_C^{(\mathrm{ext})}$ is similarly invariant because, under time reversal, also $A^{\mu(\mathrm{ext})}$ satisfies an analogous transformation property, namely,

$$TA^{\mu(\mathrm{ext})} = -A_{\mu}^{(\mathrm{ext})}. \tag{24}$$

Notice that this equation does not imply at all that $A^{\mu(\mathrm{ext})}$ is odd with respect to time-reversal transformation. Finally, since the coupling term with the EM self-field $S_C^{(\mathrm{self})}$ is by construction a symmetric functional, it is therefore T invariant, too.

These results allow one to conclude that the exact nonlocal RR equation for finite-size classical charged particles is both TCP and time-reversal invariant. This proves the microscopic reversibility of the physical system when the EM RR effect is taken into account within the framework of the variational treatment based on the nonlocal Hamilton action functional $S(r,u,[r])$.

III. ASYMPTOTIC APPROXIMATIONS OF THE RR EQUATION

In this section we address the determination of asymptotic approximations of the exact RR equation for finite-size particles and the investigation of the corresponding symmetry properties which they satisfy. More precisely, this refers to the search for asymptotic approximations holding in the case of slowly varying and smooth external EM forces (in the sense defined in Ref. [12]) and applying in validity of the *short delay-time ordering*. This is obtained formally, requiring

$$0 < \epsilon \equiv \left|\frac{s_{\mathrm{ret}}}{s}\right| \ll 1, \tag{25}$$

where $s_{\mathrm{ret}} = s - s'$ is referred to as delay time, with s and s' denoting, respectively, the present and retarded particle proper times.

In principle, two different Taylor expansions can be performed on the EM self-force. In analogy with Ref. [12], these can be obtained by expanding the self-force either in the neighborhood of s (*present-time expansion*) or s' (*retarded-time expansion*). Although qualitatively similar, the two expansions produce intrinsically different approximations for the RR equation. It is immediate to show (see Ref. [12]) that a prerequisite for both expansions is the validity of the asymptotic approximation

$$s_{\mathrm{ret}} = \sigma[1 + O(\epsilon^2)]. \tag{26}$$

In fact, employing, for example, the present-time expansion, the delay equation (5) requires

$$s_{\mathrm{ret}}^2 - \frac{1}{12}s_{\mathrm{ret}}^4\frac{du^k(s)}{ds}\frac{du_k(s)}{ds} = \sigma^2. \tag{27}$$

Hence, for consistency with Eq. (26), it must be

$$\frac{1}{24}s_{\mathrm{ret}}^2\frac{du^k(s)}{ds}\frac{du_k(s)}{ds} \sim O(\epsilon^4). \tag{28}$$

Estimating the 4-acceleration in terms of the external Lorentz force $G_{\mu}^{(\mathrm{ext})}$ given by Eq. (14) yields the *condition of boundness*:

$$\frac{1}{24}\left(\frac{\sigma}{m_o c}\right)^2\left|G_{\mu}^{(\mathrm{ext})}G^{(\mathrm{ext})\mu}\right| \sim O(\epsilon^4). \tag{29}$$

This result is important because it provides a practical definition of the dimensionless parameter ϵ. We note that, however, the parameter ϵ thus defined may become of $O(1)$ in extreme physical conditions. This might happen, for example, in laser beams characterized by extremely high intensities (e.g., $I \sim 10^{24}$ W cm^{-2}) or in exotic astrophysical environments, such as the so-called magnetars in which the magnetic field intensity can be as high as 10^{14} G. Therefore, the validity of the ordering (26) necessarily excludes these circumstances. In other words, if the condition (26) is violated, necessarily the exact RR equations (1) must be used.

When ϵ can be considered infinitesimal, as pointed out in Refs. [11,12], explicit expressions of the asymptotic approximations for the RR self-force can be determined. In particular, we stress that the requirement (26) is necessary in all cases considered here, and in particular in order to recover the customary form of the LAD equation from the exact RR equation.

We consider first the present-time expansion. In this case, one can show that it leads to an infinite-order differential equation. In particular, by truncating the Taylor expansion to first order in ϵ, the procedure recovers, apart from a mass-renormalization term, the customary expression for the well-known LAD equation [11]. The corresponding expression of the EM self-force for the resulting LAD equation is realized by the asymptotic approximation

$$G_{\mu}|_{\mathrm{LAD}} = g_{(\mathrm{LAD})\mu}[1 + O(\epsilon)], \tag{30}$$

where the leading-order term $g_{(\mathrm{LAD})\mu}$ is given by

$$g_{(\mathrm{LAD})\mu} = -m_{o\mathrm{EM}}c\frac{du_{\mu}(s)}{ds} + h_{(\mathrm{LAD})\mu}. \tag{31}$$

Here, the first term is the present-time mass-renormalization contribution, which is proportional to the EM mass $m_{o\mathrm{EM}} \equiv \frac{q^2}{2c^2\sigma}$. Instead, $h_{(\mathrm{LAD})\mu}$ is the customary LAD 4-vector

$$h_{(\mathrm{LAD})\mu} = \frac{2}{3}\frac{q^2}{c}\left[\frac{d^2u_{\mu}(s)}{ds^2} - u_{\mu}(s)u^k(s)\frac{d^2u_k(s)}{ds^2}\right], \tag{32}$$

which leads to a third-order ODE for the RR equation, in which all of the contributions are evaluated at present proper time s. The connection with the LL equation then follows by employing the one-step reduction process described in Ref. [4], which requires also neglecting mass-renormalization contributions. This procedure, however, is only applicable when the EM self-force can be considered to be a small perturbation with respect to the Lorentz force produced by the external EM field. The corresponding expression for the self-force in this approximation gives $G^{(\mathrm{self})\mu} \cong F_{\mathrm{LL}}^{\mu}$, where

$$F_{\mathrm{LL}}^{\mu} = \frac{2}{3}\frac{q^3}{m_o c^3}\left[u_k u^{\lambda}\partial_{\lambda}F^{(\mathrm{ext})\mu k} + \frac{q}{m_0 c^2}h^{\mu}\right], \tag{33}$$

with

$$h^{\mu} \equiv -F^{(\mathrm{ext})\mu k} F^{(\mathrm{ext})}_{\lambda k} u^{\lambda} + u^{\mu} F^{(\mathrm{ext})}_{\sigma\lambda} u^{\lambda} F^{(\mathrm{ext})\sigma k} u_k. \quad (34)$$

In such a case, by construction, both for LAD and LL equations, delay-time effects are ignored. As remarked in the Introduction, the resulting asymptotic equations are nonvariational and therefore non-Hamiltonian.

Consider next the retarded-time expansion. This generates a second-order delay-type ODE, in which the RR delayed contributions contain now in principle infinite-order derivatives, all evaluated at the retarded time s'. Upon truncation of the expansion to first order in ϵ, as in the case of the LAD equation, this yields the equation discovered in Ref. [12], hereon referred to as the *CT RR equation*. In this case the expression of the EM self-force gives the asymptotic approximation

$$G_{\mu}|_{\mathrm{CT}} = g_{(\mathrm{CT})\mu}[1 + O(\epsilon)], \quad (35)$$

where the leading-order term $g_{(\mathrm{CT})\mu}$ depends on the retarded proper time s' only and is given by

$$g_{(\mathrm{CT})\mu} = -m_{o\mathrm{EM}} c \frac{du_{\mu}(s')}{ds'} + h_{(\mathrm{CT})\mu}. \quad (36)$$

Here, the first term can be interpreted as a retarded-time mass-renormalization contribution, while $h_{(\mathrm{CT})\mu}$ is now the 4-vector

$$h_{(\mathrm{CT})\mu} = \frac{1}{6}\frac{q^2}{c}\frac{d^2 u_{\mu}(s')}{ds'^2} - \frac{2}{3}\frac{q^2}{c} u_{\mu}(s') u^k(s') \frac{d^2 u_k(s')}{ds'^2}. \quad (37)$$

Again, provided the RR self-force $g_{(\mathrm{CT})\mu}$ can be treated as a perturbation of the externally produced Lorentz force, the same iterative procedure invoked to reach the LL equation (starting from the LAD equation) can be invoked also here for the CT RR equation to reduce the order of the derivatives entering $h_{(\mathrm{CT})\mu}$. This yields for $h_{(\mathrm{CT})\mu}$ the controvariant iterative approximation

$$h^{\alpha}_{(\mathrm{CT})} \cong h^{\alpha}_1 + h^{\alpha}_2 + h^{\alpha}_3, \quad (38)$$

where

$$h^{\alpha}_1 \equiv \frac{1}{6}\frac{q^3}{m_o c^3}\frac{\partial \overline{F}^{(\mathrm{ext})\alpha k}}{\partial r^l} u_k(s') u^l(s'), \quad (39)$$

$$h^{\alpha}_2 \equiv -\frac{1}{6}\frac{q^4}{m_o^2 c^5} \overline{F}^{(\mathrm{ext})\alpha l} \overline{F}^{(\mathrm{ext})}_{kl} u^k(s'), \quad (40)$$

$$h^{\alpha}_3 \equiv \frac{2}{3}\frac{q^4}{m_o^2 c^5} \overline{F}^{(\mathrm{ext})}_{kl} u^l(s') \overline{F}^{(\mathrm{ext})km} u_m(s') u^{\alpha}(s'). \quad (41)$$

The resulting second-order delay-type ODE obtained implementing this approximation in the exact RR equation will be referred to as the *reduced CT RR equation*.

A notable feature of both the CT RR and the reduced CT RR equations obtained in this way is that they take into account consistently relativistic finite delay-time effects that are characteristic of the RR phenomenon and, furthermore, are variational and preserve the Hamiltonian structure of the exact RR problem. It should be also remarked that the LAD and CT approximations are intrinsically different and do not match in the limit $s' \to s$. Therefore, contrary to naive interpretations, the two approximations cannot be trivially obtained one from the other by simply exchanging s with s'. In fact, this would correspond to considering the point-particle limit. However, in such a limit, first, the mass-renormalization terms diverge, and second, as proved above, the two asymptotic expressions $h_{(\mathrm{LAD})\mu}$ and $h_{(\mathrm{CT})\mu}$ do not coincide. The present conclusion shows that the point-particle limit must be regarded as unphysical. Nevertheless, it is obvious that the two asymptotic approximations, i.e., the LAD and CT equations (respectively the LL and reduced CT equations), are mutually related by a simple Taylor expansion. In other words, for example, the LAD equation reduces to the CT equation by Taylor expanding the whole 4-vector $g_{(\mathrm{LAD})\mu}$ in the neighborhood of the proper time s'. This provides the connection between the two sets of asymptotic approximations.

To conclude this section, let us address the issue of the symmetry properties with respect to the discrete TCP and time-reversal transformations. Since in all cases the perturbative expansions are performed with respect to particle proper time s, it is obvious that both symmetries remain necessarily preserved. As a consequence it is concluded that both LAD and LL equations as well as the CT and corresponding reduced CT equations keep the correct TCP and time-reversal invariance properties of the exact RR equation. This permits to state that microscopic reversibility is preserved for the dynamics of single charges subject to EM RR also when asymptotic approximations of the RR self-force apply.

IV. THE BOLTZMANN-SHANNON ENTROPY

In this section we introduce the notion of BS entropy in the context of relativistic treatment of CSM, which extends the definition holding in the case of nonrelativistic theory. It is well known that the BS entropy follows from the concept of the ignorance function originally introduced by Shannon in information theory [17] and further developed by Jaynes [18,19]. A basic prerequisite is the axiomatic formulation of the microscopic statistical description for relativistic systems (see Ref. [12]), here briefly recalled for convenience. For definiteness we restrict our analysis to isolated relativistic single charged particles in the absence of binary collisions, as appropriate for the Vlasov-Maxwell statistical treatment. We assume that the particle is identified with a superabundant state vector $\mathbf{x} = (r^{\mu}, u^{\mu})$ spanning the extended eight-dimensional phase space Γ and with essential state variables $\mathbf{x}_1(\mathbf{x})$ spanning the six-dimensional reduced phase space Γ_1 and to be suitably defined (see below). In particular, r^{μ} and u^{μ} are, respectively, the position and velocity 4-vectors associated with a particle characterized by proper time s, so that $u^{\mu} = \frac{dr^{\mu}}{ds}$. It follows that necessarily the invariant set $\Gamma_1 = \Gamma_1(s)$ is defined as

$$\Gamma_1(s) \equiv \{\mathbf{x} : \mathbf{x} \in \Gamma,\ |u| = 1,\ ds = \sqrt{g_{\mu\nu} dr^{\mu} dr^{\nu}}\}, \quad (42)$$

where $|u| \equiv \sqrt{u^{\alpha} u_{\alpha}}$ and s is the world-line proper time uniquely related to any $\mathbf{x}$. Furthermore, we assume that particle dynamics is determined by the flow $T_{s_0,s}$ which is taken of the form

$$\begin{aligned} T_{s_0,s} : \mathbf{x}_0 \equiv \mathbf{x}(s_0) &\to \mathbf{x} \equiv \mathbf{x}(s) \\ &= \chi\left(\mathbf{x}_0, s - s_0, \{\mathbf{x}(s_1) : s_1 \in]-\infty, s_0[\}\right), \end{aligned} \quad (43)$$

with χ being a smooth real function. The flow $T_{s_0,s}$ is generated by a delay ODE of the type

$$\frac{d}{ds}\mathbf{x}(s) = \mathbf{X}\left(\mathbf{x}, \{\mathbf{x}(s_1) : s_1 \in]-\infty, s_0[\}\right), \quad (44)$$

where $\mathbf{X}$ is generally a nonconservative vector field defined as $\mathbf{X} \equiv (u^\mu, F^\mu)$, with F^μ denoting the force 4-vector. Notice that this map does not generally define a dynamical system, since for relativistic systems the state $\mathbf{x}(s)$ can depend on the whole causal past history of the particle through $\{\mathbf{x}(s_1)\}$ (see also related discussions in Ref. [11]). This permits us to introduce a probability measure on Γ_1 to be identified, for an arbitrary subset $B(s) \subseteq \Gamma_1$, with the set function

$$P(B(s)) = \int_\Gamma d\mathbf{x}\rho(\mathbf{x})\delta(|u|-1)\delta(s-s(\mathbf{x}))\delta_{B(s)}(\mathbf{x}), \qquad (45)$$

where $d\mathbf{x} = d^4rd^4u$ is the canonical measure on Γ, $\delta_{B(s)}(\mathbf{x})$ is the characteristic function of $B(s)$, and $\rho(\mathbf{x}) > 0$ is the probability density on Γ_1 and therefore defined so that $P(\Gamma_1) = 1$. In the absence of binary collisions, by assumption $P(B(s))$ must satisfy the axiom of probability conservation, namely, for all $B(s_0)$ and for all $s, s_0 \in I \subseteq \mathbb{R}$,

$$P(B(s)) = P(B(s_0)). \qquad (46)$$

It follows that the probability density $\rho(\mathbf{x})$ must satisfy the integral Vlasov (Liouville) equation

$$\left|\frac{\partial \mathbf{x}(s)}{\partial \mathbf{x}_0}\right| \rho(\mathbf{x}(s)) = \rho(\mathbf{x}(s_0)) \equiv \rho(\mathbf{x}_0), \qquad (47)$$

where $\left|\frac{\partial \mathbf{x}(s)}{\partial \mathbf{x}_0}\right|$ is the Jacobian of the map (43), which, according to the Liouville theorem, is given by

$$\left|\frac{\partial \mathbf{x}(s)}{\partial \mathbf{x}_0}\right| = \exp\left[-\int_{s_0}^{s} ds' \frac{\partial}{\partial u^\mu(s')} F^\mu(s')\right]. \qquad (48)$$

In particular, in the case in which $\mathbf{X}$ is conservative, namely, $\frac{\partial}{\partial u^\mu}F^\mu = 0$, it follows that identically $|\frac{\partial \mathbf{x}(s)}{\partial \mathbf{x}_0}| = 1$. Requiring now $\rho(\mathbf{x}_0)$ to be differentiable, the previous equation, for all $\mathbf{x} \equiv \mathbf{x}(s)$, can be equivalently cast in terms of the generally nonconservative differential Vlasov equation (see Theorem 6 in Ref. [12]):

$$\frac{d}{ds}\left[\left|\frac{\partial \mathbf{x}(s)}{\partial \mathbf{x}_0}\right| \rho(\mathbf{x}(s))\right] = u^\mu \frac{\partial \rho(\mathbf{x})}{\partial r^\mu} + \frac{\partial (F^\mu \rho(\mathbf{x}))}{\partial u^\mu} = 0. \qquad (49)$$

In order to proceed we distinguish here the cases in which $\rho(\mathbf{x})$ is, respectively, (1) a strictly positive ordinary function (stochastic PDF) and (2) a product of a distribution with an ordinary strictly positive function $w(\mathbf{x})$ (partially deterministic PDF). The latter is identified with a PDF of the general form $\rho(\mathbf{x}) = \delta(f(\mathbf{x}) - f(\mathbf{x}(s)))w(\mathbf{x})$, with $f(\mathbf{x})$ being a suitable smooth function of the particle state. An example is realized by letting $f(\mathbf{x}) = x_1$, so that $\rho(\mathbf{x}) = \delta(x_1 - x_1(s))w(\mathbf{x})$, where x_1 identifies one of the components of the particle state and $x_1(s)$ is considered a prescribed function of the proper time s. Correspondingly, we can now introduce the notion of BS entropy $S(\rho(\mathbf{x}))$ associated with $\rho(\mathbf{x})$. In the first case $S(\rho(\mathbf{x}))$ is identified with the 4-scalar

$$S(\rho(\mathbf{x})) = -\int_\Gamma d\mathbf{x}\delta(|u|-1)\delta(s-s(\mathbf{x}))\rho(\mathbf{x}) \ln \frac{\rho(\mathbf{x})}{A(\mathbf{x})}, \qquad (50)$$

with $A(\mathbf{x})$ being a suitable 4-scalar, which coincides with $A(\mathbf{x}) = 1$ in the flat space-time when $r^\mu = (r^0 = ct, \mathbf{r})$ and $\mathbf{r}$ is represented in terms of orthogonal Cartesian coordinates (r^1, r^2, r^3) (see also Ref. [20]). As a consequence it follows by construction that S is a function of s of the form $S(s) \equiv S(\rho(\mathbf{x}(s)))$. It is then immediate to show that, when $A(\mathbf{x}) = 1$, the BS entropies $S(s)$ and $S(s_0)$ are simply related by

$$S(s) = -\int_\Gamma d\mathbf{x}_0\delta(|u_0|-1)\delta(s_0 - s(\mathbf{x}_0)) \times \rho(\mathbf{x}_0) \ln\left[\rho(\mathbf{x}_0)\left|\frac{\partial \mathbf{x}_0}{\partial \mathbf{x}(s)}\right|\right]. \qquad (51)$$

Thanks to the Liouville theorem it follows that

$$S(s) = S(s_0) - \int_\Gamma d\mathbf{x}_0\delta(|u_0|-1)\delta(s_0 - s(\mathbf{x}_0)) \times \rho(\mathbf{x}_0)\int_{s_0}^{s} ds' \frac{\partial}{\partial u^\mu(s')} F^\mu(s'). \qquad (52)$$

This implies that generally, for a nonconservative F^μ, the entropy production rate $\frac{\partial}{\partial s}S(s)$ is nonzero and is given by

$$\frac{\partial}{\partial s}S(s) = -\int_\Gamma d\mathbf{x}\delta(|u|-1)\delta(s-s(\mathbf{x}))\rho(\mathbf{x})\frac{\partial F^\mu}{\partial u^\mu}. \qquad (53)$$

In the following we shall refer to the nonvanishing entropy production rate as the characteristic feature for the macroscopic irreversibility for the dynamics of the N-body system. On the other hand, in the case of a conservative 4-force F^μ one obtains the constant H-theorem

$$\frac{\partial}{\partial s}S(s) = 0, \qquad (54)$$

which is interpreted as the requirement which assures the macroscopic reversibility of the N-body dynamics. Similarly, for a partially deterministic PDF one introduces for the BS entropy the definition

$$S(s) = -\int_\Gamma d\mathbf{x}\delta(|u|-1)\delta(s-s(\mathbf{x})) \times \delta(x_1 - x_1(s))w(\mathbf{x})\ln w(\mathbf{x}), \qquad (55)$$

where we have set $A(\mathbf{x}) = 1$. Also in this case it is immediate to relate $S(s)$ with $S(s_0)$. In fact, from Eq. (47) one gets an integral equation which determines $w(\mathbf{x}(s))$, namely,

$$w(\mathbf{x}_0) = \left|\frac{\partial \mathbf{x}(s)}{\partial \mathbf{x}_0}\right| w(\mathbf{x}(s)). \qquad (56)$$

As a consequence one recovers again an expression analogous to Eq. (52) with $\rho(\mathbf{x}_0) \equiv \delta(x_{1,0} - x_1(s_0))w(\mathbf{x}_0)$.

It is worth noting that the notion of BS entropy introduced here is consistent with the physical interpretation adopted in the context of information theory. As a particular case, let $w(\mathbf{x}) = 1$ and $\delta(f(\mathbf{x}) - f(\mathbf{x}(s))) \equiv \delta(\mathbf{x} - \mathbf{x}(s))$ coincide with the eight-dimensional Dirac-delta function. It follows that $\rho(\mathbf{x})$ coincides with the deterministic PDF, while both $S(s)$ and the corresponding entropy production rate $\frac{\partial}{\partial s}S(s)$ vanish identically. This conclusion implies that indeed $S(s)$ is a measure of the ignorance on the physical system also in the case of relativistic CSM.

V. BS ENTROPY FOR THE RR EQUATION

In this section we determine the BS entropy and the corresponding entropy production rate which is obtained in the case of the RR equation for the EM RR. This corresponds to identify the 4-vector F^{μ} with

$$F^{\mu} = \eta^{\mu\nu}\left(F_{\nu}^{(\mathrm{ext})} + F_{\nu}^{(\mathrm{self})}\right), \tag{57}$$

where $F_{\nu}^{(\mathrm{ext})}$ and $F_{\nu}^{(\mathrm{self})}$ denote the EM forces acting on the charged particle and due respectively to the external and self (i.e., RR) EM fields. Two different representations are considered. The first one relies on the customary point-particle model in the framework of the LL approximation of the RR self-force. In this approximation, the external Lorentz force is identified with

$$F_{\nu}^{(\mathrm{ext})} \equiv \frac{q}{m_o c^2} u^{\alpha} F_{\alpha\nu}^{(\mathrm{ext})}(r), \tag{58}$$

where $F_{\alpha\nu}^{(\mathrm{ext})}(r) \equiv \partial_{\alpha} A_{\nu}^{(\mathrm{ext})}(r) - \partial_{\nu} A_{\alpha}^{(\mathrm{ext})}(r)$ is the Faraday tensor associated with the external EM field produced by the 4-vector potential $A_{\nu}^{(\mathrm{ext})}(r)$. Instead, the self-force is obtained by identifying $F^{(\mathrm{self})\mu}$ with the 4-vector F_{LL}^{μ} given by Eq. (33).

The second representation follows from the Hamiltonian theory presented in Ref. [12] and applies for the treatment of finite-size charge distributions. In this case the external 4-vector force $F_{\nu}^{(\mathrm{ext})}$ and the self 4-force $F_{\nu}^{(\mathrm{self})}$ are identified, respectively, with the 4-vectors $G_{\mu}^{(\mathrm{ext})}$ and $G_{\mu}^{(\mathrm{self})}$ given by Eqs. (14) and (15).

Let us now proceed with the explicit evaluation of the BS entropy and the corresponding entropy production rate in the two representations considered. We first notice that in both cases the only possibly nonvanishing contribution to the entropy production rate can only arise due to the RR self-force. In fact, due to the antisymmetric property of the Faraday tensor associated with the external EM field, in both cases the divergence $\frac{\partial F^{\mu(\mathrm{ext})}}{\partial u^{\mu}}$ vanishes identically. Let us consider next the contribution due to $F_{\mu}^{(\mathrm{self})}$. In the case of the LL theory, after straightforward algebra, one obtains

$$\frac{\partial}{\partial u^{\mu}} F_{\mathrm{LL}}^{\mu} = \frac{2}{3}\frac{q^3}{m_o c^3} u_k \partial_{\mu} F^{(\mathrm{ext})\mu k} - \frac{2}{3}\frac{q^3}{m_o^2 c^5} F^{(\mathrm{ext})\mu k} F_{\mu k}^{(\mathrm{ext})} + 2\frac{q^3}{m_o^2 c^5} u_k u^{\lambda} F^{(\mathrm{ext})k\sigma} F_{\sigma\lambda}^{(\mathrm{ext})}, \tag{59}$$

where in general each contribution on the right-hand side is nonvanishing. In particular, the first term is proportional to the external 4-current density, i.e., $\partial_{\mu} F^{(\mathrm{ext})\mu k} = -\frac{4\pi}{c} J^{(\mathrm{ext})k}$. The second term is instead proportional to the invariant EM 4-scalar $W = F^{(\mathrm{ext})\mu k} F_{\mu k}^{(\mathrm{ext})} = B^2 - E^2$. Finally, the third term is proportional to the square of the external EM force. Due to the independence of these terms, it follows that the divergence of F_{LL}^{μ} is generally nonzero. This implies that the entropy production rate (53) associated with the 1-body PDF $\rho(\mathbf{x})$ is always nonvanishing in the case of the LL treatment, implying a macroscopic irreversibility for the dynamics of the corresponding N-body system.

Let us now analyze in detail the implications of the representation for $F_{\mu}^{(\mathrm{self})}(r,[r])$ given above in Eq. (15) and holding in the case of extended particles. It is immediate to show that the corresponding entropy production rate necessarily vanishes identically. The result is straightforward and follows as a consequence of the Hamiltonian structure of the RR theory recalled in the Introduction and leading to the representation (15). In fact, we notice that, by construction, the BS entropy can be viewed as an observable, which cannot depend on the particular choice of state $\mathbf{x}$ adopted for its representation. Therefore, provided the flow (44) can be represented in canonical form, i.e., there is a diffeomorphism of the form $\mathbf{x} \to \mathbf{y}$, with $\mathbf{y} = (r^{\mu}, P_{\mu})$ being a canonical state, it follows that necessarily $S(s)$ must satisfy the constant H-theorem given by Eq. (54). In the present case the previous transformation is simply determined by the set of equations

$$r^{\mu} = r^{\mu}, \tag{60}$$

$$P_{\mu} = u_{\mu} + \frac{q}{c}\left[\overline{A}_{\mu}^{(\mathrm{ext})} + 2\overline{A}_{\mu}^{(\mathrm{self})}\right], \tag{61}$$

where $\overline{A}_{\mu}^{(\mathrm{self})}$ is the nonlocal EM 4-vector potential of the self-field given in Eq. (4). The result follows in elementary way. In fact, let us start from the expression of BS entropy (50) expressed in terms of the noncanonical state $\mathbf{x}$ and in which $A(\mathbf{x})$ is set equal to one. Then, introducing the transformation $\mathbf{x} \to \mathbf{y}$ and noting that, according to Eqs. (60) and (61), $|\frac{\partial \mathbf{x}}{\partial \mathbf{y}}| = 1$, it follows identically that

$$S(s) \equiv S(\rho(\mathbf{x}(s))) = S_1(\rho_1(\mathbf{y}(s))), \tag{62}$$

where $\rho(\mathbf{x}) = \rho_1(\mathbf{y})$ and $S_1(\rho_1(\mathbf{y}(s)))$ is defined as the BS entropy associated with $\rho_1(\mathbf{y})$, namely,

$$S_1(\rho_1(\mathbf{y}(s))) = -\int_{\Gamma_{\mathbf{y}}} d\mathbf{y}\,\delta(|u|-1)\delta(s-s(\mathbf{y})) \times \rho_1(\mathbf{y}) \ln \rho_1(\mathbf{y}). \tag{63}$$

Then, for all $s,s_0 \in I$, the integral Liouville equation (47) reduces to $\rho_1(\mathbf{y}(s)) = \rho_1(\mathbf{y}(s_0))$. This implies identically that

$$S_1(\rho_1(\mathbf{y}(s))) = S_1(\rho_1(\mathbf{y}(s_0))). \tag{64}$$

Therefore, it follows that in the framework of the Hamiltonian theory of EM RR for extended charges the BS entropy $S(s)$ recovers, as expected, the constant H-theorem (54). This is consistent with the microscopic reversibility proved in Sec. II.

VI. PHYSICAL INTERPRETATION

The results obtained in the previous section raise the problem of the existence of an apparent contradiction between the entropy production rates characterizing the two theories of EM RR considered above. The purpose of this section is to investigate in detail the meaning of the BS entropy in the two cases, determine their relationship, and discuss the physical interpretation which follows from the validity of the corresponding H-theorems. In particular, here we point out that the two results can be reconciled in the framework of the exact theory, which allows one also to get a correct interpretation of the nonvanishing entropy production rate predicted by the LL approximation (see below). In this regard we first notice that a basic difference between the LL theory and the Hamiltonian formulation lies in the realization of the corresponding 1-body PDFs and the related evolution equations. In fact, in the formulation based on the LL approximation, since the theory is non-Hamiltonian, the PDF $\rho(\mathbf{x})$ can only be expressed in terms of a noncanonical state [for example, $\mathbf{x} = (r^{\mu}, u^{\mu})$], while a transformation of the type $\mathbf{x} \to \mathbf{y}$ to a canonical state $\mathbf{y}$ remains forbidden in such a case. In contrast, in the framework of the exact Hamiltonian theory, the PDF admits a functional dependence in terms of the canonical state $\mathbf{y} = (r^\mu, P_\mu)$, which is uniquely determined by Eqs. (60) and (61). As a basic consequence, in the second case the PDF $\rho_1(\mathbf{y})$ carries an implicit contribution due to the EM self 4-potential and arising specifically through the definition of the canonical 4-momentum P_μ. In Ref. [12] it was shown that this kind of functional form is actually crucial for the consistent formulation of both kinetic and fluid theories following from the statistical description based on $\rho_1(\mathbf{y})$. We now proceed investigating how this property affects also the formulation of the H-theorem.

In order to establish the connection between the entropy production rate in the two cases, it is convenient to adopt a perturbative treatment analogous to that adopted in Ref. [12]. This requires to represent the PDF $\rho_1(\mathbf{y})$ in terms of generally noncanonical variables $\mathbf{w}$. For definiteness, let us introduce an arbitrary noncanonical phase-space diffeomorphism from Γ to $\Gamma_\mathbf{w}$, with $\Gamma_\mathbf{w}$ denoting a transformed phase space having the same dimension of Γ,

$$\mathbf{y} \equiv (r^\mu, P_\mu) \rightarrow \mathbf{w} \equiv \mathbf{w}(\mathbf{y}), \tag{65}$$

where, for example, $\mathbf{w}$ can be identified with the noncanonical state $\mathbf{x} \equiv (r^\mu, u_\mu)$. In this case the transformation follows from Eq. (61) and is realized by

$$r^\mu = r^\mu, \tag{66}$$

$$u_\mu = P_\mu - \frac{q}{c}\left[\overline{A}^{(\mathrm{ext})}_\mu + 2\overline{A}^{(\mathrm{self})}_\mu\right], \tag{67}$$

while the relativistic nonlocal RR equations are given again by Eq. (44), where now $F_\mu = \frac{\partial p_\mu}{\partial r^\nu} u^\nu - \frac{\partial u_\mu}{\partial P_\nu}\frac{\partial H_{\mathrm{eff}}}{\partial r^\nu}$ is the exact EM force, including both external and self-contributions. In the LL theory the exact EM force is expressed in terms of the asymptotic approximation

$$F^\mu = F^{(\mathrm{ext})\mu} + F^\mu_{\mathrm{LL}} + \Delta F^\mu, \tag{68}$$

where $F^{(\mathrm{ext})\mu}$ and F^μ_{LL} are defined by Eqs. (58) and (33), while ΔF^μ is the difference between the exact and approximate EM forces, which is assumed to be a small perturbation. Incidentally, this viewpoint is consistent with the assumptions underlying the LL theory, where indeed the EM self-force is treated perturbatively. As a consequence, in the leading-order approximation, the noncanonical state $\mathbf{x}$ satisfies now the asymptotic LL equation

$$\frac{dr^\mu}{ds} = u^\mu, \tag{69}$$

$$\frac{du^\mu}{ds} = F^{(\mathrm{ext})\mu} + F^\mu_{\mathrm{LL}}. \tag{70}$$

The perturbative approach requires introducing for the canonical state $\mathbf{y} = (r^\mu, P_\mu)$ the representation

$$\mathbf{y} = \mathbf{y}_{\mathrm{nc}} + \Delta\mathbf{y}, \tag{71}$$

where $\mathbf{y}_{\mathrm{nc}} \equiv (r^\mu, p_\mu)$ is the particle state whose dynamics is determined by the LL equations (69) and (70), with $p_\mu \equiv u_\mu + \frac{q}{c}A^{(\mathrm{ext})}_\mu$, while $\Delta\mathbf{y} \equiv \mathbf{y} - \mathbf{y}_{\mathrm{nc}}$ is assumed to be a small perturbation. When this condition is realized, a Taylor expansion for the PDF $\rho_1(\mathbf{y})$ becomes possible. To first order this yields

$$\rho_1(\mathbf{y}) \simeq \rho_1(\mathbf{y}_{\mathrm{nc}}) + \Delta\rho_1(\mathbf{y},\mathbf{y}_{\mathrm{nc}}), \tag{72}$$

where

$$\Delta\rho_1(\mathbf{y},\mathbf{y}_{\mathrm{nc}}) \equiv \Delta\mathbf{y}\left.\frac{\partial\rho_1(\mathbf{y})}{\partial\mathbf{y}}\right|_{\mathbf{y}=\mathbf{y}_{\mathrm{nc}}} \tag{73}$$

is the first-order term of the series. When the representation (72) is substituted in the definition (50) of BS entropy and we let $A(\mathbf{x}) = 1$, the following asymptotic approximation is obtained:

$$S_1(\rho_1(\mathbf{y})) \cong S_1(\rho_1(\mathbf{y}_{\mathrm{nc}})) + \Delta S_1(\rho_1(\mathbf{y},\mathbf{y}_{\mathrm{nc}})), \tag{74}$$

where, correct to first order in the perturbative expansion,

$$S_1(\rho_1(\mathbf{y}_{nc})) = \int_{\Gamma_\mathbf{y}} d\mathbf{y}_{\mathrm{nc}}\delta(|u|-1)\delta(s-s(\mathbf{y}_{\mathrm{nc}})) \times \rho_1(\mathbf{y}_{\mathrm{nc}})\ln\rho_1(\mathbf{y}_{\mathrm{nc}}), \tag{75}$$

$$\begin{aligned}\Delta S_1(\rho_1(\mathbf{y},\mathbf{y}_{\mathrm{nc}})) = &\int_{\Gamma_\mathbf{y}} d\mathbf{y}_{\mathrm{nc}}\delta(|u|-1)\delta(s-s(\mathbf{y}_{\mathrm{nc}}))\\ &\times\Delta\rho_1(\mathbf{y},\mathbf{y}_{\mathrm{nc}})[1+\ln\rho_1(\mathbf{y}_{\mathrm{nc}})]\\ &+\int_{\Gamma_\mathbf{y}} d\mathbf{y}_{\mathrm{nc}}\left|\frac{\partial\Delta\mathbf{y}}{\partial\mathbf{y}_{\mathrm{nc}}}\right|\delta(|u|-1)\rho_1(\mathbf{y}_{\mathrm{nc}})\\ &\times\delta(s-s(\mathbf{y}_{\mathrm{nc}}))\ln\rho_1(\mathbf{y}_{\mathrm{nc}}).\end{aligned} \tag{76}$$

One can prove that the first term of $\Delta S_1(\rho_1(\mathbf{y},\mathbf{y}_{\mathrm{nc}}))$ vanishes identically, thus indicating that to this order $\Delta\rho_1(\mathbf{y},\mathbf{y}_{\mathrm{nc}})$ does not contribute explicitly to the entropy correction ΔS_1. Hence, in this approximation the previous equation reduces to

$$\Delta S_1(\rho_1(\mathbf{y},\mathbf{y}_{\mathrm{nc}})) = \int_{\Gamma_\mathbf{y}} d\mathbf{y}_{\mathrm{nc}}\left|\frac{\partial\Delta\mathbf{y}}{\partial\mathbf{y}_{\mathrm{nc}}}\right|\delta(|u|-1)\rho_1(\mathbf{y}_{\mathrm{nc}}) \times\delta(s-s(\mathbf{y}_{\mathrm{nc}}))\ln\rho_1(\mathbf{y}_{\mathrm{nc}}). \tag{77}$$

As a consequence of Eq. (64) one finds that the entropy production rate arising in the LL approximation (for the RR self-force) is only due to the nonconservation of the phase-space volume in this description, namely, due to the Jacobian $|\frac{\partial\Delta\mathbf{y}}{\partial\mathbf{y}_{\mathrm{nc}}}|$ being generally different from unity. This provides a simple interpretation of the simultaneous validity of the two H theorems holding in the two cases discussed here. In particular, it shows that the nonvanishing entropy production rate characterizing the LL theory is intrinsically related to its non-Hamiltonian character.

The real issue is whether this conclusion is merely incidental or has an actual physical meaning related in some sense to the actual EM RR interaction. The answer to this question is of course of basic importance. Its precise formulation can again be shown to follow from the Hamiltonian theory developed in Ref. [12]. In fact, in validity of the short delay-time ordering (25), it has been shown that the adoption of the retarded-time (rather than the present-time) expansion permits one to recover a nonlocal asymptotic approximation of the exact EM RR force (CT and reduced CT equations). Such an approximation remarkably preserves the Hamiltonian structure of the exact theory in terms of the set $\{\mathbf{z}, H_{\mathrm{asym}}\}$ with canonical equations of the form

$$\frac{d\mathbf{z}}{ds} = [\mathbf{z}, H_{\mathrm{asym}}]. \tag{78}$$

Here the vector $\mathbf{z} \equiv (r^\mu, \pi_\mu)$ spanning the eight-dimensional phase space $\Gamma_\mathbf{z}$ is the canonical state defined with respect to the non-local Hamiltonian function

$$H_{\rm asym}(r,\pi,r_0') = \frac{1}{2m_o c}\left(\pi_\mu - \frac{q}{c}\overline{A}^{(\rm ext)}_\mu(r)\right) \times \left(\pi^\mu - \frac{q}{c}\overline{A}^{(\rm ext)\mu}(r)\right) + g_{(\rm CT)\mu}(r(s'))r^\mu, \tag{79}$$

while the canonical momentum is now

$$\pi_\mu = m_0 c\frac{dr_\mu(s)}{ds} + \frac{q}{c}\overline{A}^{(\rm ext)}_\mu(r). \tag{80}$$

In addition, notice that here the 4-vector $g_{(\rm CT)\mu}(r(s'))$ depends only on the extremal particle world line $r(s')$ at the retarded proper time s' and is given by Eq. (36) above. In this Hamiltonian approximation, denoting by $\rho_2(\mathbf{z})$ the PDF which evolves by means of the integral Vlasov equation,

$$\rho_2(\mathbf{z}(s)) = \rho_2(\mathbf{z}(s_0)) \equiv \rho_2(\mathbf{z}_0), \tag{81}$$

it follows that the BS entropy associated with the same PDF,

$$S_2(\rho_2(\mathbf{z})) = -\int_{\Gamma_z} d\mathbf{z}\delta(|u|-1)\delta(s-s(\mathbf{z}))\rho_2(\mathbf{z})\ln\rho_2(\mathbf{z}), \tag{82}$$

is again identically conserved (as in the exact Hamiltonian formulation). This means that the difficulty inherited by the LL approximation should be considered purely incidental and not be interpreted as a physically meaningful property.

VII. CONCLUSIONS

In the Vlasov-Maxwell statistical description of classical N-body systems, the customary approach to the EM RR problem based on the relativistic Landau-Lifschitz (LL) equation gives rise to a generally nonvanishing BS entropy production rate. In the framework of classical statistical mechanics, when binary collisions are negligible, this represents a paradox. In fact, the conclusion appears in contradiction with the microscopic reversibility of the underlying classical system. The dilemma is whether such a macroscopic irreversibility of the dynamics of the N-body system is an intrinsic feature of the EM RR or is merely a consequence of the asymptotic approximations adopted in the LL equation. To address the issue, in this paper we have first considered the symmetry properties of the RR equation with respect to both TCP and time-reversal transformations. It has been proved that these invariance features hold for the single-particle dynamics both for the exact variational and Hamiltonian RR equation as well as for its relevant asymptotic approximations, including the LL treatment. This makes possible to assure the microscopic reversibility when nonlocal EM RR effects are taken into account. Then, the notion of BS entropy and related entropy production rate have been properly extended to relativistic systems and formulated in the context of classical statistical mechanics (CSM). An axiomatic approach of the EM RR problem earlier developed has been adopted, which allows one to determine the exact Hamiltonian structure of single-particle dynamics subject to external and self EM fields. The BS entropy production rates predicted by the Hamiltonian and LL approaches have been explicitly determined and compared. In particular, in contrast to the Hamiltonian theory, the LL equation has been shown to lead to a generally nonvanishing entropy production rate. A discussion concerning the physical interpretation of the result has been provided, showing that the paradox only arises due to the intrinsic non-Hamiltonian character involved in the LL approximate treatment of the EM RR. As shown here, this behavior can be avoided by making use either of the exact Hamiltonian approach or introducing a suitable asymptotic Hamiltonian approximation.

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