Coulomb expansion: Analytical solutions

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Exact and approximate analytical solutions are presented, describing expansion of a cloud of charged particles in one, two, and three dimensions (assuming the planar, axial, and spherical symmetries, respectively). The expansion occurs in a gas or dilute plasma, where the screening is unimportant, so that particles interact with each other via Coulomb repulsive forces. It is shown that, irrespective of dimensionality, the density distribution remains homogeneous across the cloud and the velocity increases linearly towards the cloud boundary. The density evolution obeys a universal dependence, asymptotically decreasing with time as t^{-1}. It is also shown that in the presence of an inhomogeneous external field the interparticle repulsion becomes negligible at an early stage of expansion and then the density decreases with time exponentially.

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Nonequilibrium dynamical processes in ensembles of highly charged particles (charged aerosol) can evolve in very different environments such as atmosphere and interstellar media, laboratory and industrial plasmas [1–5]. The behavior of charged grains in the surrounding gas or plasma is determined by various collective mechanisms. If grains are embedded in a sufficiently dense plasma, the screening provided by free electrons and ions ensures short-range interparticle interactions, so the grain dynamics is characterized by wave modes sustained in such multispecies systems [6,7]. In contrast, in some cases the surrounding plasma can be so dilute that the screening is not important at relevant spatial scales. Then the charged grains interact with each other via long-range Coulomb forces. Experiments performed with such "dilute" complex plasmas [8,9] indicate that nonequilibrium dynamics of particle clouds in this case can be very different from the behavior of "regular" quasineutral clouds.

In this paper we study the expansion of a cloud of charged particles occurring in a gas or dilute plasma. We start the analysis with a one-dimensional (1D) plane problem and first investigate a free Coulomb expansion. Then we discuss the effect of an external field on the expansion and also present some analytical solutions for 2D and 3D cases. We neglect the pressure term in the equation of motion, naturally assuming that thermal effects are negligible. Furthermore, since the coordinate or/and density dependence of the particle charge Q makes the problem nonlinear (and presumably intractable), we shall consider the case when Q is constant (or an explicit function of time).

Planar 1D problem. To describe the 1D dynamics, we employ the Lagrangian mass coordinates (s,t_L) [10], where the Lagrangian time is $t_L = t$ and

$$s = \int_0^x dx' n(x',t) \tag{1}$$

is the coordinate expressed via the local number density $n(x,t)$. We assume that the system remains symmetric with respect to $x = 0$, i.e., $v(0,t) = 0$. Then the material and spatial derivatives are transformed from Eulerian coordinates according to the well-known rule

$$\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} = \frac{\partial}{\partial t_L}, \tag{2}$$

$$\frac{\partial}{n\partial x} = \frac{\partial}{\partial s}. \tag{3}$$

The resulting continuity and momentum equations for the density n and velocity v, as well as Poisson's equation for the self-consistent electric field E produced by charged particles, are (the subscript L for time is now omitted)

$$\frac{\partial n^{-1}}{\partial t} = \frac{\partial v}{\partial s}, \tag{4}$$

$$\frac{\partial v}{\partial t} + \nu v = \frac{QE}{M}, \tag{5}$$

$$\frac{\partial E}{\partial s} = 4\pi Q. \tag{6}$$

Here Q and M are, respectively, the charge and mass of individual particles and ν is the damping rate due to friction against a gas.

By taking the time derivative of Eq. (4), substituting $\partial^2 v/\partial s \partial t$ from Eq. (5), and using Eq. (6) we get

$$\frac{\partial^2 n^{-1}}{\partial t^2} + \nu\frac{\partial n^{-1}}{\partial t} = \frac{4\pi Q^2}{M}, \tag{7}$$

which has a general solution

$$n^{-1}(s,t) = c_1(s) + c_2(s)e^{-\nu t} + \frac{4\pi Q^2}{M\nu}t, \tag{8}$$

where $c_{1,2}(s)$ are to be determined from initial conditions. Assuming the initial density to be constant, $n(x,0) = n_0$ within the range $|x| \leqslant x_0$, we obtain $c_1 + c_2 = n_0^{-1}$. Then, substituting $\partial n^{-1}/\partial t$ in Eq. (4) and integrating it we derive $v(s,t) = (4\pi Q^2/M\nu)s - \nu e^{-\nu t}\int_0^s ds' c_2(s')$. If the particles were initially at rest, $v(s,0) = 0$, then $c_2 = 4\pi Q^2/M\nu$ and the solution of Eqs. (4)–(6) reads

$$\frac{n(t)}{n_0} = \left[1 + \frac{\omega_p^2}{\nu^2}(\nu t - 1 + e^{-\nu t})\right]^{-1}, \tag{9}$$

$$v(s,t) = \frac{\omega_p^2 s}{\nu n_0}(1 - e^{-\nu t}), \tag{10}$$

$$E(s) = 4\pi Q s, \tag{11}$$

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where $\omega_p = \sqrt{4\pi Q^2 n_0/M}$ is the initial plasma frequency of the charged cloud. We point out that the above solution is also valid when Q is an explicit function of time (e.g., due to charge fluctuations [6] or decharging processes [11,12]).

We see that the expansion is characterized by a uniform stretching—the density remains constant in space and decreases monotonically with time. Therefore, Eq. (1) is reduced to a simple relation $s = n(t)x$ and the cloud boundary x_b (> 0) is determined by the condition $s_b = n(t)x_b(t) = n_0 x_0$. At $\nu t \sim 1$, the time scaling of the density decay crosses over from $n \propto t^{-2}$ to $n \propto t^{-1}$, viz.,

$$\frac{n(t)}{n_0} \simeq \begin{cases} \left(1+\frac{1}{2}\omega_p^2 t^2\right)^{-1}, & \nu t \ll 1 \qquad (12)\\ \left(1+\omega_p^2 t/\nu\right)^{-1}, & \nu t \gg 1. \qquad (13)\end{cases}$$

Correspondingly, v increases linearly with x and attains the maximum at x_b. For $\nu t \gg 1$, the maximum velocity tends to a constant value of $v_b = x_0\omega_p^2/\nu$, which corresponds to a balance of the electric and friction forces. In the absence of friction, the velocity grows linearly with time, $v_b = x_0\omega_p^2 t$.

We note that the functions $c_{1,2}(s)$ in Eq. (8) can be derived also for a general case, when the initial density $n_0(x)$ is not constant and/or the initial velocity $v_0(x)$ is not zero. We get $c_1(s) + c_2(s) = n_0^{-1}(s)$ and $c_2(s) = 4\pi Q^2/M\nu^2 - v_0'(s)/\nu$, where the functions $n_0(s)$ and $v_0'(s) \equiv dv_0(s)/ds$ can be obtained from $n_0(x)$ and $v_0(x)$, respectively, by employing the relation $x(s)$ from Eq. (1). It is noteworthy that the asymptotic ($\nu t \gg 1$) solutions for n and v do not depend on the initial conditions and coincide with Eqs. (9) and (10) in this limit.

Role of external forces. The use of Lagrangian mass coordinates requires that velocity remains zero at $s = 0$, which implies that for a 1D problem the force $F_{\text{ext}}(x)$ associated with an external field should be an odd function of x. Let us consider a linear force, which we write in the form $F_{\text{ext}} = M\alpha x$. After the transformation from Eulerian coordinates, the external force results in an additional term $\alpha\int_0^s ds' n^{-1}(s',t)$ on the right-hand side of Eq. (5). This in turn leads to the term $-\alpha n^{-1}$ to be added to the left-hand side of Eq. (7). Obviously, when $\alpha < 0$ (which corresponds to an external potential well) the density exhibits damped oscillations, while for $\alpha > 0$ the external force accelerates the Coulomb expansion.

Let us illustrate the expansion with $\alpha > 0$. For simplicity, we consider the case of a weak damping $\sqrt{\alpha} \gg \nu$ and again assume homogeneous steady initial conditions. Then a general solution of the modified Eq. (7) is

$$n^{-1}(s,t) \simeq c_1 e^{\sqrt{\alpha}t} + c_2 e^{-\sqrt{\alpha}t} - \frac{\omega_p^2}{n_0\alpha}.$$

By employing the initial conditions we derive the following expressions for the density and velocity:

$$\frac{n(t)}{n_0} = \left[\cosh\sqrt{\alpha}t + \frac{\omega_p^2}{\alpha}(\cosh\sqrt{\alpha}t - 1)\right]^{-1}, \qquad (14)$$

$$v(s,t) = \frac{\sqrt{\alpha}s}{n_0}\left(1+\frac{\omega_p^2}{\alpha}\right)\sinh\sqrt{\alpha}t, \qquad (15)$$

while $E(s)$ remains (explicitly) unaffected. When $\alpha \to 0$, Eqs. (14) and (15) are naturally reduced to Eqs. (9) and (10), respectively (taken in the limit $\nu \to 0$). One can also see that even for $\omega_p \gg \sqrt{\alpha}$ the external field rapidly overcomes the effect of the self-consistent field (at $\sqrt{\alpha}t \gtrsim 3$) and then the density decays exponentially.

Thus an inhomogeneous external field always inhibits the Coulomb expansion: For $\alpha < 0$ the cloud remains confined in a potential trap produced by the field, while for $\alpha > 0$ the mutual repulsion rapidly becomes negligible in comparison with the external forces. The latter has been observed in the experiment of Ref. [9], where it was possible to analyze only the initial stage of expansion occurring in the plasma sheath.

Spherically and cylindrically symmetric problems. We see that the solution for a 1D case has certain characteristic features: For homogeneous steady initial conditions, the density of the expanding cloud remains homogeneous and the velocity increases linearly towards the boundary (and asymptotically, this is true for arbitrary initial conditions). One can expect that this similarity is universal and independent of dimensionality.

To verify this assumption, we write the system of governing equations in Eulerian coordinates

$$\frac{\partial n}{\partial t} + \frac{1}{r^{D-1}}\frac{\partial(r^{D-1}nv)}{\partial r} = 0, \qquad (16)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \nu v = \frac{QE}{M}, \qquad (17)$$

$$\frac{1}{r^{D-1}}\frac{\partial(r^{D-1}E)}{\partial r} = 4\pi Qn, \qquad (18)$$

where $D = 1,2,3$, and use the following ansatz for n, v, and E:

$$\frac{n(t)}{n_0} = \left(\frac{r_0}{r_b(t)}\right)^D, \qquad (19)$$

$$v(r,t) = \frac{r}{r_b(t)}\dot{r}_b(t), \qquad (20)$$

$$E(r,t) = \frac{4\pi}{D}Qn(t)r, \qquad (21)$$

with $r \leqslant r_b$ (for simplicity, we again consider a homogeneous initial distribution). This ansatz ensures that Eqs. (16) and (18) are satisfied identically. From Eq. (17) we derive the equation for the function $N(t) = r_b(t)/r_0 \equiv [n(t)/n_0]^{-1/D}$,

$$\ddot{N} + \nu\dot{N} = \frac{\omega_p^2}{D}N^{-D+1}. \qquad (22)$$

For $D = 1$, when $N = (n/n_0)^{-1}$ and Eq. (22) is reduced to Eq. (7), we obtain the solution corresponding to Eqs. (9)–(11). For $D = 2$ and 3, one cannot solve Eq. (22) analytically in a general case. However, several important cases can still be considered.

(i) One can find the asymptotic solution of Eq. (22) at $\nu t \gg 1$, when the second time derivative is negligible. This yields $N^D(t) \equiv n_0/n(t) = 1 + \omega_p^2 t/\nu$, which coincides with Eq. (14). Hence the asymptotic density evolution does not depend on D.

(ii) Equation (22) can be integrated exactly for $\nu = 0$. The first integrals are $\dot{N}^2 = \omega_p^2\ln N$ for $D = 2$ and $\dot{N}^2 = \frac{2}{3}\omega_p^2(1 - N^{-1})$ for $D = 3$ and the respective solutions are

$$\mathrm{erfi}\sqrt{\ln N} = \frac{1}{\sqrt{\pi}}\omega_p t, \quad D = 2,$$

$$\sqrt{N(N-1)} + \ln(\sqrt{N} + \sqrt{N-1}) = \sqrt{\frac{2}{3}}\omega_p t, \quad D = 3,$$

where $\mathrm{erfi}x = \frac{2}{\sqrt{\pi}}\int_0^x dx' e^{x'^2}$ is the imaginary error function. While the short-time evolution ($\omega_p t \ll 1$) is the same for all D and coincides with Eq. (12), the asymptotic behavior essentially depends on dimensionality: $n \propto (t^2 \ln t)^{-1}$ for $D = 2$ and $n \propto t^{-3}$ for $D = 3$. Here we note a very interesting link to self-similar solutions obtained for the expansion of ultracold neutral plasmas (assuming a Gaussian spatial distribution) [13–15]. In particular, these results suggest that the asymptotic behavior of the average density in expanding 3D neutral plasmas also obeys a dependence proportional to t^{-3} [13].

(iii) One can also get an approximate general solution of Eqs. (16)–(18), provided certain conditions are satisfied: For this, we transform Eqs. (16)–(18) to Lagrangian mass coordinates $s_{2D} = 2\pi \int_0^r dr' r' n(r',t)$ or $s_{3D} = 4\pi \int_0^r dr' r'^2 n(r',t)$. Also, we introduce $\tilde{v} = 2\pi r v$ and $\tilde{E} = 2\pi r E$ for a 2D problem and $\tilde{v} = 4\pi r^2 v$ and $\tilde{E} = 4\pi r^2 E$ for a 3D problem. The resulting set of equations for n, $\tilde{v}$, and $\tilde{E}$ is identical to Eqs. (4)–(6), except for the additional term $\tilde{v}^2/2\pi r^D$ in the velocity equation, which turns out to be negligible at any t provided $\nu \gtrsim \omega_p$. In this case, the solutions for $n(t)$ and $E(r,t)$ are given by Eqs. (9) and (21), respectively, whereas the general dependence for the velocity is

$$v(r,t) = \frac{\omega_p^2 r}{D\nu}\frac{n(t)}{n_0}(1 - e^{-\nu t}). \tag{23}$$

The field at the cloud boundary varies with time as $[n(t)]^{1-1/D}$, i.e., it decreases monotonically for $D = 2$ and 3. Correspondingly, the velocity at the boundary varies as $(1 - e^{-\nu t})[n(t)]^{1-1/D}$, i.e., for $D = 2$ and 3 it attains a maximum at $t_{\max} \sim \nu^{-1}\ln(\nu/\omega_p)$ (for $\nu/\omega_p \gg 1$).

A remarkable fact is that, irrespective of dimensionality and initial conditions, for any $\nu \neq 0$ the density decays asymptotically as $n \propto t^{-1}$. This implies that the expansion of "asymmetric" clouds (which do not possess the initial symmetries considered in this paper) may follow the same scaling law. Interestingly, the cloud expansion observed in the experiment of Ref. [16] was attributed by the authors to an ambipolar diffusion. The latter is characterized by exponential density decay, but Fig. 2 of that paper shows that in fact the asymptotic decay was very close to the dependence proportional to t^{-1}, suggesting the Coulomb expansion as the dominating process.

We conclude that expansion of a cloud of charged particles is described by analytical solutions such that the density distribution remains homogeneous across the cloud and the velocity increases linearly towards the cloud boundary. Such similarity is universal for planar, axially symmetric, and spherically symmetric clouds. The presented solutions can be useful for the analysis of experiments and numerical simulations investigating the dynamics of charged grain or aerosol clouds. It would be very interesting to verify whether the derived scaling laws also work for asymmetric expanding clouds.

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