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Role of delay in the mechanism of cluster formation

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We study the role of delay in phase synchronization and phenomena responsible for cluster formation in delayed coupled maps on various networks. Using numerical simulations, we demonstrate that the presence of delay may change the mechanism of the unit to unit interaction. At weak coupling values, the same parity delays are associated with the same phenomenon of cluster formation and exhibit similar dynamical evolution. Intermediate coupling values yield rich delay-induced driven cluster patterns. A Lyapunov function analysis sheds light on the robustness of the driven clusters observed for delayed bipartite networks. Our results reveal that delay may lead to a completely different relation, between dynamical and structural clusters, than that observed for the undelayed case.

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Studying the impact of network topology on dynamical processes is of fundamental importance for understanding the functioning of many real world complex networks [1]. The dynamical behavior of a system depends on the collective behavior of its individual units. One of the most fascinating emergent behaviors of interacting chaotic units is the observation of synchronization [2]. In general, synchronization may lead to more complicated patterns including clusters [3–5]. The interplay between the underlying network structure and dynamical clusters has been the prime area of focus for the past two decades [6]. Furthermore, a communication delay naturally arises in extended systems [7]. A delay gives rise to many new phenomena in dynamical systems such as oscillation death, stabilizing periodic orbits, enhancement or suppression of synchronization, chimera state, etc. [8–14].

In this Rapid Communication, we study the impact of delay on the phenomenon of phase synchronized clusters in coupled map networks. We investigate the formation of clusters on various networks, namely, one-dimensional (1D) lattice, small-world, random, scale-free, and bipartite networks [15], and provide a Lyapunov function analysis for bipartite networks to explain the possible reasons behind the role of a delay on synchronized clusters. So far, studies on delayed coupled dynamical systems have mostly concentrated on a global synchronized state, except a few recent studies which have focused on pattern formation or clustered states [4,5,16,17]. These studies have revealed that delay emulates qualitative changes in a clustered state, whereas the mechanism of delayed unit to unit interactions still needs to be investigated.

Previous studies on undelayed coupled systems have identified two different phenomena for synchronization, namely, the driven (D) and the self-organized (SO) [3]. SO (D) synchronization refer to the state when clusters are formed because of intracluster (intercluster) couplings. Here, we report that a delay can play a crucial role in the formation of clusters as well as the phenomenon behind it. The formation of delay-induced synchronized clusters may be because of intercluster couplings, instead of coupling between synchronized units [5,16]. Introduction of a delay may result in a transition from D to SO synchronization or vice versa. Furthermore, our studies demonstrate a delay-induced emergence of dynamical phase synchronized D patterns. These patterns are stable with time and are dynamical with respect to a change in τ. A delayed bipartite network leads to a transition from SO to D synchronization in an intermediate coupling range, irrespective of τ.

Here we take networks with a less average degree ($N_C \sim N$), leading to phase synchronized clusters instead of a complete synchronized state which usually spans all the nodes. We consider a network of N nodes and N_c connections between the nodes. Let each node of the network be assigned a dynamical variable $x^i, i = 1,2,\ldots,N$. The dynamical evolution is defined by the well known coupled maps [18]

$$x_i(t+1) = (1-\varepsilon)f(x_i(t)) + \frac{\varepsilon}{k_i}\sum_{j=1}^{N} A_{ij}\, g(x_j(t-\tau)). \quad (1)$$

Here A is the adjacency matrix with elements A_{ij} taking values 1 and 0 depending upon whether or not there is a connection between i and j. $k_i = \sum_{j=1}^{N} A_{ij}$ is the degree of the ith node and ε is the overall coupling constant. In the present investigation we consider a homogeneous delay τ. The function $f(x)$ defines a local nonlinear map, and $g(x)$ defines the nature of the coupling between the nodes. We consider phase synchronization as described in Ref. [19]. As the network evolves, it splits into several synchronized clusters. In order to have a clear picture of SO and D behavior, we use f_{intra} and f_{inter} measures for intra- and intercluster couplings as follows [3]: $f_{\mathrm{intra}} = N_{\mathrm{intra}}/N_c$ and $f_{\mathrm{inter}} = N_{\mathrm{inter}}/N_c$, where N_c is total number of connections in the network, and N_{intra} and N_{inter} are the numbers of intra- and intercluster couplings, respectively [20]. We evolve Eq. (1) starting from random initial conditions, and study

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the dynamical behavior of nodes after an initial transience. First let us consider the local dynamics being governed by the logistic map $f(x) = 4x(1-x)$, and the coupling function $g(x) = f(x)$.

The undelayed coupled maps on all model networks we have considered yield dominant D clusters in the range $0.16 \lesssim \varepsilon \lesssim 0.25$. For the rest of the ε values, coupled maps on 1D lattice and small-world networks exhibit no phase synchronization, except for $\varepsilon \gtrsim 0.74$ having mixed clusters with very small values of $f_{\rm inter}$ and $f_{\rm intra}$ [Figs. 1(a) and 1(b)]. In this ε range scale-free and random networks favor synchronization, yielding better cluster formation than the corresponding regular and small-world networks [Figs. 1(c) and 1(d)], while bipartite networks lead to ideal SO synchronization for $0.45 \lesssim \varepsilon \lesssim 0.85$ and ideal D synchronization for higher ε values [3]. Upon introducing a delay of $\tau = 1$ in Eq. (1), after very small ε values, for which there is no phase synchronization for the undelayed case [black in Figs. 1(a)–1(d)], we get SO clusters in the region $0.13 \lesssim \varepsilon \lesssim 0.2$, as seen from the white regions in Figs. 1(b) and 1(d). For most of the ε values in this region, the coupled dynamics exhibits a periodic evolution with a period depending upon τ. For a further increase in ε, in the middle coupling range, the 1D lattice, small-world, scale-free, and random networks lead to an increase in D synchronization in $0.4 \lesssim \varepsilon \lesssim 0.7$, whereas for complete bipartite networks, ideal D synchronization is achieved for almost all ε values in this range. For $0.85 \lesssim \varepsilon \lesssim 1.0$, the delayed case exhibits a very small (almost negligible) cluster formation compared to the undelayed case, hence indicating a suppression of synchronization for all the networks except for bipartite networks forming ideal D clusters. For $\tau = 2$, the lower ε range coerces the formation of dominant D clusters, similar to the undelayed case. As ε increases, 1D lattice and small-world networks lead to mixed clusters, whereas scale-free and random networks lead to dominant D clusters. Bipartite networks emulate ideal D synchronization. With a further increase in τ, at a lower ε range, odd τ leads to a similar behavior as for $\tau = 1$ and even τ exhibits similar behavior as for $\tau = 0$ and $\tau = 2$. For the intermediate ε range there is a suppression in synchronization. Higher ε values manifest no cluster formation, as illustrated by the black regions in Fig. 1 for all networks except the bipartite, which form ideal D clusters for $\varepsilon \gtrsim 0.4$ for all τ.

The above description boils down to the following: There is a ε region which demonstrates a change in the phenomenon of cluster formation with a change in τ. The zero and even delays imply dominant D clusters, whereas odd delays imply ideal or dominant SO clusters. Moreover, odd delays lead to SO clusters with a periodic evolution, whereas zero and even delays lead to a D cluster with periodic, quasiperiodic, or chaotic evolution [21]. Note that the measure of phase synchronization considered here satisfies the metric properties, but does not include antiphase synchronization and consequently nodes being antiphase synchronized would end up in different clusters. However, antiphase or phase shift synchronization is not the only cause behind the separation of nodes in clusters [21].

Though the nodes in various clusters display a rich dynamical evolution, a simple analysis for the periodic synchronized state, for example, bipartite networks in the lower ε region, provides a basic understanding of different behaviors indicated

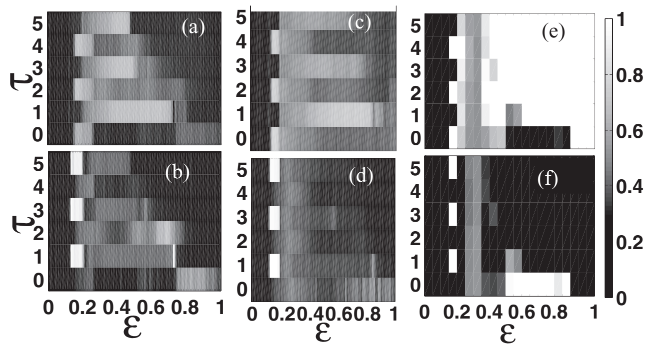

FIG. 1. Phase diagram of phase synchronization patterns in system (1) for a 1D lattice with $N = 50$, $\langle k \rangle = 4$. Grayscale encoding represents values of (a) $f_{\rm inter}$ and (b) $f_{\rm intra}$. The local dynamics is governed by a logistic map $f(x) = 4x(1-x)$ and a coupling function $g(x) = f(x)$. The figure is obtained by averaging over 20 random initial conditions. The regions, which are black in both (a) and (b), correspond to states of no cluster formation. Both subfigures with gray shades correspond to clusters having both inter- and intracouplings. The regions in (a), which are lighter as compared to the corresponding ε and τ values in (b), refer to dominant D phase synchronized clusters, and the reverse refer to dominant SO phase synchronized clusters. White regions in (a) and (b) refer to ideal D or ideal SO clusters, respectively. The regions, which are dark gray in (a) and black in (b) or vice versa, correspond to states where many less clusters are formed. (c) and (d) are for scale-free and (e) and (f) are for bipartite networks and demonstrate the same as (a) and (b), respectively.

by odd and even delays. In this ε region, the coupling term having a delay part yields

$$f(x(t-\tau)) = \begin{cases} f(p_1) & \text{if } \tau = 0 \text{ and even,} \\ f(p_2) & \text{if } \tau \text{ is odd,} \end{cases}$$

implying that the discrete time delay considered here introduces a difference in the evolution of the nodes [Eq. (1)] depending upon the parity of delay, and thus leading to a particular behavior for zero and even delays but a different behavior for odd delays.

Furthermore, a change in τ leads to a change in SO or D cluster pattern. A pattern refers to a particular phase synchronized state, containing information about all the pairs of the phase synchronized nodes distributed in the various clusters. A change in the pattern refers to the state when members of a cluster get changed as an effect of delay. For some cases we observe ideal D or SO clusters. Ideal SO synchronization refers to a state when clusters do not have any connection outside the cluster, except one. The ideal D synchronization refers to the state when clusters do not have any connections within them, and all connections are outside.

Next we focus on the ε range where the delayed evolution leads to ideal D clusters for bipartite networks, and dominant D clusters for other networks. In bipartite networks, a division of nodes into ideal D clusters is unique, whereas for other network structures there can be various possible ways in which one can distribute nodes to form ideal D (for an average degree of two) or dominant D (for larger average degree) clusters. Figure 2 plots snapshots of clusters for different τ by keeping all other parameters the same. It indicates that with a change in τ, both nodes forming clusters as well as the size of the

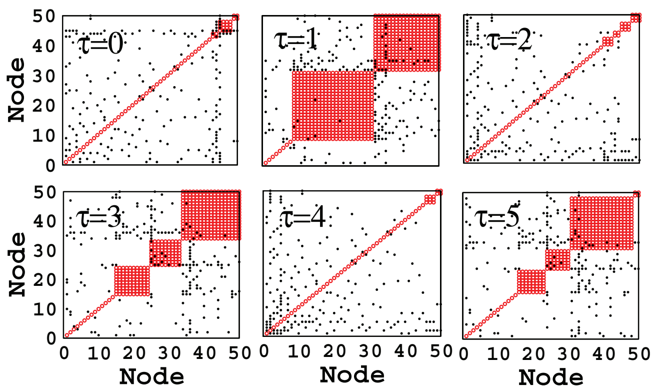

FIG. 2. (Color online) A typical behavior of coupled dynamics illustrating D patterns observed with changing τ. Squares represent clusters, diagonal dots represent freely evolved nodes, while off-diagonal dots imply that the two corresponding nodes are coupled (i.e., $A_{ij} = 1$). In each case the node numbers are reorganized so that the nodes belonging to the same cluster are numbered consecutively. The example presents a scale-free network with $N = 50$ and $\varepsilon = 0.6$. For $\tau = 0$, very few nodes are forming a cluster. For $\tau = 1$, 3, and 5, nodes form dominant D clusters, whereas $\tau = 2$ and 4 yield very few nodes forming clusters of an ideal D type.

clusters are changed. Note that the dynamical evolution here may be periodic, quasiperiodic, or chaotic. In this region, for a particular delay value, the clusters are almost stable with time evolution, with few nodes of the floating type [3]. But a change of τ has a drastic impact on cluster patterns, and may lead to entirely different sets of nodes forming clusters. Hence D patterns obtained in this range are dynamic with respect to a change in τ. However, the phenomenon behind the pattern formation does not change, and the D mechanism is mainly responsible for the cluster formation. For this ε range, a delayed evolution on a bipartite network yields ideal D clusters for all τ values we have investigated.

The aforementioned can be explained further using the example of bipartite networks. A Lyapunov function analysis can be carried out for the delayed case in a very similar fashion as for $\tau = 0$ described in Ref. [22], and for a pair of synchronized nodes on a bipartite network can be written as

$$V_{ij}(t+1) = \Bigg[(1-\epsilon)[f(x_i(t)) - f(x_j(t))] + \frac{2\varepsilon}{N} \times \sum_{j=N/2+1}^{N} g(x_j(t-\tau)) - \frac{2\varepsilon}{N}\sum_{i=1}^{N/2} g(x_i(t-\tau))\Bigg]^2.$$

For an ideal D state, the synchronization between two nodes which are not directly connected is independent of the delay terms as the coupling terms cancel out, and only depends on ε. Hence, delay does not affect synchronization between the nodes which are not directly connected [22], and only comprehends its presence for those which are directly connected. As a consequence, depending upon ε and τ, it may either enhance or destroy the synchrony between them. For instance, in the lower ε range odd delays lead to an enhancement of coordination between connected nodes, yielding a transition to SO clusters, whereas in middle ε range, delay destroys synchronization between the connected nodes, yielding a D

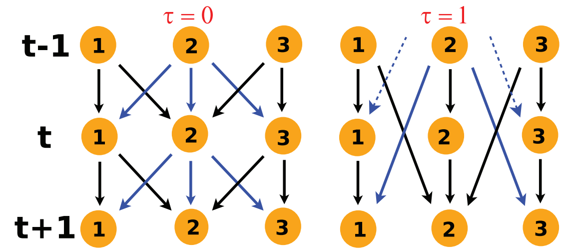

FIG. 3. (Color online) Three-node schematic diagram illustrating the impact of delay. Arrows depict the direction of information flow as governed by Eq. (1). The dashed lines show the flow of information from the $(t-2)$th time step. For $\tau = 0$, the evolution of all nodes (solid orange circles) receives information from the second node (left panel), whereas in the presence of delay, the evolution of the connected nodes at a particular time does not involve any common term (right panel). For both panels, the first and third nodes are connected with the second one, leading to the construction of the smallest possible bipartite network.

cluster state. As indicated in Fig. 3, for $\tau = 0$, the common term in the evolution equation for all the nodes may be the reason for global synchronization, whereas for $\tau > 0$ the network gets divided into two parts, with one set of nodes having completely different terms in its evolution equations than those of the second set. An important inference of our results is that in the presence of delay, the dynamical evolution on the bipartite network identifies the underlying network structure and gives rise to ideal D clusters for almost all the couplings for $\varepsilon \gtrsim 0.4$. Note that a previous result on delayed bipartite networks concludes that they would lead to the worst synchronization [9], but D clusters observed here very clearly reveal a very good synchronizing power of the same.

In order to demonstrate the robustness of the above phenomena, we also present results for coupled circle maps. In Eq. (1), the local dynamics is defined by the circle map $f(x) = x + \omega + (p/2\pi)\sin(2\pi x)$, with parameter values taken in a chaotic regime. Figure 4 plots the examples demonstrating the S-D transition, and, furthermore, different τ values are associated with a change cluster pattern as manifested by coupled logistic maps.

We have studied the effects of delay on the phenomena of phase synchronized cluster formation in coupled map

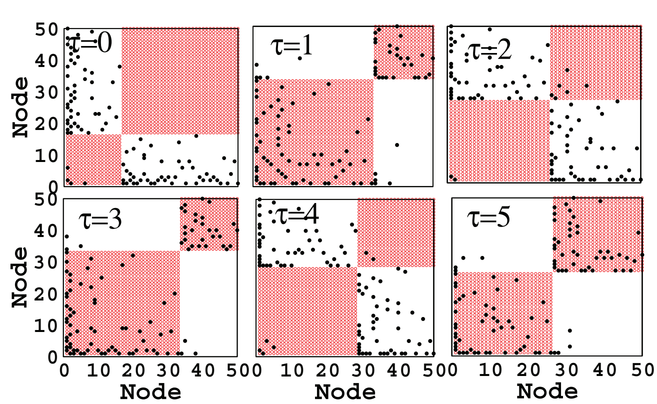

FIG. 4. (Color online) Phase synchronized patterns for coupled circle maps on scale-free networks with $N = 50$, $\langle k \rangle = 2$, $g(x) = x$, and $\varepsilon = 0.24$.

networks. Depending upon ε values, a change in τ may lead to a change in the phenomenon of cluster formation, with delays of the same parity being associated with the same phenomenon, or favoring D clusters for a delayed case over the undelayed one which in an extreme case of bipartite networks demonstrates the robustness of the D mechanism against change in τ. Furthermore, different τ values may lead to an entirely new pattern of the cluster. For example, in the middle ε range, different τ values lead to different dynamical patterns of dominant D type, whereas lower ε values produce dynamical patterns of dominant D or dominant SO type.

While an enhancement or suppression of complete synchronization as an introduction of delay was already well investigated in coupled map models, the mechanisms of the delayed unit to unit interaction were lacking. Delay may enhance the coordination among the connected nodes, leading to an enhancement of synchronization identifying the underlying connection topology, which had been the main theme of a few recent studies, but the observation of a D mechanism behind the cluster formation in delayed coupled networks is an insight suggesting that delay-induced synchronization may lead to a completely different relation between functional clusters and topology than relations observed for the undelayed evolution. Our study draws its significance in understanding synchronization in real world networks such as neural networks, where clusters are formed due to delayed interactions between neurons [23] and may be of D type [24]. An analysis presented for bipartite and periodic cases helps in discerning a possible impact of τ on the coupled evolution in such systems. Moreover, a change in patterns of neural activities has been found to be related with brain disorders such as Alzheimer's [25]. Research in the dimension of delay-induced patterns might propagate a finer apprehension of the origin and treatment of these diseases. At a fundamental level, a study of phase shift synchronization [2], based on the phase synchronization measure considered here, is an aspect to explore in future [26].

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[20] In N_{inter}, the coupling between two isolated nodes is not included.
[21] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.87.030902 for a discussion on the largest Lyapunov exponent of coupled dynamics and dynamical evolution of nodes in phase synchronized clusters.
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[26] Phase shift synchronization in maps can be defined as follows: maxima (minima) of a pair of nodes following a constant time difference for $t > t_0$.