

Fluctuations in multiplicative systems with jumps

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Fluctuation properties of the Langevin equation including a multiplicative, power-law noise and a quadratic potential are discussed. The noise has the Lévy stable distribution. If this distribution is truncated, the covariance can be derived in the limit of large time; it falls exponentially. Covariance in the stable case, studied for the Cauchy distribution, exhibits a weakly stretched exponential shape and can be approximated by the simple exponential. The dependence of that function on system parameters is determined. Then we consider a dynamics which involves the above process and obeys the generalized Langevin equation, the same as for the Gaussian case. The resulting distributions possess power-law tails, which fall similarly to those for the driving noise, whereas central parts can assume the Gaussian shape. Moreover, a process with the covariance $1/t$ at large time is constructed, and the corresponding dynamical equation is solved. Diffusion properties of systems for both covariances are discussed.

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I. INTRODUCTION

Trajectories encountered in complex systems often reveal discontinuities, and the probability distributions are not governed by equations with local operators. Those distributions differ from the Gaussian and contain slowly falling tails. Lévy stable distributions with tails $\sim |x|^{-1-\alpha}$, where $\alpha \in (0, 2)$, are distinguished due to the generalized central limit theorem. However, the divergent variance makes the Lévy stable distributions problematic in some physical applications; it may imply, for example, infinite kinetic energy. Similarly, covariance functions for the Lévy stable processes with $\alpha < 2$ do not exist. The above difficulties do not emerge if the tails, being still of the power form, fall faster than for the Lévy stable distributions. In fact, tails of the form $|x|^{-\beta}$, where $\beta \geq 3$, are frequently observed. This is the case for the financial market that possesses typical characteristics of the complex system, and then some of its properties may be universal. Analyses of returns of stock indices show that a cumulative distribution of returns is power law with $3 < \beta < 4$ [1,2] and such values of β are required by the optimal market strategy [3]. The minority game implies a similar value, $\beta = 3.9$ [4]. Since large jumps represent extreme events, one can expect that first passage time probability should obey the Weibull distribution. However, a phenomenological analysis of the empirical data demonstrates that it is the case only for small returns; the large ones are of the power form with $\beta = 3.32$ [5]. Moreover, fast falling power-law tails result from a multifractal analysis of the extreme events [6], characterizing the hydraulic conductivity in the porous media and the atmospheric turbulence [7].

The variance becomes finite when we modify the asymptotics of the Lévy stable distribution by introducing either a simple cutoff or some fast-falling tail. Such truncated distributions very slowly [8] converge with time to the normal distribution. Moreover, dynamical systems stimulated by the Lévy stable noise possess finite moments of the stationary distribution if the particle is trapped inside a potential well with a sufficiently large slope [9]. The Langevin equation with a multiplicative Lévy noise η and a linear deterministic force also predicts finite moments, if interpreted in the Stratonovich sense; this property was demonstrated by numerical simula-

tions [10,11]. On the other hand, we can define the colored noise $\eta_c(t)$ by the generalized Ornstein-Uhlenbeck process:

$$d\eta_c(t) = -\gamma_n \eta_c(t) dt + \gamma_n dL(t), \quad (1)$$

where the increments of $L(t)$ have the stable Lévy distribution and take the white-noise limit, $\eta(t) = \lim_{\gamma_n \rightarrow \infty} \eta_c(t)$. The stochastic equation with the white noise $\eta(t)$, given by the above expression, allows us to change the variable in the usual way and obtain the Stratonovich result [12]. The generalized Wiener process with that noise as a driving force is characterized by the subdiffusive motion. Taking into account that the mass in the Langevin equation is finite modifies the slope of the tails: it diminishes with the inertia and finally converges to the result of the Itô interpretation for the infinite mass. Multiplicative non-Gaussian white noises serve to describe population and ecological problems: a dynamics of two competing species [13,14] and a population density in terms of the Verhulst model [15,16].

Long tails of the distributions in complex systems are often accompanied by a long memory [4,17], and then time-dependence of the fluctuations becomes nontrivial. The well-known multifractal structure of the financial time series [18] is attributed to both fat tails, that fall as a power law but faster than for the stable Lévy, and power-law correlations [19]. A slow decay of the correlations, even slower than a power law, is regarded as a necessary condition of the multifractality in any complex system [20]. In the present paper, we demonstrate that the generalized Ornstein-Uhlenbeck process $\xi(t)$, driven by $\eta(t)$, possesses a well-determined covariance and fat tails of the distribution. Moreover, trajectories $\xi(t)$ preserve a typical feature of the Lévy flights: smooth segments interrupted by large, rare jumps.

In the presence of the memory effects, a description in terms of the standard Langevin equation with a colored noise is problematic since a response of the system to the stochastic stimulation is not instantaneous and, as a consequence, a retarded friction must be introduced. The fluctuation-dissipation relation requires that the equipartition energy rule is satisfied, i.e., the temperature is well defined, and a retarded friction kernel is uniquely determined by the noise covariance; then

$\xi(t)$ is called the internal noise. Obviously, that relation does not hold for the Lévy noise $L(t)$ due to the infinite covariance [21] and $L(t)$ can be regarded only as an external noise. The dynamical equation with the retarded friction, the generalized Langevin equation (GLE), is well known for the Gaussian noise. Then, for a given noise covariance, it allows us to determine all the fluctuations, which, on the other hand, is not possible for non-Gaussian noises like $\xi(t)$. This process is interesting due to its jumping structure, convergent variance, and nontrivial distributions resulting from GLE driven by $\xi(t)$. In this paper, we discuss those distributions for two different memory kernels: exponential and $1/t$.

The paper is organized as follows. In Sec. II, the autocorrelation function for the Ornstein-Uhlenbeck process with the multiplicative Lévy noise is derived both for the stable and truncated distribution. Section III is devoted to GLE driven by that process for the Cauchy distribution: the probability density distributions are simulated and the resulting fluctuations are compared with general analytical predictions. A similar analysis is performed for the case of the power-law covariance. Results are summarized in Sec. IV.

II. AUTOCORRELATION FUNCTION FOR THE MULTIPLICATIVE ORNSTEIN-UHLENBECK PROCESS

Dynamics of a massless particle subjected to a stochastic force and a linear deterministic force is determined by the following Langevin equation:

$$\dot{\xi}(t) = -\gamma\xi(t) + G(\xi)\eta(t), \quad (2)$$

where the ξ -dependent noise intensity $G(\xi)$ accounts for a nonhomogeneous form of the stochastic activation. We assume that increments of the stochastic force, $\eta(t)$, possess the stable and symmetric Lévy distribution defined by a characteristic function $\exp(-K^\alpha |k|^\alpha)$, where α ($0 < \alpha \leq 2$) is a Lévy index and $G(\xi) = |\xi|^{-\theta}$. The system Eq. (2) resolves itself to the ordinary Ornstein-Uhlenbeck process if $\theta = 0$ and $\alpha = 2$. In the multiplicative case, we must settle the stochastic integral interpretation that is decisive for the existence of the second moment: in the Itô interpretation the asymptotic distribution of ξ is the same as for η , whereas in the Stratonovich one the multiplicative factor essentially modifies the tail [10]. The latter interpretation applies when the white noise is regarded as a limit of the correlated noise and the inertia is small [12]. Then the Langevin equation (2) can be reduced to an equation with the additive noise by a simple change of the variable:

$$y = \frac{1}{K(1+\theta)} |\xi|^{1+\theta} \text{sgn}(\xi). \quad (3)$$

The Fokker-Planck equation in the new variable takes the form

$$\frac{\partial}{\partial t} p(y,t) = \gamma(1+\theta) \frac{\partial}{\partial y} [yp(y,t)] + \frac{\partial^\alpha}{\partial |y|^\alpha} p(y,t), \quad (4)$$

and its solution, after transformation to the original variable, reads

$$p(\xi,t) = \frac{1+\theta}{\alpha|\xi|} H_{2,2}^{1,1} \left[\frac{|\xi|^{1+\theta}}{K(1+\theta)\sigma(t)^{1/\alpha}} \middle| \begin{matrix} (1, 1/\alpha), (1, 1/2) \\ (1, 1), (1, 1/2) \end{matrix} \right], \quad (5)$$

where

$$\sigma(t) = \frac{1 - \exp[-\gamma\alpha(1+\theta)t]}{\gamma\alpha(1+\theta)} \quad (6)$$

and the initial condition $p(\xi,0) = \delta(\xi)$ has been assumed. Asymptotics of Eq. (5) is a power law: $p(\xi,t) \sim |\xi|^{-1-\alpha-\alpha\theta}$ ($|\xi| \gg 1$). In the limit $t \rightarrow \infty$, the system reaches a stationary state which is characterized by the variance

$$\langle \xi^2 \rangle = -\frac{2}{\pi} K^{2/(1+\theta)} \alpha^{-1-2/\alpha(1+\theta)} (1+\theta)^{(2\alpha-2)/\alpha(1+\theta)} \gamma^{-2/\alpha(1+\theta)} \times \Gamma\left(-\frac{2/\alpha}{1+\theta}\right) \Gamma\left(1 + \frac{2}{1+\theta}\right) \sin\left(\frac{\pi}{1+\theta}\right) \quad (7)$$

if $\theta > 2/\alpha - 1$ ($\alpha < 2$).

For $\alpha = 2$, a formal expression for the correlation function can be derived by means of an expansion of the general Fokker-Planck equation solution into its eigenfunctions [22]. The asymptotic behavior appears exponential, and the rate is given by the lowest eigenvalue. However, this conclusion may be wrong if a continuous spectrum is not negligible. This happens, for example, for the linear problem ($\theta = -1$), and then the exponential is modified by an algebraic term [23]. The system Eq. (2) for $\alpha < 2$ and with $\theta = 0$ has a finite relaxation time, but its quantification in terms of the covariance function is possible only after either a modification of this quantity [24] or by introducing a cutoff in the distribution. In the case of the multiplicative noise, for $\theta > 2/\alpha - 1$, the autocorrelation function exists and can be expressed by the integral

$$\begin{aligned} C(t) &= \langle \xi(0)\xi(t) \rangle = \iint \xi_1 \xi_2 p(\xi_2, t; \xi_1, 0) d\xi_1 d\xi_2 \\ &= \iint \xi_1 \xi_2 p(\xi_2, t | \xi_1, 0) p(\xi_1) d\xi_1 d\xi_2, \end{aligned} \quad (8)$$

where $p(\xi) = \lim_{t \rightarrow \infty} p(\xi, t)$. In terms of the transformed variables, $C(t)$ assumes the following form:

$$\begin{aligned} C(t) &= [K(1+\theta)]^{\frac{2}{1+\theta}} \iint |y_1|^{\frac{1}{1+\theta}} |y_2|^{\frac{1}{1+\theta}} \text{sgn}(y_1) \text{sgn}(y_2) \\ &\quad \times p(y_2, t | y_1, 0) p(y_1) dy_1 dy_2. \end{aligned} \quad (9)$$

The conditional probability in Eq. (9) is given by

$$\begin{aligned} p(y_2, t | y_1, 0) &= \frac{1}{K\alpha\sigma(t)^{1/\alpha}} H_{2,2}^{1,1} \left[\frac{1}{K\sigma(t)^{1/\alpha}} |y_2 - y_1 e^{-\gamma_\theta t}| \right. \\ &\quad \left. \times \begin{matrix} (1 - 1/\alpha, 1/\alpha), (1/2, 1/2) \\ (0, 1), (1/2, 1/2) \end{matrix} \right], \end{aligned} \quad (10)$$

where $\gamma_\theta = \gamma(1+\theta)$, whereas

$$\begin{aligned} p(y_1) &= \frac{(\alpha\gamma_\theta)^{1/\alpha}}{K\alpha} H_{2,2}^{1,1} \\ &\quad \times \left[\frac{(\alpha\gamma_\theta)^{1/\alpha}}{K} |y_1| \middle| \begin{matrix} (1 - 1/\alpha, 1/\alpha), (1/2, 1/2) \\ (0, 1), (1/2, 1/2) \end{matrix} \right]. \end{aligned} \quad (11)$$

The integral Eq. (9) can be estimated in the long-time limit; details of the derivation are presented in Appendix A. The final expression reads

$$C(t) \approx 4K^{\frac{3}{1+\theta}} (1+\theta)^{\frac{2}{1+\theta}} \alpha^{\frac{1}{\alpha} \frac{\theta}{1+\theta} - 1} \gamma_\theta^{-\frac{1}{\alpha} \frac{1}{1+\theta}} \frac{\Gamma(\frac{\theta/\alpha}{1+\theta})}{\Gamma(\frac{\theta/2}{1+\theta})} I e^{-\gamma_\theta t}, \quad (12)$$

where I is given by Eq. (A4). However, the expansion Eq. (A1) contains infinite terms if $\alpha < 2$; in particular, the asymptotic expansion of the Fox function in Eq. (A4) produces an integrand tail of the form $y_1^{1/(1+\theta)-\alpha}$ and then I diverges for any θ if $\alpha \leq 1$. Therefore, the approximation Eq. (12) is not valid for the general stable distributions. However, one can argue that in many systems very long jumps do not emerge and it is reasonable to introduce a truncation of the distribution. Such a truncation can be realized as a simple cutoff or by inserting a fast-falling tail; typical forms are the exponential [25] and a power law [26,27]. Systems involving the multiplicative Lévy noise were considered from that point of view in [28]. Convergence to the normal distribution, expected in this case, is so slow that it is not observed in the numerical simulations. The cutoff at some value of $|\eta|$ implies a finite upper integration limit in Eq. (A4), and I becomes convergent. Then the autocorrelation function falls exponentially with time; the rate does not depend on α and rises with θ . The above result is valid also for the additive noise, $\theta = 0$.

The following analysis is restricted to the Cauchy distribution of the noise $\eta(t)$ ($\alpha = 1$). The integral Eq. (9) for the case without the truncation has been evaluated numerically. Inserting the conditional probability

$$p(y_2, t | y_1, 0) = \frac{K^2}{\pi \gamma \theta} \frac{1 - \exp(-\gamma \theta t)}{[y_2 - y_1 \exp(-\gamma \theta t)]^2 + K^4 [1 - \exp(-\gamma \theta t)]^2 / \gamma \theta^2} \quad (13)$$

to Eq. (9) yields the expression for $\mathcal{C}(t)$; it is presented in Fig. 1 for some values of θ . The figure reveals a stretched exponential shape, $\exp(-\lambda t^\beta)$, and the parameter β rises monotonically from 1.040 for $\theta = 1.5$ to 1.081 for $\theta = 5$. Since β is close to 1, deviation from the simple exponential emerges only for very small values of $\mathcal{C}(t)$ (large t) and/or large θ . Therefore,

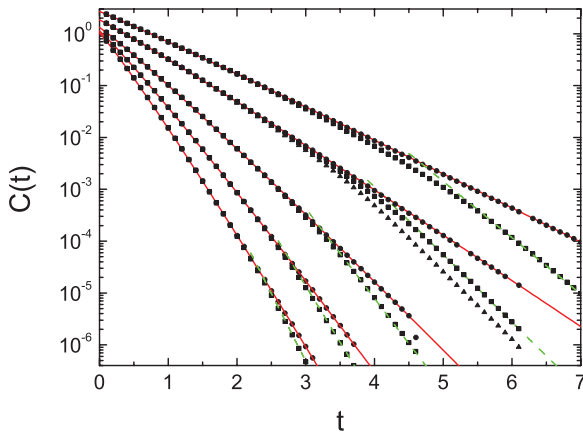


FIG. 1. (Color online) The autocorrelation function of $\xi(t)$ for $\alpha = 1$ and $\gamma = 1$, calculated from Eq. (9), for the following values of θ : 1.5, 2, 3, 4, and 5 (points, from right to left). Results for the distribution truncated at 10^6 and 10^5 are marked by squares and triangles, respectively. The red solid line represents the stretched exponential function, and the dashed green line has the shape $e^{-\gamma t}$.

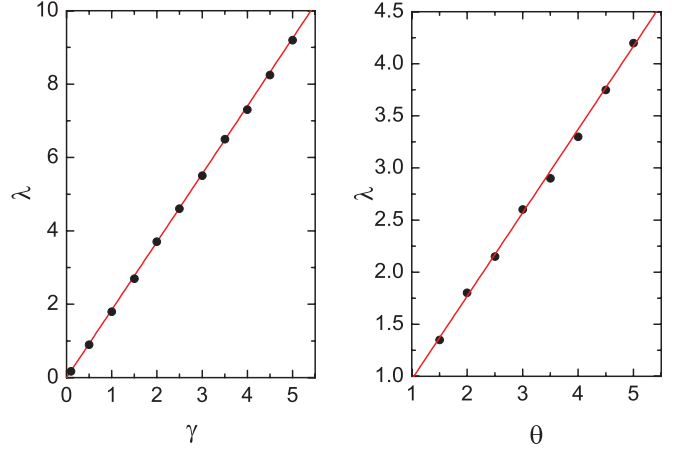


FIG. 2. (Color online) Dependence $\lambda(\gamma)$ and $\lambda(\theta)$ (points). The straight lines mark the functions 1.85γ and $0.17 + 0.8\theta$ for the left and right panel, respectively.

$\mathcal{C}(t)$ can be reasonably approximated by the dependence

$$\mathcal{C}(t) = \frac{(K/\gamma)^{2/(1+\theta)}}{\cos[\pi/(1+\theta)]} e^{-\lambda t}, \quad (14)$$

where $\mathcal{C}(0)$ follows from Eq. (7) and λ is a parameter. Results for the truncated distributions, also presented in the figure, exhibit the fast-falling exponential tail, in agreement with Eq. (12), and they coincide with the stable case at small t .

By rescaling the noise $\eta(t)$ we get rid of K and then λ is completely determined by θ and γ . Figure 2 demonstrates that dependence on both parameters is simple; the expression for $\lambda(\theta, \gamma)$ can be uniquely determined from those results:

$$\lambda = 0.80(\theta + 0.31)\gamma \equiv c_\theta \gamma. \quad (15)$$

The formula Eq. (14) with λ from Eq. (15) and $\langle \xi^2 \rangle$ from Eq. (7) will be applied in Sec. III as the approximation to Eq. (9) if time is not very large.

III. MEMORY EFFECTS IN THE DYNAMICS

When the dynamics proceeds in a medium of a nonhomogeneous structure, one can expect nonlinear effects and non-Gaussian distributions. For example, a Langevin description for the case of a Brownian particle interacting with a general non-Gaussian thermal bath resolves itself to a nonlinear, multiplicative Langevin equation with a non-Gaussian white noise and nonlinear friction term, which follows from the detailed balance symmetry [29]. If the equilibrium state of a stochastic system results from an interplay between an internal noise and damping, the noise intensity and the dissipation have to be mutually related (the Einstein relation). For a correlated noise and a linear coupling in the thermal bath, that relation requires a retarded friction in the Langevin equation, which then becomes a linear integro-differential equation [30,31]. Memory effects are important also for processes involving the Lévy stable noise. It has been demonstrated in [12] that the external noise relaxation time modifies a slope of the power-law density distribution. The memory makes the friction term nonlocal in time. The fractional Langevin equation was introduced by Lutz [32] for the Gaussian noise, which is distinguished due to the central limit theorem. For more

general cases, e.g., in complex systems, an ordinary central limit theorem is no longer valid and the effective random force may assume a form different from the Gaussian even if a coupling within the thermostat is linear. GLE may be applied to such non-Gaussian processes [33], but in this case higher moments cannot be expressed by the first and second moments. We assume that dynamics is governed by a Langevin equation with the retarded friction and driven by the effective random force $\xi(t)$, defined by Eq. (2). It satisfies the second fluctuation-dissipation theorem (FDT) [34] to ensure a proper thermal equilibrium. Such a description is possible since FDT requires the existence of only first and second moments. Then we consider GLE in the form

$$m \frac{dv(t)}{dt} = -m \int_0^t K(t-\tau)v(\tau)d\tau + \xi(t), \quad (16)$$

where $v(t)$ is a velocity. FDT implies that the memory kernel has the same form as the noise covariance, $K(t) = \mathcal{C}(t)/mT$, where T is the temperature and the Boltzmann constant is set at 1. The equipartition energy rule is satisfied: $\langle v(\infty)^2 \rangle = T/m$. Applying the Laplace transformation yields the solution

$$v(t) = R(0)v_0 + m^{-1} \int_0^t R(t-\tau)\xi(\tau)d\tau, \quad (17)$$

with the initial condition $v(0) = v_0$, where the Laplace transform of the resolvent $R(t)$ is given by the equation

$$\tilde{R}(s) = 1/[s + \tilde{K}(s)]. \quad (18)$$

All the fluctuations, if they exist, are determined by the resolvent $R(t)$. The energy equipartition rule follows from Eq. (17) in the limit $t \rightarrow \infty$. The resolvent $R(t)$ has an interpretation of the velocity autocorrelation function, $C_v = \langle v_0 v(t) \rangle = (T/m)R(t)$, and it determines a speed of the relaxation to the equilibrium [35]:

$$\langle [v(t) - R(t)v_0]^2 \rangle = \frac{T}{m} [1 - R^2(t)]. \quad (19)$$

We assume that the driving noise $\xi(t)$ is given by Eq. (2) and approximate its autocorrelation function by the exponential dependence Eq. (14). Then a straightforward calculation yields

$$R(t) = \begin{cases} 1/(2\sqrt{-\Delta}) (B e^{-At} - A e^{-Bt}) (\Delta < 0) \\ e^{-\lambda t/2} / \sqrt{\Delta} (\frac{\lambda}{2} \sin \sqrt{\Delta} t + \sqrt{\Delta} \cos \sqrt{\Delta} t) (\Delta > 0) \end{cases}, \quad (20)$$

where $A = \lambda/2 - \sqrt{-\Delta}$, $B = \lambda/2 + \sqrt{-\Delta}$, and $\Delta = \langle \xi^2 \rangle / (mT) - \lambda^2/4$. One could expect that the probability density distribution, $p_v(v, t)$, converges to the normal distribution due to the finite variance. According to Eq. (17), the velocity is a linear combination of the weighted values of the noise, $R(t-t_i)\xi(t_i)\Delta t$, where Δt is a constant integration step. The subsequent components are not independent, but, since the autocorrelation $\mathcal{C}(t)$ falls with time, terms corresponding to times larger than some relaxation time of $\xi(t)$, $t_r \sim 1/\gamma$, can be regarded as independent and assumptions of the central limit theorem are satisfied. More precisely, for a sufficiently large t the sums $S_n = \sum_{i=0}^n R(t-i\Delta t - nt_r)\xi(i\Delta t + nt_r)$, where $n = [t_r/\Delta t]$, are independent stochastic variables of finite variance. The variable $\sum_{n=1}^{\lfloor t/t_r \rfloor + 1} S_n$ may converge with t to the normal distribution if both t/t_r is large for a nonzero

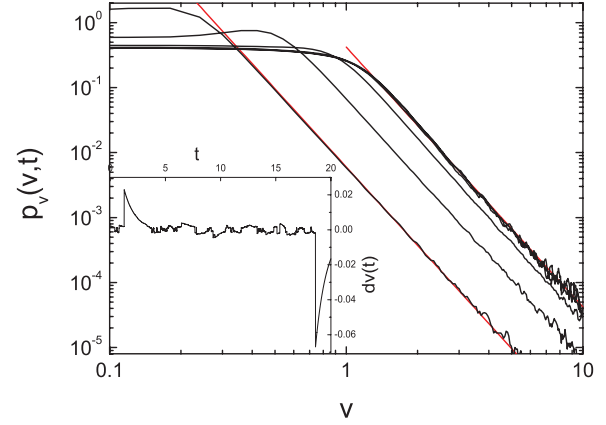


FIG. 3. (Color online) The velocity distribution for $\theta = 2$ and $\gamma = 1$ at $t = 0.2, 0.5, 1, 2, 5,$ and 10 (from left to right), calculated from GLE, Eq. (16). The straight red lines mark the dependence v^{-4} . Inset: velocity increments for a single trajectory.

$R(t)$ and fluctuations of S_n are small. The latter condition emphasizes the importance of higher moments of $\xi(t)$. According to the Berry-Esséen theorem [36], a distribution of a sum of m mutually independent variables differs from the Gaussian by $33\langle \xi^3 \rangle / (4\sigma^3 \sqrt{m})$, where σ is the standard deviation, providing the third moment is finite. Therefore, convergence to the normal distribution is not ensured if $\theta \leq 2$, and even for a larger θ it may be very slow. Deviations from the Gaussian are especially pronounced for large values of $|v|$ which correspond to jumps, and such events are usually a result of single stochastic activations. In this case a distribution is similar to $p(\xi)$ and tails have the form $|v|^{-2-\theta}$. On the other hand, events that produce small $|v|$ correspond to the trajectories consisting of many small segments, similar to the ordinary Brownian motion, and fluctuations are small; then we may expect the Gaussian shape.

The Monte Carlo simulations confirm the presence of the power-law tails. Time evolution of $p_v(v, t)$, presented in Fig. 3 for the case of the infinite third moment, indicates no trace of a convergence with time to the Gaussian; the distribution apparently reaches a stationary state near $t = 2$, and the power-law dependence, $|v|^{-2-\theta}$, dominates the distribution. Trajectories reveal a jumping structure typical for the Lévy flights. This structure is clearly visible when we plot the velocity increments for the discretized integral Eq. (17) (Fig. 3). Note the finite jump relaxation time.

Speed of the equilibration is governed by the parameter A in Eq. (20). Dependence on the parameters follows from Eqs. (7) and (15); an estimation for large γ yields $A \sim \gamma^{-2/(1+\theta)-1}/mT$. We conclude that the equilibration time ($\sim 1/A$) rises with γ because the noise intensity declines. Figure 4 presents equilibration of the variance for different sets of the parameters. The equilibrium value, $\langle v^2 \rangle = T/m$, is reached at short time for small both γ and θ since then the noise intensity is large. Differences between results of the Monte Carlo calculations and Eq. (19) are due to the approximation of the exact $\mathcal{C}(t)$ by the exponential; the equilibrium value is slightly overestimated compared to the energy equipartition rule.

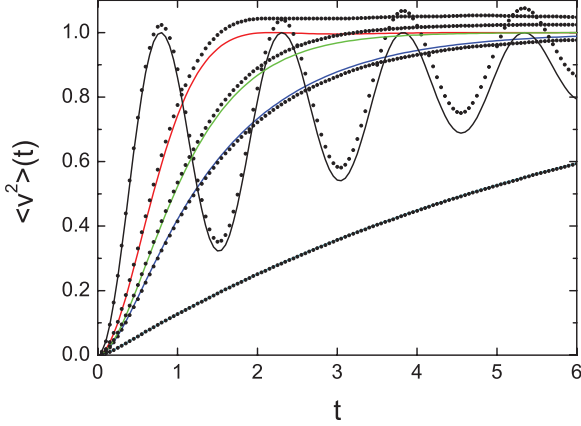


FIG. 4. (Color online) Time evolution of the variance calculated from the Monte Carlo simulations (points) and from Eq. (19) (lines) for $T = 1$ and $m = 1$ with the initial condition $v_0 = 0$. The following cases are presented (from left to right): (1) $\theta = 3$ and $\gamma = 0.1$ (black), (2) $\theta = 2$ and $\gamma = 1$ (red), (3) $\theta = 3$ and $\gamma = 1$ (green), (4) $\theta = 2$ and $\gamma = 2$ (blue), and (5) $\theta = 2$ and $\gamma = 5$ (cyan).

The position x is given by an expression similar to Eq. (17) but with the resolvent $R_x(t) = \int_0^t R(t')dt'$. The distribution of x , $p_x(x,t)$ is presented in Fig. 5 as a function of time; we assumed the initial condition $x(0) = 0$. Two limits, discussed above, are clearly visible if time is sufficiently long: $p_x(x,t)$ assumes the Gaussian shape for $|x| < 20$ ($t = 30$), whereas the tail is of the form $|x|^{-2-\theta}$. The position variance directly follows from the identity:

$$\langle x^2 \rangle(t) = 2 \int_0^t (t-t')C_v(t')dt', \quad (21)$$

where the averaging is performed over the equilibrium state. A direct evaluation for the case $\Delta > 0$ yields

$$\begin{aligned} \langle x^2 \rangle(t) = & \frac{2mT^3}{\langle \xi^2 \rangle^2} \left\{ \frac{\langle \xi^2 \rangle}{mT} - \lambda^2 + \frac{\lambda \langle \xi^2 \rangle}{mT} t + e^{-\lambda t/2} \right. \\ & \times \left[\left(\lambda^2 - \frac{\langle \xi^2 \rangle}{mT} \right) \cos(\sqrt{\Delta}t) + \frac{\lambda}{2\sqrt{\Delta}} \right. \\ & \left. \left. \times \left(\frac{\lambda^2}{4} - 3 \frac{\langle \xi^2 \rangle}{mT} \right) \sin(\sqrt{\Delta}t) \right] \right\}. \quad (22) \end{aligned}$$

We omit the analogous expression for $\Delta < 0$. In the limit of large time the variance rises linearly with time and the diffusion coefficient $D = \lim_{t \rightarrow \infty} \langle x^2(t) \rangle / 2t = T^2 \lambda / \langle \xi^2 \rangle = T^2 c_\theta \cos[\pi/(1+\theta)] \gamma^{1+2/(1+\theta)}$. The position variance as a function of time is presented in Fig. 6. The Monte Carlo results, obtained by integration of Eq. (17), reveal a slightly stronger time dependence than the linear growth predicted by Eq. (21): t^β with $\beta = 1.02, 1.04$, and 1.05 for $\theta = 1.5, 2$, and 3 , respectively.

In many physical problems the observed covariance functions are not exponential. The power-law form of the memory function was discussed in connection with a frictional resistance [37] and in the hydrodynamics [38]. In particular, diffusion in the dense liquids requires the memory function falling like $t^{-3/2}$ [39]. GLE for systems with power-law kernels, $|t|^{-\beta}$ ($\beta \neq 1$), takes a form of the fractional Langevin equation, where the damping term is expressed by the Riemann-Louville

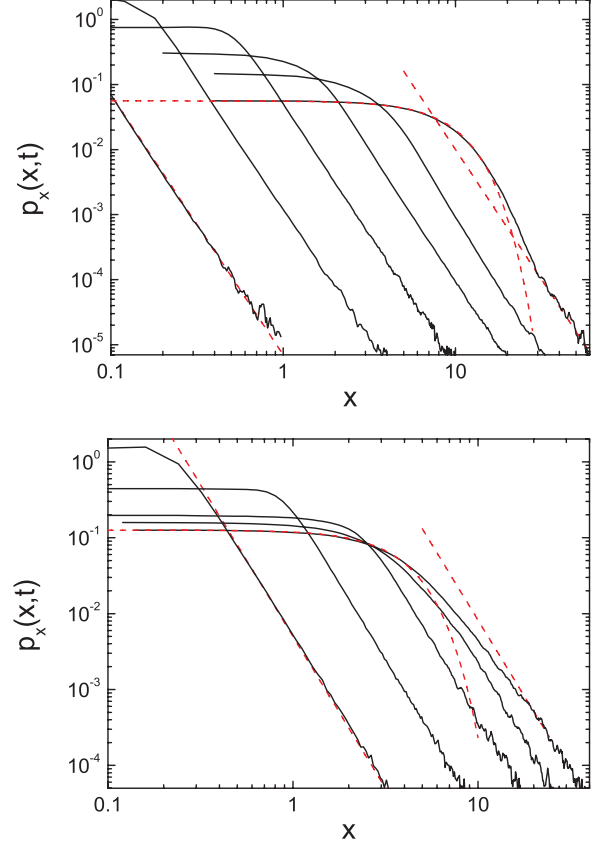


FIG. 5. (Color online) Upper part: the position distribution for $\theta = 2$ and $\gamma = 1$ at $t = 0.2, 0.5, 1, 2, 5$, and 30 (from left to right). The red dashed lines mark the dependence x^{-4} and the Gaussian. Lower part: the same but for the covariance Eq. (23) and the following times: $1, 2, 5, 20$, and 50 .

operator, ${}_0D_t^{\beta-1}$ [32]; power-law kernels are present in the fractional Brownian motion theory [40]. Long-time correlations are observed in the complex systems that usually possess non-Gaussian distributions with power-law tails. A very slow falling covariance, corresponding to a $1/f$ noise, was found in an analysis of absolute returns in the US market [41]. The

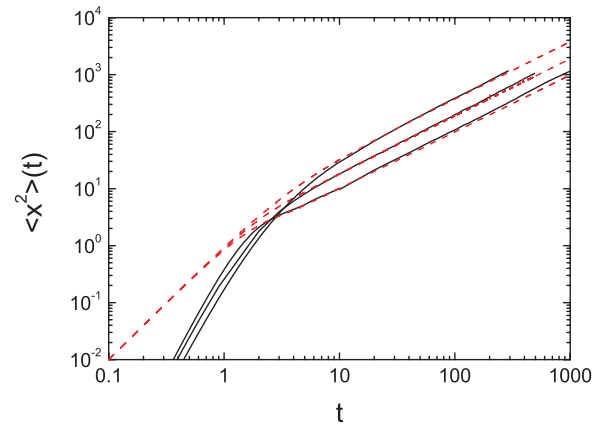


FIG. 6. (Color online) Position variance obtained from integration of Eq. (17) with the initial condition $v_0 = x(0) = 0$ for $\gamma = 1$ and $\theta = 1.5, 2$, and 3 (solid lines from bottom to top). Results marked by dashed red lines follow from Eq. (21).

autocorrelation function of the displacement may even rise with time; this effect was experimentally demonstrated for the diffusion in dusty plasma liquids for which the corresponding probability distributions exhibit fat tails [42]. The covariance $1/t$, in turn, was observed in connection with the noise-induced Stark broadening [43] and obtained, for a two-dimensional system, from the Navier-Stokes equations [44]. The presence of this form of the autocorrelation function may be related to a specific topology of the medium: it emerges when the trajectory has a structure of long straight-line intervals, like for the Lorentz gas [45], and may be encountered in the nuclear reactions [46]. In this paper, we solve GLE with the memory function in the form $1/t$. More precisely, we assume

$$\mathcal{C}(t) = (1 - e^{-Lt})/Lt, \quad (23)$$

where $L = \text{const} > 0$.

According to the results of Sec. II, any process $\xi(t)$, given by Eq. (2), is characterized by the exponential covariance and the rate is uniquely determined by λ . Assuming that $\xi(t)$ is an elementary process $\xi_\lambda(t)$, we can construct a compound process by a superposition of $\xi_\lambda(t)$ where the parameter λ is regarded as a stochastic variable. Therefore, we may obtain an arbitrary, *a priori* assumed covariance by averaging, with a weight $\psi(\lambda)$, over an ensemble of trajectories corresponding to a fixed value of θ and different values of γ . Since, for a given θ , $\mathcal{C}(t) = \int_0^\infty \langle \xi^2 \rangle \psi(\lambda) e^{-\lambda t} d\lambda \sim \int_0^\infty \lambda^{-2/(1+\theta)} \psi(\lambda) e^{-\lambda t} d\lambda$, the distribution of λ can be evaluated for any $\mathcal{C}(t)$ by inversion of the Laplace transform:

$$\psi(\lambda) \sim \lambda^{2/(1+\theta)} \mathcal{L}^{-1}[\mathcal{C}(t)]. \quad (24)$$

Equation (23) corresponds to the following normalized distribution:

$$\psi(\lambda) = \frac{L^{2/(1+\theta)}}{2/(1+\theta) + 1} c_\theta^{-2/(1+\theta)} \cos \frac{\pi}{1+\theta} \lambda^{2/(1+\theta)} \quad (25)$$

for $\lambda \in (0, L)$ and 0 elsewhere. The distribution of all the elementary processes $\xi_\lambda(t)$ has the same asymptotic form, $|\xi|^{-2-\theta}$, with the same slope of the tails since θ is fixed in the statistical ensemble. γ , in turn, influences a relative intensity of the noises $\xi_\lambda(t)$. Solution of GLE is given by Eq. (17), where transform of the memory kernel, $\tilde{K}(s) = [\ln(s/L + 1) - \ln(s/L)]/L$, follows from Eq. (18). Equation (21) yields $\mathcal{D} = \lim_{t \rightarrow \infty} d\langle x^2(t) \rangle / dt = \int_0^\infty \mathcal{C}_v(t) dt \sim \tilde{R}(0) = 0$ for any L . Inversion of the transform yields

$$R(t) = e^{at} [c_1 \cos(bt) + c_2 \sin(bt)] - L \int_0^L \frac{e^{-tx} dx}{[Lx - \ln(L/x - 1)]^2 + \pi^2}. \quad (26)$$

Details of the derivation and values of the coefficients, as well as some remarks about the numerics, are presented in Appendix B.

The shape of the stationary velocity distribution for the covariance Eq. (23) is similar to that for the exponential covariance case, but the dependence $1/|v|^{2+\theta}$ of the tails for $\theta = 2$ shifts to the relatively large $|v|$ and the equilibration time is larger. The damping parameter a , given by Eq. (B1), nonmonotonically depends on L , but it becomes very small for large L , and the time needed to reach the stationary state is then extremely long. The parameter θ does not influence

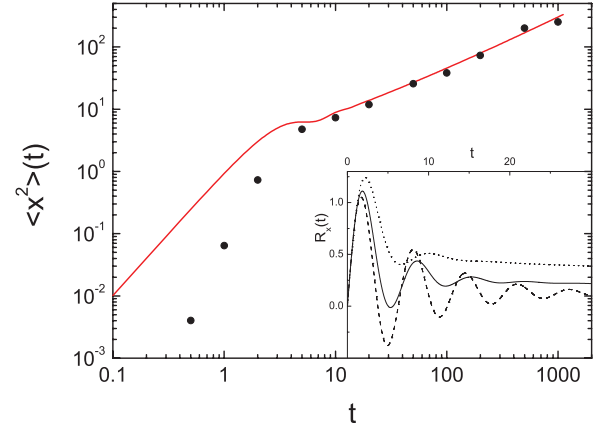


FIG. 7. (Color online) Position variance obtained from integration of Eq. (17) with the initial condition $v_0 = x(0) = 0$ for the covariance Eq. (23), calculated with the parameters $L = 1$ and $\theta = 2$ (points). Results marked by the red line follow from Eq. (21). Inset: the resolvent $R_x(t)$ for $L = 0.5$ (dashed line), 1 (solid line), and 2 (dotted line).

the equilibration time but strongly modifies the distribution tail. For example, the tail assumes the shape $1/v^8$ for $\theta = 4$; i.e., it falls stronger than the noise distribution. Anyway, a convergence to normal distribution is not observed.

Distribution of the position was calculated by means of the integrated resolvent $R_x(t)$, and results are presented in Fig. 5. The x dependence of $p_x(x, t)$, in particular the asymptotics, is similar to the case of the exponential covariance. The main difference consists in the expansion speed: the distribution for the covariance Eq. (23) widens much slower. Time dependence of the variance is given by Eq. (21), where the integral in Eq. (26) has to be estimated numerically. The shape of the curve, shown in Fig. 7, reveals an apparent shape $t^{0.8}$ at long time, which indicates the sublinear behavior. As expected from the equation $\tilde{R}(s=0) = 0$, the system is subdiffusive and simulations agree with Eq. (21) in the stationary limit. However, asymptotics of the variance is in fact not a power law. According to a conjecture in [47], the position variance should behave in the limit of long time like $t/\ln(t)$, the form of which has been interpreted by the authors as an analogy to critical exponents in a phase transition. One obtains a similar dependence when $\mathcal{D}(t)$ is estimated by establishing lower and upper limits of the integral [48]. The expression $\langle x^2 \rangle(t) = 0.91t/\ln(t)$ agrees with the exact result, Eq. (21), for $t > 30$. The time-dependent diffusion coefficient is given by the resolvent $\mathcal{D}(t) \sim R_x(t)$, and it is also presented in Fig. 7. It appears very sensitive on L : oscillations, being strong for small L , vanish quickly if L is large.

IV. SUMMARY AND CONCLUSIONS

The overdamped Langevin equation with the quadratic potential and the multiplicative Lévy stable noise describes a process that comprises a jumping structure of trajectories and convergent moments: variance and covariance. We have demonstrated that the autocorrelation function $\mathcal{C}(t)$ for the truncated distribution falls exponentially, in the limit of a long time, with the α -independent rate $\gamma(1 + \theta)$. Correlations were

studied in detail for the Cauchy distribution. It has been found that for the stable case $\mathcal{C}(t)$ obey the stretched-exponential form but can be reasonably approximated by the simple exponential. The rate has been uniquely determined as a function of the system parameters: it rises linearly with γ and θ . Higher moments may also be convergent if one chooses a sufficiently large θ . The exponential decay for the truncated case is faster than that for the stable distribution, a conclusion that emphasizes a role of very long jumps in preserving the memory in the system. One may construct a stochastic process characterized by an arbitrary form of the covariance by a superposition of trajectories with different γ , i.e., by assuming the parameter γ as a stochastic variable. Moreover, one can reproduce an arbitrary slope of the distribution tail since that is governed solely by the parameter θ . The above properties of the process $\xi(t)$ suggest its applicability to problems which require both fat tails and long correlation time.

If a stochastic force that obeys the above properties is balanced by the damping force, the fluctuation-dissipation theorem and the equipartition energy rule are satisfied; then the process obeys GLE, a fact that is well known for the Gaussian case. We applied GLE to the case for which the driving force is given by $\xi(t)$. The equation predicts tails of both velocity and position distribution, dominated by single jumps, of the same form as the driving noise. The central part of the distribution, in turn, results from many small stochastic activations and for $p_x(x, t)$ converges to the Gaussian, whereas the intermediate region assumes the fast-falling power law. Similar distributions may be observed in many complex systems since they are characterized by a substantial memory and the thermal equilibration is accompanied by rare but spectacular events. Transport properties of the system described by GLE follow directly from the noise covariance; the position variance rises linearly for the exponential covariance and sublinearly for the covariance $\sim 1/t$. Numerical trajectory simulations involving the process $\xi(t)$ confirm that general result. Therefore, jumps and power-law tails of the distribution may coexist with the thermal equilibrium.

APPENDIX A

We derive the expression for $\mathcal{C}(t)$ in a limit of large t , Eq. (12). First, we expand the conditional probability, Eq. (10), in powers of $\epsilon = e^{-\gamma\theta t}$ to the first order. Expansion of the first term in Eq. (10) yields $(1 - \epsilon^\alpha)^{-1/\alpha} = 1 + \epsilon^\alpha/\alpha + \dots$. The Fox function is given by the series

$$H\left[\frac{1}{K\sigma^{1/\alpha}}|y_2 - y_1\epsilon|\right] = H\left[\frac{(\alpha\gamma\theta)^{1/\alpha}}{K}|y_2|\right] + \frac{\partial}{\partial\epsilon}H(\epsilon = 0)\epsilon + \dots, \quad (A1)$$

where the coefficients are dropped. The derivative involves the Fox function of the higher order [49,50]:

$$\frac{\partial H}{\partial\epsilon} = -\frac{y_1}{y_2 - y_1\epsilon}H_{3,3}^{1,2}\left[\frac{1}{K\sigma^{1/\alpha}}|y_2 - y_1\epsilon|\right. \\ \left.\times\left[\begin{matrix} (0,1), (1 - 1/\alpha, 1/\alpha), (1/2, 1/2) \\ (0,1), (1/2, 1/2), (1,1) \end{matrix}\right]\right]. \quad (A2)$$

Next, we insert the above expansions to Eq. (9) and neglect terms of a higher order than ϵ . The first component vanishes because the double integral can be factorized and both integrands are odd. The integral over y_2 resolves itself to a Mellin transform:

$$\int_0^\infty y_2^{-\theta/(1+\theta)}H_{3,3}^{1,2}\left[\frac{(\alpha\gamma\theta)^{1/\alpha}}{K}y_2\right]dy_2 \\ = K^{1/(1+\theta)}(\alpha\gamma\theta)^{-1/\alpha(1+\theta)}\chi\left(-\frac{1}{1+\theta}\right) \\ = -K^{1/(1+\theta)}(\alpha\gamma\theta)^{-1/\alpha(1+\theta)}\frac{\Gamma\left(\frac{\theta/\alpha}{1+\theta}\right)}{\Gamma\left(\frac{\theta/2}{1+\theta}\right)}, \quad (A3)$$

where χ stands for the Mellin transform from $H_{3,3}^{1,2}$. Elimination of the algebraic factor in the integral over y_1 yields

$$I = \int_0^\infty H_{2,2}^{1,1}\left[x\left|\begin{matrix} \left(1 + \frac{1}{\alpha(1+\theta)}, \frac{1}{\alpha}\right), \left(1 + \frac{1}{2}\frac{1}{1+\theta}, \frac{1}{2}\right) \\ \left(\frac{2+\theta}{1+\theta}, 1\right), \left(1 + \frac{1}{2}\frac{1}{1+\theta}, \frac{1}{2}\right) \end{matrix}\right.\right]dx. \quad (A4)$$

APPENDIX B

We derive the expression Eq. (26) where the integral $\int_{\sigma-i\infty}^{\sigma+i\infty}\tilde{R}(z)e^{tz}dz$ is to be evaluated. The contour consists of a straight line parallel to the imaginary axis at $\sigma > 0$, a large half circle in the left half plane, and a cut along the real segment $(-L, 0)$. Roots of the equation

$$Lz + \ln(z/L + 1) - \ln(z/L) = 0 \quad (B1)$$

are of the form $z_{1,2} = a \pm bi$, and they have to be found numerically for a given L . After a straightforward evaluation of the sum over residues, we obtain the first component of Eq. (26), where the coefficients are $c_1 = 2[a^4 + 2a^3L + a(2b^2 - 1)L + a^2(2b^2 + L^2 - 1) + b^2(b^2 + L^2 + 1)]/A$, $c_2 = 2b(2a + L)/A$, and $A = 1 + a^4 + b^4 + 2a^3L + 2a(b^2 - 1)L + b^2(2 + L^2) + a^2(2b^2 + L^2 - 2)$. Contribution from both branches along the cut resolves itself to the integral in Eq. (26).

Trajectory numerical simulations require a value of $R(t)$ for each integration step. Approximation of the asymptotics is easy to determine. For example, for $\theta = 2$ and $L = 1$ we get $R(t) = -0.113t^{-1.292}(t > 60)$, a formula that coincides with the numerical integration up to at least $t = 2000$. $R(t)$ for small t was evaluated with a step 0.001 and stored.

[1] H. E. Stanley, *Physica A* **318**, 279 (2003).
 [2] V. Plerou and H. E. Stanley, *Phys. Rev. E* **77**, 037101 (2008).
 [3] X. Gabaix, P. Gopikrishnan, V. Plerou, and H. E. Stanley, *Nature (London)* **423**, 267 (2003).

[4] F. Ren, B. Zheng, T. Qiu, and S. Trimper, *Phys. Rev. E* **74**, 041111 (2006).
 [5] J. Perelló, M. Gutiérrez-Roig, and J. Masoliver, *Phys. Rev. E* **84**, 066110 (2011).

- [6] J. F. Muzy, E. Bacry, and A. Kozhemyak, *Phys. Rev. E* **73**, 066114 (2006).
- [7] D. Schertzer, M. Larchevêque, J. Duan, V. V. Yanovsky, and S. Lovejoy, *J. Math. Phys.* **42**, 200 (2001).
- [8] R. N. Mantegna and H. E. Stanley, *Phys. Rev. Lett.* **73**, 2946 (1994).
- [9] A. Chechkin, V. Gonchar, J. Klafter, R. Metzler, and L. Tanatarov, *Chem. Phys.* **284**, 233 (2002).
- [10] T. Srokowski, *Phys. Rev. E* **80**, 051113 (2009).
- [11] T. Srokowski, *Phys. Rev. E* **81**, 051110 (2010).
- [12] T. Srokowski, *Phys. Rev. E* **85**, 021118 (2012).
- [13] A. La Cognata, D. Valenti, A. A. Dubkov, and B. Spagnolo, *Phys. Rev. E* **82**, 011121 (2010).
- [14] A. La Cognata, D. Valenti, B. Spagnolo, and A. A. Dubkov, *Eur. Phys. J. B* **77**, 273 (2010).
- [15] A. A. Dubkov and B. Spagnolo, *Eur. Phys. J. B* **65**, 361 (2008).
- [16] A. Dubkov, *Acta Phys. Pol. B* **43**, 935 (2012).
- [17] R. Segev, M. Benveniste, E. Hulata, N. Cohen, A. Palevski, E. Kapon, Y. Shapira, and E. Ben-Jacob, *Phys. Rev. Lett.* **88**, 118102 (2002).
- [18] P. Oświęcimka, J. Kwapien, and S. Drożdż, *Phys. Rev. E* **74**, 016103 (2006).
- [19] J. W. Kantelhardt, S. A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde, and H. E. Stanley, *Physica A* **316**, 87 (2002).
- [20] A. Saichev and D. Sornette, *Phys. Rev. E* **74**, 011111 (2006).
- [21] B. J. West and V. Seshardi, *Physica A* **113**, 203 (1982).
- [22] A. Schenzle and H. Brand, *Phys. Rev. A* **20**, 1628 (1979).
- [23] R. Graham and A. Schenzle, *Phys. Rev. A* **25**, 1731 (1982).
- [24] G. Samrodintsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes* (Chapman and Hall, London, 1994).
- [25] I. Koponen, *Phys. Rev. E* **52**, 1197 (1995).
- [26] I. M. Sokolov, A. V. Chechkin, and J. Klafter, *Physica A* **336**, 245 (2004).
- [27] A. V. Chechkin, V. Yu. Gonchar, R. Gorenflo, N. Korabel, and I. M. Sokolov, *Phys. Rev. E* **78**, 021111 (2008).
- [28] T. Srokowski, *Physica A* **388**, 1057 (2009).
- [29] A. A. Dubkov, P. Hänggi, and I. Goychuk, *J. Stat. Mech.: Theory Exp.* (2009) P01034.
- [30] H. Mori, *Prog. Theor. Phys.* **33**, 423 (1965); **34**, 399 (1965).
- [31] M. H. Lee, *J. Math. Phys.* **24**, 2512 (1983).
- [32] E. Lutz, *Phys. Rev. E* **64**, 051106 (2001).
- [33] W. T. Coffey, Yu. P. Kalmykov, and J. T. Waldron, *The Langevin Equation* (World Scientific, Singapore, 2004).
- [34] R. Kubo, *Rep. Prog. Phys.* **29**, 255 (1966).
- [35] S. A. Adelman, *J. Chem. Phys.* **64**, 124 (1976).
- [36] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2 (Wiley, New York, 1966).
- [37] J. Boussinesq, *Théorie Analytique de la Chaleur*, Vol. 2 (Gauthiers-Villars, Paris, 1903).
- [38] R. M. Mazo, *J. Chem. Phys.* **54**, 3712 (1971).
- [39] I. Oppenheim, K. Shuler, and G. Weiss, *Stochastic Processes in Chemical Physics: The Master Equation* (MIT Press, Cambridge, 1977).
- [40] W. H. Deng and E. Barkai, *Phys. Rev. E* **79**, 011112 (2009).
- [41] I. Gvozdanovic, B. Podobnik, D. Wang, and H. E. Stanley, *Physica A* **391**, 2860 (2012).
- [42] C.-W. Io and L. I., *Phys. Rev. E* **85**, 026407 (2012).
- [43] A. Brissaud and U. Frisch, *J. Quant. Spectrosc. Radiat. Transfer* **11**, 1767 (1971).
- [44] B. J. Alder and T. E. Wainwright, *Phys. Rev. A* **1**, 18 (1970).
- [45] T. Geisel, A. Zacherl, and G. Radons, *Z. Phys. B* **71**, 117 (1988).
- [46] T. Srokowski and M. Płoszajczak, *Phys. Rev. Lett.* **75**, 209 (1995).
- [47] R. M. S. Ferreira, M. V. S. Santos, C. C. Donato, J. S. Andrade, Jr., and F. A. Oliveira, *Phys. Rev. E* **86**, 021121 (2012).
- [48] A. Kamińska and T. Srokowski, *Phys. Rev. E* **67**, 061114 (2003).
- [49] A. M. Mathai and R. K. Saxena, *The H-Function with Applications in Statistics and Other Disciplines* (Wiley, New Delhi, 1978).
- [50] H. M. Srivastava, K. C. Gupta, and S. P. Goyal, *The H-Functions of One and Two Variables with Applications* (South Asian Publishers, New Delhi, 1982).