Relativistic hydrodynamics from the projection operator method

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(Received 7 October 2012; published 13 February 2013)

We study relativistic hydrodynamics in the linear regime, based on Mori's projection operator method. In relativistic hydrodynamics, it is considered that an ambiguity about the fluid velocity occurs from the choice of a local rest frame: the Landau and Eckart frames. We find that the difference of the frames is not the choice of the local rest frame, but rather that of dynamic variables in the linear regime. We derive hydrodynamic equations in both frames by the projection operator method. We show that the natural derivation gives the linearized Landau equation. Also we find that, even for the Eckart frame, the slow dynamics is actually described by the dynamic variables for the Landau frame.

DOI: 10.1103/PhysRevE.87.023007 PACS number(s): 47.75.+f, 05.40.−a, 47.10.−g

I. INTRODUCTION

Relativistic hydrodynamics has been widely applied for studying relativistic nonequilibrium phenomena. For examples, it describes hadron spectra and elliptic flow in heavy ion physics [1,2] and jets in astrophysics [3,4]. The hydrodynamic equations applied to these systems are mainly those for perfect fluids. One of the reasons is that the dissipative effects in relativistic hydrodynamics are not fully understood, e.g., some pathological problems arise from the dissipative effects: the acausal propagation and the instability of the equilibrium state [5]. Although many hydrodynamic equations have been proposed to resolve these problems [6–11], it is still not obvious which equation describes the correct behavior of the relativistic dissipative fluid. Namely, even the basic equation has not been established in relativistic hydrodynamics.

The relativistic hydrodynamic equations are generally given as the following conservation laws:

$$\partial_\mu j^\mu = 0, \tag{1}$$

$$\partial_\mu T^{\mu\nu} = 0. \tag{2}$$

Here j^μ is the particle current and $T^{\mu\nu}$ is the energy-momentum tensor. They are decomposed into

$$j^\mu = nu^\mu + \nu^\mu, \tag{3}$$

$$T^{\mu\nu} = hu^\mu u^\nu - Pg^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \tau^{\mu\nu}, \tag{4}$$

where n is the particle density, $h = e + P$ the enthalpy density, P the pressure, e the energy density, and u^μ the fluid four-velocity. The dissipative terms ν^μ, q^μ, and $\tau^{\mu\nu}$ denote the particle and energy diffusions and the viscous stress tensor, respectively. The explicit expressions of these terms are not unique, but depend on the equations considered. This ambiguity comes from the choice of local rest frames of the fluid.

To see this ambiguity let us classify the hydrodynamic equations into two groups: the Eckart and Landau frames [12,13]. In the Eckart frame, the local rest frame is defined as that of the particle current, i.e., the fluid velocity is proportional to the particle current,

$$u_E^\mu \propto j^\mu. \tag{5}$$

In this frame the particle diffusion is absent $\nu^\mu = 0$. In contrast, in the Landau frame, the fluid velocity is proportional to the energy current

$$u_L^\mu \propto u_L^\nu T^\mu_\nu. \tag{6}$$

In contrast to the Eckart frame, the energy diffusion is absent $q^\mu = 0$. We note that nonrelativistic hydrodynamics do not have these ambiguity. In the nonrelativistic limit, the energy current is identical to the particle current because the mass energy dominates the energy of fluids. Actually, the Navier-Stokes equation does not have the same ambiguity as the frames. This difference between the frames is considered just by the references frames and is apparent. However, several differences that are not apparent, actually exist. For example, the Eckart frame has the instability of the global equilibrium state at the rest frame, but the Landau frame does not.

To discuss the difference between the Landau and Eckart frames we consider fluctuations from the global equilibrium state, namely, the linear nonequilibrium regime. The merit of this fluctuating state is that we can observe the state at the same rest frame for the energy and particle currents. We note that, at the equilibrium state, the particle and energy currents rest: $u_L^\mu = u_E^\mu = (1,\mathbf{0})$. In the fluctuating state, we also have the same reference frame for the Landau and Eckart frames because the state considered is just perturbed from the equilibrium one; moreover, we need not be concerned about what local equilibrium and local rest are for the relativistic system. Therefore, in this paper we focus on the linear fluctuations from the thermal equilibrium state at the rest frame.

To see relativistic hydrodynamics in the linear regime, let us consider the Landau and Eckart equations as examples. For the Landau equation the dissipative terms read

$$\nu^\mu = \lambda\left(\frac{nT}{h}\right)^2 \partial_\perp^\mu(\beta\mu), \tag{7}$$

$$q^\mu = 0, \tag{8}$$

$$\tau^{\mu\nu} = \eta\left[\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}(\partial_\perp\cdot u)\right] + \zeta\Delta^{\mu\nu}(\partial_\perp\cdot u), \tag{9}$$

where λ, η, and ζ are the thermal conductivity, the shear viscosity, and the bulk viscosity, respectively; $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is a projection; and $\partial_\perp^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ is the spacelike derivative.

Now we linearize the Landau equation about fluctuations from the equilibrium state. Let us write $n(x) = n_0 + \delta n(x)$,

$e(x) = e_0 + \delta e(x)$, $P(x) = P_0 + \delta P(x)$, $(\beta\mu)(x) = (\beta\mu)_0 + \delta(\beta\mu)(x)$, and $u^\mu(x) = u_0^\mu + \delta u^\mu(x)$. Here the symbols with the prefix δ denote the fluctuations. The equilibrium values are denoted by the suffix 0. Hereafter we employ variables with the suffix and the prefix as the equilibrium values and fluctuations, respectively. For simplicity, let us choose the rest frame as the reference frame: $u_0^\mu = (1,\mathbf{0})$. By the relation in the linear regime $u_0^\mu \delta u_\mu = 0$, the fluid-velocity fluctuation is written as

$$\delta u^\mu = (0,\delta \boldsymbol{v}_L). \tag{10}$$

In consequence, the Landau equation is linearized as

$$\partial_0 \delta n = -n_0 \nabla \cdot \delta \boldsymbol{v}_L + \lambda \left(\frac{n_0 T_0}{h_0}\right)^2 \nabla^2 \delta(\beta\mu), \tag{11}$$

$$\partial_0 \delta e = -h_0 \nabla \cdot \delta \boldsymbol{v}_L, \tag{12}$$

$$\partial_0 (h_0 \delta \boldsymbol{v}_L) = -\nabla(\delta P) + \left(\zeta + \frac{1}{3}\eta\right) \nabla(\nabla \cdot \delta \boldsymbol{v}_L) + \eta \nabla^2 \delta \boldsymbol{v}_L. \tag{13}$$

Let us move on to the Eckart equation. The dissipative terms of the Eckart equation are

$$\nu^\mu = 0, \tag{14}$$

$$q^\mu = \lambda(\partial_\perp^\mu T - T D u^\mu), \tag{15}$$

$$\tau^{\mu\nu} = \eta \left[\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}(\partial_\perp \cdot u)\right] + \zeta \Delta^{\mu\nu}(\partial_\perp \cdot u), \tag{16}$$

where $D \equiv u^\mu \partial_\mu$ is the timelike derivative. The linearized equations are

$$\partial_0 \delta n = -n_0 \nabla \cdot \delta \boldsymbol{v}_E, \tag{17}$$

$$\partial_0 \delta e = -h_0 \nabla \cdot \delta \boldsymbol{v}_E + \lambda(\nabla^2 \delta T + T_0 \partial_0 \nabla \cdot \delta \boldsymbol{v}_E), \tag{18}$$

$$\partial_0 (h_0 \delta \boldsymbol{v}_E) = -\nabla(\delta P) + \eta \nabla^2 \delta \boldsymbol{v}_E + \left(\zeta + \frac{1}{3}\eta\right) \nabla(\nabla \cdot \delta \boldsymbol{v}_E) + \lambda\left(\nabla \partial_0 \delta T + T_0 \partial_0^2 \delta \boldsymbol{v}_E\right). \tag{19}$$

We note that the linearized Landau and Eckart equations have different forms even in the same rest frame, as previously mentioned.

To investigate relativistic hydrodynamics in the linear regime, we use here Mori's projection operator method [14]. Mori's projection operator method is a powerful tool for extracting slow dynamics. This method is widely applied and successful in condensed matter physics [15–17]. Actually, various slow dynamics, e.g., the Navier-Stokes, Langevin, and Boltzmann equations and equations for Nambu-Goldstone bosons, are derived [16,18,19]. The merit of the projection operator method is that we can derive slow dynamics only by choosing slow variables and commutation relations of those without microscopic details. We note that dynamics on a macroscopic scale can be described by much fewer degrees of freedom than those on a microscopic scale. Such degrees of freedom are called slow variables (or gross variables). The slow variables are degrees of freedom that label a macroscopic state and describe its long-time behavior.

From the projection operator method we find that the choice of slow variables is important for hydrodynamics. From the linearized Landau and Eckart equations (11)–(13) and (17)–(19) we see that the dynamic variables are given as the energy and particle densities and fluid-velocity fluctuations[1]

$$\left\{\delta e, \delta n, \delta v^i_{L,E}\right\}. \tag{20}$$

The fluid velocities are different, depending on the frames,

$$\delta v_E^i = n_0^{-1} j^i, \tag{21}$$

$$\delta v_L^i = h_0^{-1} T^{0i}. \tag{22}$$

Namely, the difference of the frames is the choice of slow variables.

The important point about the choices is that j^i is not essentially slow because it is not a conserved charge. In contrast, T^{0i} is a conserved charge and slow because the energy current is equivalent to the momentum for the relativistic system: $T^{0i} = T^{i0}$. Actually, we shall find that a slow part in j^i originates from T^{0i}. We will provide details on this point in Secs. II C and V. We will apply the projection operator method for the choices

$$\{\delta e, \delta p^i, \delta n\}, \tag{23}$$

$$\{\delta e, \delta p^i, \delta n, \delta j^i\}, \tag{24}$$

where $\delta p^i \equiv T^{0i}$ and $\delta j^i \equiv j^i$. We will show that the first set of slow variables (23) gives the linearized Landau equations (11)–(13). Furthermore, we will derive the equations for the second set (24), which include the Landau and Eckart frames. Then we will derive the linearized Eckart equations by eliminating δp^i from the equations for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$. We note that, to correctly treat the slow part of δj^i coming from δp^i, we have to choose both of them as slow variables.

After that we will discuss that the Landau frame is natural for the slow dynamics. In particular, we shall show that the equations for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$, which include both the Landau and Eckart frames, have the same slow modes as the Landau equation. Namely, the slow modes are determined only by $\{\delta e, \delta p^i, \delta n\}$ and the current density δj^i contains an irrelevant part for the slow dynamics. Moreover, we will illustrate that, even for the Eckart equation, the slow dynamics is actually described by the Landau variables $\{\delta e, \delta p^i, \delta n\}$.

Our study is an attempt to base the derivation on the projection operator method. Earlier studies [7–10] assume the relativistic Boltzmann equation as an underlying microscopic theory. In contrast, we stress that our derivation is independent of microscopic details.

This paper is organized as follows. In Sec. II we briefly review Mori's projection operator method for readers unfamiliar with it. Also we explain conserved charges as the slow variables and discuss those for relativistic hydrodynamics. In Sec. III we determine equal-time correlations of the slow variables to determine the properties of the equilibrium state. We note that here we consider the fluctuations from the equilibrium state. Our determination is based on thermodynamics and Lorentz symmetry on a microscopic scale, thus it is independent of the microscopic detail. In Sec. IV we derive the slow dynamics by the projection operator method for the

[1] We note that the intensive variables, such as $\delta(\beta\mu)$ and δT, turn out to be functions of δn and δe by thermodynamic relations.

sets of equations (23) and (24). In Sec. V we discuss that the Landau frame is natural for relativistic hydrodynamics. In particular, we study slow modes of the equations for $\{\delta e,\delta p^i,\delta n\}$ and $\{\delta e,\delta p^i,\delta n,\delta j^i\}$. We explicitly show that these equations have the same modes. Furthermore, we consider the Onsager reciprocal relation in the Eckart equation to illustrate the slow part of $\delta \boldsymbol{j}$ coming from $\{\delta e,\delta p^i,\delta n\}$.

II. MORI'S PROJECTION OPERATOR METHOD

In this section we provide Mori's projection operator method [14,16,17,20,21]. We can formally extract slow dynamics from microscopic Hamiltonian dynamics by this method. On the microscopic scale, an operator at time t, $\hat{\mathcal{O}}(t) = e^{i\hat{H}t}\hat{\mathcal{O}}(0)e^{-i\hat{H}t}$, evolves by the Heisenberg equation

$$\partial_0\hat{\mathcal{O}}(t) = i[\hat{H},\hat{\mathcal{O}}(t)] \equiv i\hat{\mathcal{L}}\hat{\mathcal{O}}(t), \tag{25}$$

where $\hat{\mathcal{L}}$ is the Liouville operator. In the following we decompose this time evolution equation into slow and fast ones.

A. Projection operator

First we introduce basic ingredients for the projection operator. Let us consider a many-body system at finite temperature. As an equilibrium distribution we assume the grand-canonical one. Then the density matrix is given as

$$\hat{\rho}_{\text{eq}} \equiv \frac{e^{-\beta(\hat{H}-\mu\hat{N})}}{\text{tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}}, \tag{26}$$

where $\hat{H}$ is the Hamiltonian, $\hat{N}$ is the number operator, μ is the chemical potential, and the inverse temperature $\beta = 1/T$. With the density matrix, the thermal average of $\hat{\mathcal{O}}(t)$ is defined as

$$\langle\hat{\mathcal{O}}(t)\rangle_{\text{eq}} \equiv \text{tr}\,\hat{\rho}_{\text{eq}}\hat{\mathcal{O}}(t) = \langle\hat{\mathcal{O}}(0)\rangle_{\text{eq}}. \tag{27}$$

Also we define the inner product of $\hat{A}$ and $\hat{B}$ as

$$\begin{aligned}(\hat{A},\hat{B}) &\equiv \frac{1}{\beta}\int_0^\beta d\tau\langle e^{\tau(\hat{H}-\mu\hat{N})}\hat{A}e^{-\tau(\hat{H}-\mu\hat{N})}\hat{B}^\dagger\rangle_{\text{eq}}\\ &= \frac{1}{\beta}\int_0^\beta d\tau\langle\hat{A}(-i\tau)\hat{B}^\dagger\rangle_{\text{eq}}.\end{aligned} \tag{28}$$

Moreover, we introduce a set of slowly varying operators (slow variables) $\{\hat{A}_n(t,\boldsymbol{x})\} = \{\hat{A}_1,\hat{A}_2,\ldots,\hat{A}_n\}$. If we can separate the time scale into long- and short-time scales, such operators exist and describe the slow dynamics. Let us consider the slow operators at the initial time $t = 0$, $\{\hat{A}_n(0,\boldsymbol{x})\}$. In general, they are not orthogonal to each other. We introduce a metric to consider the orthogonal basis:

$$g_{nm}(\boldsymbol{x}-\boldsymbol{y}) \equiv (\hat{A}_n(0,\boldsymbol{x}),\hat{A}_m(0,\boldsymbol{y})). \tag{29}$$

The orthogonal operators, represented with an upper index, are defined as

$$\hat{A}^n(t,\boldsymbol{x}) \equiv \int d^3y\, g^{nm}(\boldsymbol{x}-\boldsymbol{y})\hat{A}_m(t,\boldsymbol{y}), \tag{30}$$

where $g^{nm}(\boldsymbol{x}-\boldsymbol{y})$ is the inverse of $g_{nm}(\boldsymbol{x}-\boldsymbol{y})$. These quantities are orthogonal to those with lower indices,

$$(\hat{A}_n(0,\boldsymbol{x}),\hat{A}^m(0,\boldsymbol{y})) = \delta_n^m\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{31}$$

$$\sum_m\int d^3y\, g_{nm}(\boldsymbol{x}-\boldsymbol{y})g^{ml}(\boldsymbol{y}-\boldsymbol{z}) = \delta_n^l\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{32}$$

We have prepared the basic ingredients. Let us introduce the projection operator acting on any operators $\hat{B}(t,\boldsymbol{x})$ as

$$\mathcal{P}\hat{B}(t,\boldsymbol{x}) \equiv \sum_n\int d^3y\, \hat{A}_n(0,\boldsymbol{y})(\hat{B}(t,\boldsymbol{x}),\hat{A}^n(0,\boldsymbol{y})). \tag{33}$$

The projection operator extracts the slowly varying part of $\hat{B}$, which is determined only by the slow variables $\{\hat{A}_n\}$. We also define the orthogonal projector as $\mathcal{Q} \equiv 1-\mathcal{P}$ for later use.

B. Generalized Langevin equation

In this section we derive the so-called the generalized Langevin equation. This equation is given by decomposing the Heisenberg equation into slow and fast parts. For the decomposition, we use the following operator identity:

$$\begin{aligned}\partial_0 e^{i\hat{\mathcal{L}}t} &= e^{i\hat{\mathcal{L}}t}\mathcal{P}i\hat{\mathcal{L}} + \int_0^t ds\, e^{i\hat{\mathcal{L}}(t-s)}\mathcal{P}i\hat{\mathcal{L}}\, e^{\mathcal{Q}i\hat{\mathcal{L}}s}\mathcal{Q}i\hat{\mathcal{L}}\\ &\quad + e^{\mathcal{Q}i\hat{\mathcal{L}}t}\mathcal{Q}i\hat{\mathcal{L}},\end{aligned} \tag{34}$$

which is valid for arbitrary $\hat{\mathcal{L}}$ and $\mathcal{P}$ [15].

Let us derive this identity. First we consider the following decomposition:

$$\partial_0 e^{i\hat{\mathcal{L}}t} = e^{i\hat{\mathcal{L}}t}i\hat{\mathcal{L}} = e^{i\hat{\mathcal{L}}t}\mathcal{P}i\hat{\mathcal{L}} + e^{i\hat{\mathcal{L}}t}\mathcal{Q}i\hat{\mathcal{L}}. \tag{35}$$

Next we consider the Laplace transform of $\exp(i\hat{\mathcal{L}}t)$,

$$\int_0^\infty dt\, e^{-zt}e^{i\hat{\mathcal{L}}t} = \frac{1}{z-i\hat{\mathcal{L}}}. \tag{36}$$

Then we decompose Eq. (36) into

$$\begin{aligned}\frac{1}{z-i\hat{\mathcal{L}}} &= \frac{1}{z-i\hat{\mathcal{L}}}(z-\mathcal{Q}i\hat{\mathcal{L}})\frac{1}{z-\mathcal{Q}i\hat{\mathcal{L}}}\\ &= \frac{1}{z-i\hat{\mathcal{L}}}(z-i\hat{\mathcal{L}}+\mathcal{P}i\hat{\mathcal{L}})\frac{1}{z-\mathcal{Q}i\hat{\mathcal{L}}}\\ &= \frac{1}{z-\mathcal{Q}i\hat{\mathcal{L}}} + \frac{1}{z-i\hat{\mathcal{L}}}\mathcal{P}i\hat{\mathcal{L}}\frac{1}{z-\mathcal{Q}i\hat{\mathcal{L}}}.\end{aligned} \tag{37}$$

Performing the inverse Laplace transform, we find the identity

$$e^{i\hat{\mathcal{L}}t} = e^{\mathcal{Q}i\hat{\mathcal{L}}t} + \int_0^t ds\, e^{i\hat{\mathcal{L}}(t-s)}\mathcal{P}i\hat{\mathcal{L}}\, e^{\mathcal{Q}i\hat{\mathcal{L}}s}. \tag{38}$$

Substituting Eq. (38) into the second term of Eq. (35), we obtain the operator identity (34).

Multiplying Eq. (34) by an initial value of the slow operator, we obtain the decomposed equation of motion for $\hat{A}_n(t) = e^{i\hat{\mathcal{L}}t}\hat{A}_n(0)$:

$$\begin{aligned}\partial_0\hat{A}_n(t,\boldsymbol{x}) &= \int d^3y\, i\Omega_n{}^m(\boldsymbol{x}-\boldsymbol{y})\hat{A}_m(t,\boldsymbol{y})\\ &\quad - \int_0^\infty ds\, d^3y\, \Phi_n{}^m(t-s,\boldsymbol{x}-\boldsymbol{y})\hat{A}_m(s,\boldsymbol{y}) + \hat{R}_n(t,\boldsymbol{x}),\end{aligned} \tag{39}$$

without any approximations. Here we introduced the following functions and operator:

$$i\Omega_n{}^m(\boldsymbol{x}-\boldsymbol{y}) \equiv (i\hat{\mathcal{L}}\hat{A}_n(0,\boldsymbol{x}),\hat{A}^m(0,\boldsymbol{y})) = -\frac{1}{\beta} i\langle[\hat{A}_n(0,\boldsymbol{x}),\hat{A}^{m\dagger}(0,\boldsymbol{y})]\rangle_{\rm eq}, \tag{40}$$

$$\Phi_n{}^m(t-s,\boldsymbol{x}-\boldsymbol{y}) \equiv -\theta(t-s)(i\hat{\mathcal{L}}\hat{R}_n(t,\boldsymbol{x}),\hat{A}^m(s,\boldsymbol{y})), \tag{41}$$

$$\hat{R}_n(t,\boldsymbol{x}) \equiv e^{it\mathcal{Q}\hat{\mathcal{L}}}\mathcal{Q}i\hat{\mathcal{L}}\hat{A}_n(0,\boldsymbol{x}), \tag{42}$$

where $\theta(t-s)$ in Eq. (41) is the Heaviside step function.

Equation (39) is the generalized Langevin equation and has the following properties.

(i) Equation (39) is the operator identity.

(ii) The first and second terms on the right-hand side represent the slow motions.

(iii) The first term corresponds to a time-reversible change.

(iv) The second term corresponds to a time-irreversible change. Also this term depends on a past time value $\hat{A}_m(s)$ for $s<t$. Here $\Phi_n{}^m(t-s,\boldsymbol{x}-\boldsymbol{y})$ is called the memory function.

(v) The last term is the noise term corresponding to the fast motion. For hydrodynamics this term is usually neglected, whereas for the Langevin dynamics we treat this term as a random noise.

It is useful to rewrite Eq. (39) as an equation in momentum space

$$\partial_0\hat{A}_n(t,\boldsymbol{k}) = i\Omega_n{}^m(\boldsymbol{k})\hat{A}_m(t,\boldsymbol{k}) - \int_0^\infty ds\,\Phi_n{}^m(t-s,\boldsymbol{k})\hat{A}_m(s,\boldsymbol{k}) + \hat{R}_n(t,\boldsymbol{k}). \tag{43}$$

For the time component we perform the Laplace transform

$$\hat{A}_n(z,\boldsymbol{k}) = \int dt\int d^3x\, e^{-zt}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{A}_n(t,\boldsymbol{x}). \tag{44}$$

Then Eq. (43) becomes

$$z\hat{A}(z,\boldsymbol{k}) = i\Omega(\boldsymbol{k})\hat{A}(z,\boldsymbol{k}) - \Phi(z,\boldsymbol{k})\hat{A}(z,\boldsymbol{k}) + \hat{R}(z,\boldsymbol{k}) + \hat{A}(t=0,\boldsymbol{k}) \tag{45}$$

in the Laplace momentum space. Here $\hat{A}(t=0,\boldsymbol{k})$ is the initial value and we used matrix notation.

C. Conserved charges as slow variables

Here we explain why conserved charges are slow and discuss the dynamic variables for the Landau and Eckart frames. The key is that conserved charge densities generally satisfy conservation laws

$$\partial_0\hat{j}^0 = -\partial_i\hat{j}^i, \tag{46}$$

where $\hat{j}^0$ is a conserved charged density and $\hat{j}^i$ is its current. In the momentum space the conservation law becomes

$$\partial_0\hat{j}^0 = ik_i\hat{j}^i. \tag{47}$$

We note that the time change rate of $\hat{j}^0$ is proportional to the wave number, so the low-wave-number components turn out to be slow. Therefore, the change of the conserved charge densities is necessarily slow in the low-wave-number region, i.e., on the macroscopic scale.

Now let us consider the case of relativistic hydrodynamics. We have the three conservation laws in Eqs. (1) and (2). From those we obtain the three conserved charges, i.e., the particle number, the energy, and the momentum:

$$\partial_0\hat{j}^0 = ik_i\hat{j}^i, \tag{48}$$

$$\partial_0\hat{T}^{00} = ik_i\hat{T}^{0i}, \tag{49}$$

$$\partial_0\hat{T}^{i0} = ik_j\hat{T}^{ij}. \tag{50}$$

We note that the above quantities are the slow variables for the Landau equation. The important point is that the particle current is not conserved:

$$\partial_0\hat{j}^i \neq ik_j\hat{\Pi}^{ij}. \tag{51}$$

Thus the time change rate is not proportional to the wave number. Namely, the particle current is not essentially slow, although it is proportional to the fluid velocity for the Eckart frame.

Nevertheless, we note that the particle current has a slow part coming from the conserved charges because $\hat{j}^i$ is not orthogonal to $\{\hat{j}^0,\hat{T}^{00},\hat{T}^{0i}\}$. In other words, the projection of $\hat{j}^i$ on those does not vanish,

$$\mathcal{P}\hat{j}^i \neq 0, \tag{52}$$

and gives the slow part. From this slow part we can derive the linearized Eckart equation, as we will show in Sec. IV. Here we stress that the slow dynamics is essentially determined by the conserved charges $\{\hat{j}^0,\hat{T}^{00},\hat{T}^{0i}\}$ even for the Eckart equation.

III. METRIC AND THERMODYNAMIC QUANTITIES

In this section we discuss relations between the metric g_{nm} and thermodynamic quantities [15,16,22]. As discussed in Sec. I, we employ the fluctuations of conserved charges as slow variables, i.e., $\hat{A}_n = \{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$, where $\delta\hat{e} \equiv \hat{T}^{00}-e_0$, $\delta\hat{p}^i \equiv \hat{T}^{0i}$, $\delta\hat{n} \equiv \hat{j}^0 - n_0$, and $\delta\hat{j}^i \equiv \hat{j}^i$ with $e_0 \equiv \langle\hat{T}^{00}\rangle_{\rm eq}$ and $n_0 \equiv \langle\hat{j}^0\rangle_{\rm eq}$. We assume that the density matrix at thermal equilibrium is invariant under a time-reversal transformation, i.e., $\mathcal{T}\hat{\rho}_{\rm eq}\mathcal{T}^{-1} = \hat{\rho}_{\rm eq}$, where $\mathcal{T}$ is the time-reversal operator. The slow variables transform under $\mathcal{T}$ as

$$\mathcal{T}\delta\hat{e}(t,\boldsymbol{x})\mathcal{T}^{-1} = \delta\hat{e}(-t,\boldsymbol{x}), \quad \mathcal{T}\delta\hat{n}(t,\boldsymbol{x})\mathcal{T}^{-1} = \delta\hat{n}(-t,\boldsymbol{x}), \tag{53}$$

$$\mathcal{T}\delta\hat{p}^i(t,\boldsymbol{x})\mathcal{T}^{-1} = -\delta\hat{p}^i(-t,\boldsymbol{x}), \quad \mathcal{T}\delta\hat{j}^i(t,\boldsymbol{x})\mathcal{T}^{-1} = -\delta\hat{j}^i(-t,\boldsymbol{x}), \tag{54}$$

where $\delta\hat{e}$ and $\delta\hat{n}$ ($\delta\hat{p}^i$ and $\delta\hat{j}^i$) are even (odd) operators, so that $\delta\hat{n}$ and $\delta\hat{e}$ does not mix $\delta\hat{p}^i$ and $\delta\hat{j}^i$, i.e., $g_{ep}(\boldsymbol{k}) = g_{ej}(\boldsymbol{k}) = g_{np}(\boldsymbol{k}) = g_{nj}(\boldsymbol{k}) = 0$.

Since we are interested in the low-energy behavior of slow variables, we apply the derivative expansion. The metric is expanded as a power series of k^i,

$$g_{nm}(\boldsymbol{k}) = g_{nm} + g^{(1)}_{nm;i}k^i + g^{(2)}_{nm;ij}k^ik^j + \cdots, \tag{55}$$

where we assumed that $g_{nm}(\boldsymbol{k})$ is analytic at $\boldsymbol{k}=\boldsymbol{0}$; in other words, there are no long-range correlations. The only leading terms g_{nm} contribute to the linearized hydrodynamic equations at first order, so we will not consider contributions from the higher-order terms.

A. Susceptibilities g_{nn}, g_{ee}, and g_{en}

First, we focus on g_{nn}, g_{ee}, and g_{en}, which are nothing but susceptibilities,

$$g_{ee} = \int d^3x(\delta\hat{e}(\boldsymbol{x}),\delta\hat{e}(\boldsymbol{0})) = \frac{1}{V}\langle(\delta\hat{H})^2\rangle_{\rm eq}, \quad (56)$$

$$g_{nn} = \int d^3x(\delta\hat{n}(\boldsymbol{x}),\delta\hat{n}(\boldsymbol{0})) = \frac{1}{V}\langle(\delta\hat{N})^2\rangle_{\rm eq}, \quad (57)$$

$$g_{en} = g_{ne} = \int d^3x(\delta\hat{e}(\boldsymbol{x}),\delta\hat{n}(\boldsymbol{0})) = \frac{1}{V}\langle\delta\hat{H}\delta\hat{N}\rangle_{\rm eq}, \quad (58)$$

where V is the volume, $\delta\hat{H} = \hat{H} - \langle\hat{H}\rangle_{\rm eq}$, and $\delta\hat{N} = \hat{N} - \langle\hat{N}\rangle_{\rm eq}$. We also used the following relations:

$$\int d^3x\, e^{\tau(\hat{H}-\mu\hat{N})}\delta\hat{e}(\boldsymbol{x})e^{-\tau(\hat{H}-\mu\hat{N})} = \int d^3x\, \delta\hat{e}(\boldsymbol{x}) = \delta\hat{H}, \quad (59)$$

$$\int d^3x\, e^{\tau(\hat{H}-\mu\hat{N})}\delta\hat{n}(\boldsymbol{x})e^{-\tau(\hat{H}-\mu\hat{N})} = \int d^3x\, \delta\hat{n}(\boldsymbol{x}) = \delta\hat{N}. \quad (60)$$

Using the grand partition function $Z \equiv \mathrm{tr}\exp[-\beta(\hat{H}-\mu\hat{N})]$, we can rewrite these susceptibilities as

$$g_{ee} = \frac{1}{V}\left(\frac{\partial^2\ln Z}{\partial\beta^2}\right)_{\beta\mu} = -\left(\frac{\partial e}{\partial\beta}\right)_{\beta\mu,V}, \quad (61)$$

$$g_{nn} = \frac{1}{V}\left(\frac{\partial^2\ln Z}{\partial(\beta\mu)^2}\right)_{\beta} = \left(\frac{\partial n}{\partial(\beta\mu)}\right)_{\beta,V}, \quad (62)$$

$$g_{en} = \frac{1}{V}\left(\frac{\partial^2\ln Z}{\partial\beta\partial(\beta\mu)}\right) = \left(\frac{\partial e}{\partial(\beta\mu)}\right)_{\beta,V} = -\left(\frac{\partial n}{\partial\beta}\right)_{\beta\mu,V}. \quad (63)$$

From these equation, the inverse matrices for e and n are obtained as

$$g^{ee} = -\left(\frac{\partial\beta}{\partial e}\right)_n, \quad (64)$$

$$g^{nn} = \left(\frac{\partial(\beta\mu)}{\partial n}\right)_e, \quad (65)$$

$$g^{ne} = g^{en} = -\left(\frac{\partial\beta}{\partial n}\right)_e = \left(\frac{\partial(\beta\mu)}{\partial e}\right)_n. \quad (66)$$

Then the slow variables with an upper index are

$$\hat{A}^e = -\left(\frac{\partial\beta}{\partial e}\right)_n\delta\hat{e} - \left(\frac{\partial\beta}{\partial n}\right)_e\delta\hat{n} \equiv -\delta\hat{\beta}, \quad (67)$$

$$\hat{A}^n = \left(\frac{\partial(\beta\mu)}{\partial e}\right)_n\delta\hat{e} + \left(\frac{\partial(\beta\mu)}{\partial n}\right)_e\delta\hat{n} \equiv \delta(\widehat{\beta\mu}), \quad (68)$$

where we introduced the operators for the fluctuations of β and $\beta\mu$. Namely, those are orthonormal to $\delta\hat{e}$ and $\delta\hat{n}$ in the leading order of the derivative expansion:

$$(\delta\hat{e}(t,\boldsymbol{r}),\delta(\widehat{\beta\mu})(t,\boldsymbol{r}')) = (\delta\hat{n}(t,\boldsymbol{r}),\delta\hat{\beta}(t,\boldsymbol{r}')) = 0, \quad (69)$$

$$-(\delta\hat{e}(t,\boldsymbol{r}),\delta\hat{\beta}(t,\boldsymbol{r}')) = (\delta\hat{n}(t,\boldsymbol{r}),\delta(\widehat{\beta\mu})(t,\boldsymbol{r}')) = \delta(\boldsymbol{r}-\boldsymbol{r}'). \quad (70)$$

For later use, we also introduce the operators for the pressure and the temperature fluctuations as

$$\delta\hat{P} \equiv \left(\frac{\partial P}{\partial e}\right)_n\delta\hat{e} + \left(\frac{\partial P}{\partial n}\right)_e\delta\hat{n}, \quad (71)$$

$$\delta\hat{T} \equiv \left(\frac{\partial T}{\partial e}\right)_n\delta\hat{e} + \left(\frac{\partial T}{\partial n}\right)_e\delta\hat{n}. \quad (72)$$

These operators satisfy the usual thermodynamic relations, e.g., the Gibbs-Duhem relation

$$\delta\hat{P} = \frac{h_0}{T_0}\delta\hat{T} + T_0 n_0\delta(\widehat{\beta\mu}), \quad (73)$$

because we can apply the thermodynamic relations to the coefficients, such as $(\partial P/\partial e)_n$, in the definitions.

B. Quantities $g_{p^ip^j}$, $g_{p^ij^j}$, and $g_{j^ij^j}$

Next we consider $g_{p^ip^j}$, $g_{p^ij^j}$, and $g_{j^ij^j}$. We show that two of them $g_{p^ip^j}$ and $g_{p^ij^j}$ are expressed as the enthalpy and the number density

$$g_{p^ip^j} = \int d^3x(\hat{T}^{0i}(0,\boldsymbol{x}),\hat{T}^{0j}(0,\boldsymbol{0})) = \delta^{ij}T_0h_0, \quad (74)$$

$$g_{p^ij^j} = \int d^3x(\hat{T}^{0i}(0,\boldsymbol{x}),\hat{j}^i(0,\boldsymbol{0})) = \delta^{ij}T_0n_0. \quad (75)$$

The same relations are obtained in Ref. [22]. These relations are derived from the Lorentz symmetry underlying the theory. For an arbitrary Hermitian operator $\hat{\mathcal{O}}$, the following identity is satisfied:

$$\int d^3x(\hat{T}^{0i}(0,\boldsymbol{x}),\hat{\mathcal{O}}) = (\hat{P}^i,\hat{\mathcal{O}}) = i([\hat{H},\hat{K}^i],\hat{\mathcal{O}}) = -iT\langle[\hat{K}^i,\hat{\mathcal{O}}]\rangle_{\rm eq}, \quad (76)$$

where $\hat{K}^i$ is the boost operator, $[\hat{H},\hat{K}^i] = -i\hat{P}^i$, and the following Kubo's identity is employed:

$$([\hat{H},\hat{A}],\hat{B}) = -T\langle[\hat{A},\hat{B}^\dagger]\rangle_{\rm eq}. \quad (77)$$

Since $\hat{T}^{\mu\nu}(x)$ and $\hat{j}^\mu(x)$ are the Lorentz tensor and vector, respectively, these transform under the Lorentz transformation as

$$[\hat{L}^{\mu\nu},\hat{T}^{\lambda\rho}(x)] = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\hat{T}^{\lambda\rho}(x) - i[\eta^{\mu\lambda}\hat{T}^{\nu\rho}(x) - \eta^{\nu\lambda}\hat{T}^{\mu\rho}(x) + \eta^{\mu\rho}\hat{T}^{\lambda\nu} - \eta^{\nu\rho}\hat{T}^{\lambda\mu}(x)], \quad (78)$$

$$[\hat{L}^{\mu\nu},\hat{j}^\rho(x)] = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\hat{j}^\rho(x) - i[\eta^{\mu\rho}\hat{j}^\nu(x) - \eta^{\nu\rho}\hat{j}^\mu(x)], \quad (79)$$

where $\hat{L}^{\mu\nu}$ is the charge of the Lorentz symmetry and $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the (inverse) Minkowski metric. For the Lorentz boost $\hat{K}^i = \hat{L}^{i0}$ they obey

$$[\hat{K}^i,\hat{T}^{0j}(x)] = -i(x^0\partial^i - x^i\partial^0)\hat{T}^{0j}(x) + i\hat{T}^{ij}(x) - i\eta^{ij}\hat{T}^{00}(x), \quad (80)$$

$$[\hat{K}^i,\hat{j}^j(x)] = -i(x^0\partial^i - x^i\partial^0)\hat{j}^j(x) - i\eta^{ij}\hat{j}^0(x). \quad (81)$$

Therefore, the thermal averages for these commutators satisfy

$$\langle[\hat{K}^i,\hat{T}^{0j}(x)]\rangle_{\rm eq} = i\langle\hat{T}^{ij}(x) - \eta^{ij}\hat{T}^{00}(x)\rangle_{\rm eq} = i\delta^{ij}h_0, \quad (82)$$

$$\langle[\hat{K}^i,\hat{j}^j(x)]\rangle_{\rm eq} = -i\langle\eta^{ij}\hat{j}^0(x)\rangle_{\rm eq} = i\delta^{ij}n_0. \quad (83)$$

Inserting Eqs. (82) and (83) into Eq. (76), we arrive at Eqs. (74) and (75). These identities enable us to relate two-point functions to one-point functions.

IV. APPLICATION OF MORI'S PROJECTION OPERATOR METHOD TO RELATIVISTIC HYDRODYNAMICS

In this section we apply Mori's projection operator method to relativistic hydrodynamic systems and derive equations of motion for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$ and $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$. We first show that the set $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$ gives the linearized Landau equation. For the Eckart equations, we introduce the current of the conserved charge $\delta\hat{j}^i$, which is proportional to the fluid velocity in the Eckart frame, in addition to $\delta\hat{p}^i$ and $\delta\hat{n}$. We employ the derivative expansion and keep the spatial and time derivative to the second order, i.e., ∂_0, ∇, ∇^2, $\partial_0\nabla$, and ∂_0^2. We will drop the noise term $\hat{R}_n(t,\boldsymbol{x})$ in the equation of motion. This term is irrelevant in the time evolution of the expectation value. If one is interested in stochastic hydrodynamics, one may keep the noise term [23].

A. Linearized Landau equation

First we derive the linearized Landau equation. For this purpose we choose $\delta\hat{e}$, $\delta\hat{p}$, and $\delta\hat{n}$ as slow variables. Since $\delta\hat{p}^i$ is chosen as a slow variable, the equation for $\partial_0\delta\hat{e}$ does not contain dissipative terms,

$$\partial_0\delta\hat{e} = -\nabla\cdot\delta\hat{\boldsymbol{p}} = -h_0\nabla\cdot\delta\hat{\boldsymbol{v}}_L, \tag{84}$$

where we defined the fluid velocity $\delta\hat{\boldsymbol{v}}_L \equiv \delta\hat{\boldsymbol{p}}/h_0$. This equation is nothing but the energy conservation law (12). This can be confirmed by the the following calculation:

$$\begin{aligned} i\Omega_e^p(\boldsymbol{k}) &= \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(i\hat{\mathcal{L}}\delta\hat{e}(\boldsymbol{x}),\delta\hat{p}^j(\boldsymbol{0}))g^{p^jp^i}(\boldsymbol{k}) \\ &= \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(-\nabla^j\delta\hat{p}^j(\boldsymbol{x}),\delta\hat{p}^i(\boldsymbol{0}))g^{p^jp^i}(\boldsymbol{k}) \\ &= -ik^l g_{p^lp^j}(\boldsymbol{k})g^{p^jp^i}(\boldsymbol{k}) \\ &= -ik^i. \end{aligned} \tag{85}$$

Therefore, the reversible term becomes $-\nabla\cdot\delta\hat{\boldsymbol{p}}$. The memory function vanishes because $i\hat{\mathcal{L}}\delta\hat{e}$ turns out to be $-i\boldsymbol{k}\cdot\delta\hat{\boldsymbol{p}}$ and then $\mathcal{Q}i\hat{\mathcal{L}}\delta\hat{e} = 0$ [see Eqs. (41) and (42)].

Let us move onto the equation for $\partial_0\delta\hat{n}$. For the reversible part, the only $i\Omega_{np^i}$ survives in Ω from time-reversal symmetry, which is

$$\begin{aligned} i\Omega_{np^i}(\boldsymbol{k}) &= \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(i\hat{\mathcal{L}}\delta\hat{n}(\boldsymbol{x}),\delta\hat{p}^i(\boldsymbol{0})) \\ &= \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(-\nabla^j\delta\hat{j}^j(\boldsymbol{x}),\delta\hat{p}^i(\boldsymbol{0})) \\ &= -ik^j g_{j^jp^i} + \mathcal{O}(k^3) \\ &= -ik^iT_0n_0 + \mathcal{O}(k^3). \end{aligned} \tag{86}$$

For the memory function, we keep it up to order k^2. Here Φ_{ne} vanishes as before due to $\mathcal{Q}i\hat{\mathcal{L}}\delta\hat{e} = 0$. Further, Φ_{np} is of order k^3 from the tensor structure, which can be neglected. Therefore, we may consider only Φ_{nn}. Since we are interested in slow dynamics, we also expand the memory function in terms of z in addition to k:

$$\begin{aligned} \Phi_{nn}(z,\boldsymbol{k}) &= \int_0^\infty dt \int d^3x\, e^{-zt}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(e^{it\mathcal{Q}\hat{\mathcal{L}}}\mathcal{Q}i\hat{\mathcal{L}}\delta\hat{n}(0,\boldsymbol{x}),i\hat{\mathcal{L}}\delta\hat{n}(0,\boldsymbol{0})) \\ &= k^ik^j\int_0^\infty dt \int d^3x\, e^{-zt}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(e^{it\mathcal{Q}\hat{\mathcal{L}}}\mathcal{Q}\delta\hat{j}^i(0,\boldsymbol{x}),\delta\hat{j}^j(0,\boldsymbol{0})) \\ &\simeq \boldsymbol{k}^2\int_0^\infty dt \int d^3x \left(\delta\hat{j}^i(t,\boldsymbol{x}) - \frac{n_0}{h_0}\delta\hat{p}^i(t,\boldsymbol{x}),\delta\hat{j}^i(0,\boldsymbol{0}) - \frac{n_0}{h_0}\delta\hat{p}^i(0,\boldsymbol{0})\right) \\ &= \boldsymbol{k}^2\lambda\left(\frac{n_0T_0}{h_0}\right)^2 \equiv \boldsymbol{k}^2\tilde{\lambda}, \end{aligned} \tag{87}$$

where $\simeq$ denotes the approximation of order k^2 and z^0. The approximation of order z^0 corresponds to the Markov approximation, i.e., in the coordinate space $\Phi_{nn}(t,\boldsymbol{x}) \simeq -\tilde{\lambda}\nabla^2\delta(t)\delta^{(3)}(\boldsymbol{x})$. We defined the thermal conductivity λ as

$$\lambda \equiv \left(\frac{h_0}{n_0T_0}\right)^2\int_0^\infty dt\int d^3x\left(\delta\hat{j}^i(t,\boldsymbol{x}) - \frac{n_0}{h_0}\delta\hat{p}^i(t,\boldsymbol{x}),\delta\hat{j}^i(0,\boldsymbol{0}) - \frac{n_0}{h_0}\delta\hat{p}^i(0,\boldsymbol{0})\right) \tag{88}$$

and used

$$\mathcal{Q}\delta\hat{j}^i(t,\boldsymbol{x}) \simeq \delta\hat{j}^i(t,\boldsymbol{x}) - \frac{n_0}{h_0}\delta\hat{p}^i(t,\boldsymbol{x}) \tag{89}$$

in the leading order of the derivative expansion. We note that the second term is important to remove the contribution of the zero mode from $\delta\hat{j}^i$. As a result, we arrive at the equation for $\partial_0\delta\hat{n}$ as

$$\partial_0\delta\hat{n} = -n_0\nabla\cdot\delta\hat{\boldsymbol{v}}_L + \tilde{\lambda}\nabla^2\delta(\widehat{\beta\mu}). \tag{90}$$

This equation coincides with Eq. (11).

Similarly, for $\delta\hat{p}^i$, the reversible terms $i\Omega_{pe} = -ik^iT_0h_0$ and $i\Omega_{pn} = -ik^iT_0n_0$, so

$$\begin{aligned} i\Omega_{pe}\hat{A}^e + i\Omega_{pn}\hat{A}^n &= -ik^i[-T_0h_0\delta\hat{\beta} + T_0n_0\delta(\widehat{\beta\mu})] \\ &= -ik^i\delta\hat{P}, \end{aligned} \tag{91}$$

where we used the Gibbs-Duhem relation (73). For the dissipative terms, Φ_{pe} vanishes and $\Phi_{pn} \sim k^3$ from tensor structure can be neglected as before. Therefore, only Φ_{pp}

survives in the leading order, which is evaluated as

$$\Phi_{p^i p^k}(z,\boldsymbol{k}) \simeq k^j k^l \int dt \int dx (\hat{T}^{ij}(t,\boldsymbol{x}),\hat{T}^{kl}(0,\boldsymbol{x})) = T_0\left(\zeta + \frac{1}{3}\eta\right)k^i k^k + T_0\eta \boldsymbol{k}^2\delta^{ik}, \quad (92)$$

where we used the same approximation as in Eq. (87). The shear and bulk viscosities are defined by the Kubo formula as

$$\eta = \beta_0 \int_0^\infty dt \int d^3x (\hat{T}^{12}(t,\boldsymbol{x}),\hat{T}^{12}(0,\boldsymbol{0})), \quad (93)$$

$$\zeta - \frac{2}{3}\eta = \beta_0 \int_0^\infty dt \int d^3x (\hat{T}^{11}(t,\boldsymbol{x}),\hat{T}^{22}(0,\boldsymbol{0})). \quad (94)$$

Noting that $\hat{A}^{p^i} = -\beta_0 \delta \hat{v}_L^i = -(\beta_0/h_0)\delta\hat{p}^i$, we obtain

$$\partial_0 \delta\hat{\boldsymbol{p}} = -\nabla\delta\hat{P} + \eta\nabla^2\delta\hat{\boldsymbol{v}}_L + \left(\zeta + \tfrac{1}{3}\eta\right)\nabla(\nabla\cdot\delta\hat{\boldsymbol{v}}_L), \quad (95)$$

which coincides with Eq. (13). We have shown that the linearized Landau equations (84), (90), and (95) are derived by choosing $\delta\hat{e}$, $\delta\hat{n}$, and $\delta\hat{p}^i$ as slow variables.

Before closing this section, let us consider the detail of $\Phi_{nn}(z,\boldsymbol{k})$. The memory function can be written as

$$\Phi(z,\boldsymbol{k}) = -[\ddot{\Xi}(z,\boldsymbol{k}) - i\Omega(\boldsymbol{k})\dot{\Xi}(z,\boldsymbol{k})]\frac{1}{1+\dot{\Xi}(z,\boldsymbol{k})}, \quad (96)$$

where we defined

$$\Xi(t) \equiv (\hat{A}_n(t),\hat{A}^m), \quad (97)$$

$$\dot{\Xi}(t) \equiv (i\hat{\mathcal{L}}\hat{A}_n(t),\hat{A}^m), \quad (98)$$

$$\ddot{\Xi}(t) \equiv ((i\hat{\mathcal{L}})^2\hat{A}_n(t),\hat{A}^m). \quad (99)$$

For the derivation of Eq. (96), see Appendix B. Since the time derivative of a conserved charge variable is slow, $\dot{\Xi}$ is of order k; then we can estimate $1/(1+\dot{\Xi}) = 1 + \mathcal{O}(k)$. Then $\Phi_{nn}(z,\boldsymbol{k})$ becomes

$$\Phi_{nn}(z,\boldsymbol{k}) \simeq -[\ddot{\Xi}(z,\boldsymbol{k}) - i\Omega(\boldsymbol{k})\dot{\Xi}(z,\boldsymbol{k})]_{nn} \simeq \boldsymbol{k}^2\left(\Xi_{jj}(z,\boldsymbol{0}) - \frac{n_0}{h_0}\Xi_{pj}(z,\boldsymbol{0})\right). \quad (100)$$

From $\dot{\Xi} = -1 + z\Xi$,

$$\Xi_{pj}(z,\boldsymbol{0}) = -\frac{1}{z}g_{pj} = -\frac{1}{z}n_0 T_0, \quad (101)$$

where we use $\dot{\Xi}_{pj}(z,\boldsymbol{0}) = 0$. Therefore, we can write

$$\Phi_{nn}(z,\boldsymbol{k}) \simeq \boldsymbol{k}^2\left(\Xi_{jj}(z,\boldsymbol{0}) - \frac{n_0^2 T_0}{z h_0}\right) \equiv \boldsymbol{k}^2\tilde{\lambda}(z), \quad (102)$$

where we defined the frequency-dependent thermal conductivity $\tilde{\lambda}(z)$. At $z=0$, $\tilde{\lambda}(0)$ coincides with $\tilde{\lambda}$. This expression will be used in the following section to derive the linearized Eckart equation.

B. Linearized Eckart equation

Here we derive the linearized Eckart equation. The charge does not dissipate in the Eckart equation, which implies that the fluid velocity is chosen as $\delta\hat{v}_E \equiv \delta\hat{j}^i/n_0$. Therefore, we choose $\delta\hat{j}^i$ as a slow variable in addition to $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$. In order to derive the Eckart equation we first derive the equations of motion for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$ and then remove the degrees of freedom of $\delta\hat{p}^i$.

The equation of time evolution for $\delta\hat{e}$ and $\delta\hat{n}$ are trivial due to the conservation laws,

$$\partial_0\delta\hat{e} = -\nabla\cdot\delta\hat{\boldsymbol{p}}, \quad (103)$$

$$\partial_0\delta\hat{n} = -\nabla\cdot\delta\hat{\boldsymbol{j}}. \quad (104)$$

As usual, we calculate the reversible terms

$$i\Omega_{ep^i}(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(i\hat{\mathcal{L}}\delta\hat{e}(\boldsymbol{x}),\delta\hat{p}^i(\boldsymbol{0})) = -ik^i T_0 h_0 + \mathcal{O}(k^3), \quad (105)$$

$$i\Omega_{ej^i}(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(i\hat{\mathcal{L}}\delta\hat{n}(\boldsymbol{x}),\delta\hat{j}^i(\boldsymbol{0})) = -ik^i T_0 n_0 + \mathcal{O}(k^3), \quad (106)$$

$$i\Omega_{nj^i}(\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(i\hat{\mathcal{L}}\delta\hat{n}(\boldsymbol{x}),\delta\hat{j}^i(\boldsymbol{0})) = -ik^j g_{j^j j^i} + \mathcal{O}(k^3). \quad (107)$$

Here the explicit form of $g_{j^j j^i}$ is not obtained; however, it is irrelevant in the leading order of the fluid equations, as will be seen later. The reversible term for $\partial_0\delta\hat{p}^i$ is the same as that of the Landau equation (91). The reversible term for $\partial_0\delta\hat{j}^i$ becomes

$$i\Omega_{jn}\hat{A}^n + i\Omega_{je}\hat{A}^e = -ik^i[g_{jj}\delta(\widehat{\beta\mu}) - T_0 n_0\delta\hat{\beta}]. \quad (108)$$

Since $\delta\hat{j}^i$ and $\delta\hat{p}^i$ are chosen as slow variables, the inverse metric contains mixing terms

$$\begin{pmatrix} g^{pp} & g^{pj} \\ g^{jp} & g^{jj}\end{pmatrix} = \frac{1}{g_{pp}g_{jj} - g_{pj}g_{jp}}\begin{pmatrix} g_{jj} & -g_{pj} \\ -g_{jp} & g_{pp}\end{pmatrix} = \frac{\beta_0}{h_0\beta_0 g_{jj} - n_0^2}\begin{pmatrix} \beta_0 g_{jj} & -n_0 \\ -n_0 & h_0\end{pmatrix}. \quad (109)$$

Then the conjugate variables for $\delta\hat{j}^i$ and $\delta\hat{p}^i$ are

$$\hat{A}^{j^i} = \frac{\beta_0}{h_0\beta_0 g_{jj} - n_0^2}(-n_0\delta\hat{p}^i + h_0\delta\hat{j}^i) = -n_0 g^{jj}(\delta\hat{v}_L^i - \delta\hat{v}_E^i), \quad (110)$$

$$\hat{A}^{p^i} = \frac{\beta_0}{h_0\beta_0 g_{jj} - n^2}(\beta_0 g_{jj}\delta\hat{p}^i - n_0\delta\hat{j}^i) = \beta_0\delta\hat{v}_E^i + \beta_0 g_{jj}h_0\hat{A}^{j^i} = \beta_0\delta\hat{v}_L^i - \frac{n_0}{h_0}\hat{A}^{j^i}, \quad (111)$$

where we defined the fluid velocity in the Eckart frame as $\delta\hat{v}_E^i \equiv \delta\hat{j}^i/n_0$. The memory functions appearing in $\partial_0\delta\hat{p}^i$ are Φ_{pp} and Φ_{pj}, which are both of order k^2. Here we assume that $\hat{A}^j$ is of order k, which will be checked later. Then $\Phi_{pj}\hat{A}^j$ can be neglected because it is of order k^3. Similarly, $\Phi_{pp}\hat{A}^p \simeq -\beta\Phi_{pp}\delta\hat{\boldsymbol{v}}_L$. Therefore,

$$\partial_0\delta\hat{\boldsymbol{p}} = -\nabla\delta\hat{P} + \eta\nabla^2\delta\hat{\boldsymbol{v}}_L + \left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\delta\hat{\boldsymbol{v}}_L). \quad (112)$$

This is the same as the Landau equation.

Finally, let us consider the equation for $\partial_0\delta\hat{j}^i$, which is written in the Laplace space

$$\begin{aligned} zn_0\delta\hat{v}_E^i &= i\Omega_{jn}\hat{A}^n + i\Omega_{je}\hat{A}^e - \Phi_{jj}\hat{A}^j \\ &\quad - \Phi_{jp}\hat{A}^p + n_0\delta\hat{v}_E^i(t=0) \\ &= -g_{jj}\nabla^i\delta(\widehat{\beta\mu}) - \frac{n_0}{T_0}\nabla^i\delta\hat{T} - \Phi_{jj}\hat{A}^j \\ &\quad - \Phi_{jp}\hat{A}^p + n_0\delta\hat{v}_E^i(t=0), \end{aligned} \tag{113}$$

where $\delta\hat{\boldsymbol{v}}_E(t=0)$ is the initial value of the fluctuation. The important point is that Φ_{jj} is not slow and gives a contribution of order k^0. This equation will give the relation between these fluid velocities. Since we are interested in the first-order hydrodynamic equation, it is enough to take into account the difference of the fluid velocity up to order k^1. In this order we can neglect Φ_{jp} because it is of order k^2.

Let us now estimate Φ_{jj}. From Eq. (96) we obtain

$$\begin{aligned} \Phi_{jj}(z,\boldsymbol{k}) &= \left[-[\ddot{\Xi}(z,\boldsymbol{k}) - i\Omega(\boldsymbol{k})\dot{\Xi}(z,\boldsymbol{k})]\frac{1}{1+\dot{\Xi}(z,\boldsymbol{k})}\right]_{jj} \\ &= \left[-z + \frac{1}{\Xi(z,\boldsymbol{k})} + i\Omega(\boldsymbol{k})\right]_{jj} \\ &= -zg_{jj} + \left[\frac{1}{\Xi(z,\boldsymbol{k})}\right]_{jj}, \end{aligned} \tag{114}$$

where we used $\ddot{\Xi} = z\dot{\Xi} - i\Omega$, $\dot{\Xi} = z\Xi - 1$, and $i\Omega_{jj}(\boldsymbol{k}) = 0$. We may estimate $\Phi_{jj}(z,\boldsymbol{k})$ at $\boldsymbol{k}=\boldsymbol{0}$ in the leading order. First we consider $\Xi_{ep^i}(z,\boldsymbol{k})$. The energy conservation provides $z\Xi_{ep^i}(z,\boldsymbol{k}) = -ik^j\Xi_{p^jp^i}(z,\boldsymbol{k})$. Thus, at $\boldsymbol{k}=\boldsymbol{0}$, $\Xi_{ep^i}(z,\boldsymbol{0}) = 0$. Similarly, one can show that $\Xi_{ej}(z,\boldsymbol{0}) = \Xi_{np}(z,\boldsymbol{0}) = \Xi_{nj}(z,\boldsymbol{0}) = 0$. Therefore, we may consider the terms with $\delta\hat{p}^i$ or $\delta\hat{j}^i$. They are estimated at $\boldsymbol{k}=\boldsymbol{0}$ as

$$\begin{aligned} \begin{pmatrix} \Xi_{pp}(z,\boldsymbol{0}) & \Xi_{pj}(z,\boldsymbol{0}) \\ \Xi_{jp}(z,\boldsymbol{0}) & \Xi_{jj}(z,\boldsymbol{0}) \end{pmatrix} &= \frac{1}{z}\begin{pmatrix} g_{pp} & g_{pj} \\ g_{jp} & z\Xi_{jj}(z,\boldsymbol{0}) \end{pmatrix} \\ &= \frac{1}{z}\begin{pmatrix} h_0T_0 & n_0T_0 \\ n_0T_0 & \frac{n_0^2T_0}{h_0} + z\tilde{\lambda}(z) \end{pmatrix}, \end{aligned} \tag{115}$$

where we used $\dot{\Xi}(z,\boldsymbol{0}) = -1 + z\Xi(z,\boldsymbol{0}) = 0$ and Eq. (102). Taking into account the metric, we find

$$\begin{aligned} \Phi_{jj}(z,\boldsymbol{0}) &= \frac{\left(g_{jj}h_0 - n_0^2T_0\right)^2}{h_0^2\tilde{\lambda}(z)} - \frac{1}{h_0}\left(hg_{jj} - n_0^2T_0\right)z \\ &= \frac{1}{(g^{jj})^2\tilde{\lambda}(z)} - \frac{1}{g^{jj}}z. \end{aligned} \tag{116}$$

Then the equation of motion becomes

$$\begin{aligned} n_0z\delta\hat{\boldsymbol{v}}_E &= -g_{jj}\nabla\delta(\beta\mu) - \frac{n_0}{T_0}\nabla\delta T + \frac{n_0}{g^{jj}\tilde{\lambda}(z)}(\delta\hat{\boldsymbol{v}}_L - \delta\hat{\boldsymbol{v}}_E) \\ &\quad - n_0z(\delta\hat{\boldsymbol{v}}_L - \delta\hat{\boldsymbol{v}}_E) + n_0\delta\hat{\boldsymbol{v}}_E(t=0). \end{aligned} \tag{117}$$

From Eq. (117) we obtain

$$\begin{aligned} h_0\delta\hat{\boldsymbol{v}}_L &= h_0\delta\hat{\boldsymbol{v}}_E + \frac{h_0}{n_0}g^{jj}\tilde{\lambda}(z)\left(n_0z\delta\hat{\boldsymbol{v}}_L + g_{jj}\nabla\delta(\beta\mu) + \frac{n_0}{T_0}\nabla\delta\hat{T} - n_0\delta\hat{\boldsymbol{v}}_E(t=0)\right) \\ &= h_0\delta\hat{\boldsymbol{v}}_E - \lambda(z)(T_0z\delta\hat{\boldsymbol{v}}_L + \nabla\delta\hat{T}) + \lambda(z)T_0^2g^{pp}(h_0z\delta\hat{\boldsymbol{v}}_L + \nabla\delta\hat{P}) - g^{jj}\frac{n_0^2T_0^2}{h_0}\lambda(z)\delta\hat{\boldsymbol{v}}_E(t=0). \end{aligned} \tag{118}$$

The third term in the last line is estimated as $h_0z\delta\hat{\boldsymbol{v}}_L + \nabla\delta\hat{P} = h_0\delta\hat{\boldsymbol{v}}_L(t=0) + \mathcal{O}(\nabla^2)$ from the equation of motion (112). Then

$$\begin{aligned} h_0\delta\hat{\boldsymbol{v}}_L &= h_0\delta\hat{\boldsymbol{v}}_E - \lambda(z)(T_0z\delta\hat{\boldsymbol{v}}_L + \nabla\delta\hat{T}) + \lambda(z)\left(T_0^2g^{pp}h_0\delta\hat{\boldsymbol{v}}_L(t=0) - g^{jj}\frac{n_0^2T_0^2}{h_0}\delta\hat{\boldsymbol{v}}_E(t=0)\right) \\ &= h_0\delta\hat{\boldsymbol{v}}_E - \lambda(z)[T_0z\delta\hat{\boldsymbol{v}}_L + \nabla\delta\hat{T} - T_0\delta\hat{\boldsymbol{v}}_L(t=0)] - \frac{T_0^2n_0}{h_0}\lambda(z)\hat{A}^j(t=0). \end{aligned} \tag{119}$$

This equation is exact for an arbitrary z in the first order of the derivative expansion of the spatial coordinate. We note that Eqs. (103), (104), (112), and (119) are those of motion for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$. These equations are derived in this paper by the projection operator method. We also note that these include the Landau and Eckart frames under the concept of the projection operator method.

Now let us derive the Eckart equation. We assume that there is no conjugate variable for j^i at the initial time, i.e., $\hat{A}^j(t=0) = 0$. We also assume that the change of the variable is so slow that the z expansion is applicable:

$$h_0\delta\hat{\boldsymbol{v}}_L = h_0\delta\hat{\boldsymbol{v}}_E - \lambda[T_0z\delta\hat{\boldsymbol{v}}_L + \nabla\delta T - \delta\hat{\boldsymbol{v}}_L(t=0)]. \tag{120}$$

In the coordinate space, this equation becomes

$$h_0\delta\hat{\boldsymbol{v}}_L = h_0\delta\hat{\boldsymbol{v}}_E - \lambda(T_0\partial_0\delta\hat{\boldsymbol{v}}_L + \nabla\delta\hat{T}). \tag{121}$$

From this equation one can confirm $\hat{A}^{j^i} = -n_0g^{jj}(\delta\hat{v}_L^i - \delta\hat{v}_E^i) = n_0g^{jj}(T_0\partial_0\delta\hat{v}_L^i + ik^i\delta\hat{T})/h_0 \sim k$. Therefore, the assumption that $\hat{A}^j$ is of order k is consistent for slow dynamics. Solving Eq. (121) for $h_0\delta\hat{\boldsymbol{v}}_L$, we obtain

$$\begin{aligned} h_0\delta\hat{\boldsymbol{v}}_L &= \frac{1}{1+\lambda\frac{T_0}{h_0}\partial_0}(h_0\delta\hat{\boldsymbol{v}}_E - \lambda\nabla\delta\hat{T}) \\ &\simeq h_0\delta\hat{\boldsymbol{v}}_E - \lambda(T_0\partial_0\delta\hat{\boldsymbol{v}}_E + \nabla\delta\hat{T}), \end{aligned} \tag{122}$$

where we dropped $\partial_0\nabla T$ in the last line because it is higher order. Inserting Eq. (118) into Eq. (112), we find

$$\begin{aligned} &\partial_0[h_0\delta\hat{\boldsymbol{v}}_E - \lambda(T_0\partial_0\delta\hat{\boldsymbol{v}}_E + \nabla\delta\hat{T})] \\ &\quad = -\nabla\delta\hat{P} + \eta\nabla^2\delta\hat{\boldsymbol{v}}_E + \left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\delta\hat{\boldsymbol{v}}_E). \end{aligned} \tag{123}$$

This equation is equal to the linearized Eckart equation (19).

V. DISCUSSION

Here we discuss how the slow dynamics is determined by the Landau variables $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$ and the Landau frame is natural for the hydrodynamics. First we study modes of the equations for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$, which include both the Landau and Eckart frames in the projection operator method. We will show that those modes are the same as those of the Landau equation. Namely, the slow modes are determined only by the Landau variables and the current density $\delta\hat{j}^i$ contains an irrelevant part for the slow dynamics.

Furthermore, we discuss the dynamic variables of the Eckart equation. The Eckart equation has $\{\delta\hat{e},\delta\hat{n},\delta\hat{j}^i\}$ as the dynamic variables, apparently. Nevertheless, we shall show that the slow part of $\delta\hat{j}^i$ is determined by $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$, actually. To illustrate this fact, we consider the Onsager reciprocal relation in the Eckart equation. If we assume that the time-reversal property of $\delta\hat{j}^i$ is odd, the reciprocal relation seems to be violated. In contrast, if we regard $\delta\hat{j}^i$ as a dependent variable of $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$, we will find that the relation is satisfied. Namely, $\delta\hat{j}^i$ in the Eckart equation is projected on the Landau variables and its time-reversal property is not odd.

A. Modes of the equations for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$

Here we study modes of the equations for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$, which has the dynamic variables for both the Landau and Eckart frames. Then we shall find that those modes are the same as those of the Landau equation.

In the Fourier space, the equations for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$ [Eqs. (103), (104), (112), and (119)] are written in the following matrix form:

$$M\begin{pmatrix}\delta\hat{e}(\omega,\boldsymbol{k})\\ \delta\hat{n}(\omega,\boldsymbol{k})\\ \delta\hat{p}_\parallel(\omega,\boldsymbol{k})\\ \delta\hat{j}_\parallel(\omega,\boldsymbol{k})\\ \delta\hat{\boldsymbol{p}}_\perp(\omega,\boldsymbol{k})\\ \delta\hat{\boldsymbol{j}}_\perp(\omega,\boldsymbol{k})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}, \tag{124}$$

where we decomposed $\delta\hat{\boldsymbol{p}}$ and $\delta\hat{\boldsymbol{j}}$ into the longitudinal and transverse components

$$\delta\hat{p}_\parallel=\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\cdot\delta\hat{\boldsymbol{p}},\quad \delta\hat{\boldsymbol{p}}_\perp=\delta\hat{\boldsymbol{p}}-\delta\hat{p}_\parallel\frac{\boldsymbol{k}}{|\boldsymbol{k}|}, \tag{125}$$

$$\delta\hat{j}_\parallel=\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\cdot\delta\hat{\boldsymbol{j}},\quad \delta\hat{\boldsymbol{j}}_\perp=\delta\hat{\boldsymbol{j}}-\delta\hat{j}_\parallel\frac{\boldsymbol{k}}{|\boldsymbol{k}|}. \tag{126}$$

We introduced the matrix

$$M\equiv\begin{pmatrix}-i\omega & 0 & ik & 0 & 0 & 0\\ 0 & -i\omega & 0 & ik & 0 & 0\\ ik\alpha_{Pe} & ik\alpha_{Pn} & -i\omega+k^2\Gamma_\parallel & 0 & 0 & 0\\ ikik\alpha_{Te} & ik\alpha_{Tn} & -i\omega+\Gamma_j & -(h_0/n_0)\Gamma_j & 0 & 0\\ 0 & 0 & 0 & 0 & -i\omega+k^2(\eta/h_0) & 0\\ 0 & 0 & 0 & 0 & -i\omega+\Gamma_j & -(h_0/n_0)\Gamma_j\end{pmatrix}, \tag{127}$$

where we defined

$$\alpha_{pe}\equiv\left(\frac{\partial P}{\partial e}\right)_n,\quad \alpha_{pn}\equiv\left(\frac{\partial P}{\partial n}\right)_e, \tag{128}$$

$$\alpha_{Te}\equiv\beta_0h_0\left(\frac{\partial T}{\partial e}\right)_n,\quad \alpha_{Tn}\equiv\beta_0h_0\left(\frac{\partial T}{\partial n}\right)_e, \tag{129}$$

$$\Gamma_\parallel=\frac{1}{h_0}\left(\zeta+\frac{4}{3}\eta\right),\quad \Gamma_j=\frac{\beta_0h_0}{\lambda}. \tag{130}$$

We neglected here the frequency dependence of the thermal conductivity. We can obtain dispersion relations from $\det M=0$. Those are given as the following, to second order in k:

$$\omega\sim-ik^2\Gamma_t, \tag{131}$$

$$\omega\sim-ik^2\Gamma_s\pm kc_s, \tag{132}$$

$$\omega=-ik^2(\eta/h_0), \tag{133}$$

where we introduced the thermal and sound diffusion constants, and the sound velocity

$$\Gamma_t=\frac{\lambda}{n_0c_P}, \tag{134}$$

$$\Gamma_s=\frac{1}{2}\left\{\Gamma_\parallel+\Gamma_t\left[n_0c_P\left(\frac{T_0n_0}{h_0}\right)^2\left(\frac{\partial(\beta\mu)}{\partial n}\right)_e-1\right]\right\}, \tag{135}$$

$$c_s=\left(\frac{\partial P}{\partial e}\right)_{s/n}. \tag{136}$$

Here c_P is the specific heat at constant pressure. From Eqs. (131)–(133) we see that the equations for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$ have the usual hydrodynamic modes: the thermal diffusion and sound and viscous diffusion modes. Moreover, these dispersions are the same as for the Landau equation,[2] i.e., the slow modes are described only by $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$ and the current density $\delta\hat{j}^i$ is irrelevant for slow dynamics.

B. Independent variables of the Eckart equation

Here we discuss that, even for the Eckart equation, the slow part of $\delta\hat{j}^i$ is determined by the Landau variables $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$.

[2]The sound diffusion constant (135) is apparently different from that in Ref. [24]. In Ref. [24], the Landau equation is solved by choosing $\delta\hat{n}$ and $\delta\hat{T}$ as the thermodynamic variables, whereas $\delta\hat{n}$ and $\delta\hat{e}$ are chosen in this paper. The difference is due to the choices and is apparent.

For this purpose, we consider the Onsager reciprocal relation in the Eckart equation. We will see that the relation is not satisfied if $\delta\hat{j}^i$ is assumed as an independent slow variable, while it is satisfied if $\delta\hat{j}^i$ is a dependent variable of $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$.

Now we consider the correlations $(\partial_0\delta\hat{e},\delta\hat{n})$ and $(\partial_0\delta\hat{n},\delta\hat{e})$. These correlations must satisfy the relation

$$(\partial_0\delta\hat{e},\delta\hat{n}) = (\partial_0\delta\hat{n},\delta\hat{e}), \tag{137}$$

which comes from the time-reversal properties of $\delta\hat{n}$ and $\delta\hat{e}$ and the equilibrium state. We note that this relation is equivalent to the Onsager reciprocal relation in the linear regime [25–28]. From the Eckart equations (17)–(19) the correlations are written as

$$(\partial_0\delta\hat{n}(t,\boldsymbol{r}),\delta\hat{e}(t,\boldsymbol{r}')) = -n_0\nabla\cdot(\delta\hat{\boldsymbol{v}}_E(t,\boldsymbol{r}),\delta\hat{e}(t,\boldsymbol{r}')), \tag{138}$$

$$\begin{aligned}(\partial_0\delta\hat{e}(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}'))
&= -h_0\nabla\cdot(\delta\hat{\boldsymbol{v}}_E(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}')) + \lambda[\nabla^2(\delta\hat{T}(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}'))\\
&\quad + T_0\nabla\cdot(\partial_0\delta\hat{\boldsymbol{v}}_E(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}'))].\end{aligned} \tag{139}$$

Now, to eliminate the time derivative in the last term of Eq. (139), we use the derivative expansion. Namely, we approximate

$$\lambda T_0\nabla\cdot\partial_0\delta\hat{\boldsymbol{v}}_E = \lambda T_0\nabla\cdot h_0^{-1}(-\nabla\delta\hat{P}+\cdots) \tag{140}$$

$$\simeq -\frac{\lambda T_0}{h_0}\nabla^2\delta\hat{P}, \tag{141}$$

where the ellipsis denotes the second- and higher-order terms about the derivative, such as $\eta\nabla^2\delta\hat{\boldsymbol{v}}_E$. Here we first used Eq. (123) for $\partial_0\delta\hat{\boldsymbol{v}}_E$ and next neglect the second-order terms, which finally yields third-order terms. Thus we obtain

$$\begin{aligned}(\partial_0\delta\hat{e}(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}')) &\simeq -h_0\nabla\cdot(\delta\hat{\boldsymbol{v}}_E(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}'))\\
&\quad -\lambda\left(\frac{n_0T_0^2}{h_0}\right)\nabla^2(\delta(\widehat{\beta\mu})(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}')),\end{aligned} \tag{142}$$

where we also used the Gibbs-Duhem relation (73). Let us here estimate the equal-time correlations $(\delta\hat{\boldsymbol{v}}_E,\delta\hat{n})$, $(\delta\hat{\boldsymbol{v}}_E,\delta\hat{e})$, and $(\delta(\widehat{\beta\mu}),\delta\hat{n})$. We assume that the time-reversal property of $\delta\hat{\boldsymbol{v}}_E$ ($\delta\hat{\boldsymbol{j}}$) is odd, according to Eq. (54). Then $\delta\hat{\boldsymbol{v}}_E$ is orthogonal to $\delta\hat{n}$,

$$(\delta\hat{\boldsymbol{v}}_E,\delta\hat{e}) = (\delta\hat{\boldsymbol{v}}_E,\delta\hat{n}) = 0, \tag{143}$$

by the time-reversal symmetry. In contrast, $(\delta(\widehat{\beta\mu}),\delta\hat{n})$ turns out to be $\delta(\boldsymbol{r}-\boldsymbol{r}')$ from Eq. (70). Finally, we obtain the correlations

$$(\partial_0\delta\hat{n}(t,\boldsymbol{r}),\delta\hat{e}(t,\boldsymbol{r}')) = 0, \tag{144}$$

$$(\partial_0\delta\hat{e}(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}')) = -\lambda\left(\frac{n_0T_0^2}{h_0}\right)\nabla^2\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{145}$$

We see that the Onsager relation (137) seems to be violated. This violation comes from the assumption in Eq. (143), as will been seen in the following.

Next, let us regard $\delta\hat{\boldsymbol{v}}_E$ as a function of $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$. Namely, we consider $\delta\hat{\boldsymbol{p}}$ as an independent variable instead of $\delta\hat{\boldsymbol{j}}$. We now use Eq. (122), which gives the relation between $\delta\hat{\boldsymbol{j}}$ and $\delta\hat{\boldsymbol{p}}$:

$$\begin{aligned}\delta\hat{\boldsymbol{p}} &= h_0\delta\hat{\boldsymbol{v}}_E - \lambda(T\partial_0\delta\hat{\boldsymbol{v}}_L + \nabla\delta\hat{T})\\
&\simeq h_0\delta\hat{\boldsymbol{v}}_E - \lambda\left(\frac{n_0T_0^2}{h_0}\right)\nabla\delta(\widehat{\beta\mu}),\end{aligned} \tag{146}$$

where we used the derivative expansion and the Gibbs-Duhem relation in the second line. Solving the above relation about $\delta\hat{\boldsymbol{v}}_E$, we obtain $\delta\hat{\boldsymbol{v}}_E$ as a function of $\{\delta\hat{n},\delta\hat{e},\delta\hat{\boldsymbol{p}}\}$:

$$\delta\hat{\boldsymbol{v}}_E(\delta\hat{n},\delta\hat{e},\delta\hat{\boldsymbol{p}}) = \frac{1}{h_0}\delta\hat{\boldsymbol{p}} + \lambda\left(\frac{n_0T_0^2}{h_0^2}\right)\nabla\delta(\widehat{\beta\mu}). \tag{147}$$

We note that the time-reversal property of $\delta\hat{\boldsymbol{v}}_E(\delta\hat{e},\delta\hat{p}^i,\delta\hat{n})$ is not odd because those of $\delta\hat{\boldsymbol{p}}$ and $\delta(\widehat{\beta\mu})$ are odd and even, respectively. Substituting Eq. (147) into Eqs. (138) and (142), we find

$$\begin{aligned}(\partial_0\delta\hat{n}(t,\boldsymbol{r}),\delta\hat{e}(t,\boldsymbol{r}')) &= -\frac{n_0}{h_0}\nabla\cdot(\delta\hat{\boldsymbol{p}}(t,\boldsymbol{r}),\delta\hat{e}(t,\boldsymbol{r}'))\\
&\quad -\lambda\left(\frac{n_0T_0}{h_0}\right)^2\nabla^2(\delta(\widehat{\beta\mu})(t,\boldsymbol{r}),\delta\hat{e}(t,\boldsymbol{r}')),\end{aligned} \tag{148}$$

$$(\partial_0\delta\hat{e}(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}')) = (\delta\hat{\boldsymbol{p}}(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}')). \tag{149}$$

We note that $\delta(\widehat{\beta\mu})$ is orthogonal to $\delta\hat{e}$ by Eq. (69). Then, if we assume that the time-reversal property of $\delta\hat{\boldsymbol{p}}$ is odd, we see that the Onsager relation is satisfied:

$$(\partial_0\delta\hat{n}(t,\boldsymbol{r}),\delta\hat{e}(t,\boldsymbol{r}')) = (\partial_0\delta\hat{e}(t,\boldsymbol{r}),\delta\hat{n}(t,\boldsymbol{r}')) = 0. \tag{150}$$

The reason why we cannot regard $\delta\hat{\boldsymbol{p}}$ as a function of $\{\delta\hat{n},\delta\hat{e},\delta\hat{\boldsymbol{j}}\}$ is that $\delta\hat{\boldsymbol{j}}$ is not essentially slow and the slow motion of that turns out to be described by the actual slow variables. Thus the time-reversal property differs from the original operator at the beginning of Sec. III. Actually, we can show that the projection of $\delta\hat{\boldsymbol{j}}$ on $\{\delta\hat{n},\delta\hat{e},\delta\hat{\boldsymbol{p}}\}$ gives Eq. (147). The derivation is given in Appendix C. Namely, the actual expression of Eq. (147) is given as

$$\begin{aligned}\mathcal{P}(\delta\hat{\boldsymbol{j}}(t)) &= \frac{n_0}{h_0}\delta\hat{\boldsymbol{p}}(t) - \lambda\left(\frac{n_0T_0}{h_0}\right)^2\nabla\left[\left(\frac{\partial(\beta\mu)}{\partial n}\right)_e\delta\hat{n}(t)\right.\\
&\quad \left.+\left(\frac{\partial(\beta\mu)}{\partial e}\right)_n\delta\hat{e}(t)\right],\end{aligned} \tag{151}$$

where $\mathcal{P}$ is the projector on $\{\delta\hat{n},\delta\hat{e},\delta\boldsymbol{p}\}$, i.e., $\delta\hat{\boldsymbol{j}}$ ($\delta\hat{\boldsymbol{v}}_E$) in the Eckart equation is projected on and differs from the original operator. In consequence, the slow variables for the Landau frame actually describe the slow dynamics even for the Eckart frame.

VI. SUMMARY

We studied relativistic hydrodynamics by Mori's projection operator method and focused on linear fluctuations around the thermal equilibrium at the rest frame. From the projection operator method, we discussed that the difference of the frames does not result from the choices of the reference frames but rather from those of the slow variables. We also found that the slow variables for the Landau frame are the conserved charges, whereas those for the Eckart frame include the current of the conserved charge, which is not essentially slow. In fact, we derived the slow dynamics by the projection operator method for the sets $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$ and $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$ as the slow variables. We first showed that the natural choice (23) gives the linearized Landau equations. We derived the equations of motion for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$, which include the Landau and Eckart frames under the concept of the projection operator method. We then derived the linearized Eckart equation by eliminating $\delta\hat{p}^i$ from the above equations.

We also discussed that the slow dynamics is determined only by $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$ and the Landau frame is natural. In particular, we showed that the equations for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n},\delta\hat{j}^i\}$ have the same modes as for $\{\delta\hat{e},\delta\hat{p}^i,\delta\hat{n}\}$. Thus we found that the slow modes are determined only by the Landau variables. Furthermore, by considering the Onsager relation, we illustrated that the slow part of the particle current $\delta\hat{j}^i$ is determined by the Landau variables even for the Eckart equation. Recently, it was pointed out that the Landau frame is natural for the relativistic hydrodynamics, based on the renormalization-group method [29].

This study bases the derivation of the relativistic hydrodynamics on the projection operator method. We stress that our derivation is independent of the microscopic details; we determine the metric from the Lorentz symmetry on the microscopic scale and the thermodynamics. Earlier studies [7–10] assume that the relativistic Boltzmann equation is the underlying microscopic theory, which is valid, however, only for a weakly correlated system such as a dilute gas. Furthermore, we note that our study is independent of the local equilibrium and the local rest in the relativistic fluids because we considered the fluctuations from a global equilibrium state. Instead, our study is restricted to the linear fluctuations from the equilibrium state at rest. We comment on the Lorentz covariance of linearized hydrodynamics. The linearized Landau equations (11)–(13) are not Lorentz covariant. The reason is the following: We considered here the fluctuations in the background medium. For such a system, the Lorentz transformation boosts the fluctuations, but not the background. Then the boosted system differs from that before the boost. Thus the linearized equations are valid only for the rest frame of the medium and not the Lorentz covariant. Actually, for the same reason, the Navier-Stokes equation is Galilei covariant, whereas the linearized one is not covariant.

In this paper we used the linear projection operator and our study was restricted to the linear regime. It is interesting to derive relativistic hydrodynamics by the nonlinear projection operator [30]. We note that, for nonrelativistic fluids, the full Navier-Stokes equation is derived by the nonlinear one [31]. Furthermore, here we focused on and discussed the Landau and Eckart frames. We concluded that the Landau frame is natural for the slow dynamics. However, other problems of relativistic hydrodynamics, such as the acausal propagation (although it is studied from the macroscopic point of view [32]), are not well understood from the underlying microscopic theory yet. It would be interesting to discuss these under the concept of the projection operator method. The projection operator method may give insight into these problems, as well as that of the frames.

ACKNOWLEDGMENTS

Y.H. thanks Teiji Kunihiro for useful discussions. This work was supported by JSPS KAKENHI Grants No. 23340067 and No. 24740184.

APPENDIX A: PROPERTIES OF THE INNER PRODUCT

Here we summarize properties of the inner product defined in Eq. (28). For Hermitian operators,

$$(\hat{A}(t),\hat{B}(0)) = (\hat{A}(t),\hat{B}(0))^* = (\hat{B}(0),\hat{A}(t)), \tag{A1}$$

$$(i\hat{\mathcal{L}}\hat{A}(t),\hat{B}(0)) = -(\hat{A}(t),i\hat{\mathcal{L}}\hat{B}(0)), \tag{A2}$$

$$(i\hat{\mathcal{L}}\hat{A}(t),\hat{B}(0)) = -\frac{i}{\beta}\langle(\hat{A}(t),\hat{B}(0))\rangle_{\text{eq}}, \tag{A3}$$

$$(\hat{A}(t),\hat{B}(0)) = \epsilon_{\hat{A}}\epsilon_B(\hat{A}(-t),\hat{B}(0)) = \epsilon_A\epsilon_B(\hat{B}(t),\hat{A}(0)) \tag{A4}$$

are satisfied. Here ϵ_A and ϵ_B denote the sign associated with time-reversal transformation, which is defined with the time-reversal operator $\mathcal{T}$ as $\mathcal{T}^{-1}\hat{A}(t)\mathcal{T} = \epsilon_A\hat{A}(-t)$.

APPENDIX B: MEMORY FUNCTION

In this Appendix we derive the full expression of the memory function following Ref. [33]. Let us first define the key functions

$$\dot{\Xi}(t) \equiv (i\hat{\mathcal{L}}\hat{A}_n(t),\hat{A}^m), \tag{B1}$$

$$\ddot{\Xi}(t) \equiv ((i\hat{\mathcal{L}})^2\hat{A}_n(t),\hat{A}^m). \tag{B2}$$

The reversible term can be expressed by $i\Omega = \dot{\Xi}(0)$. Here $\ddot{\Xi}(t)$ coincides with the memory function obtained by replacing $\mathcal{Q}$ with unity. Expanding $\exp(t\mathcal{Q}i\hat{\mathcal{L}})$ in terms of $\mathcal{P}i\hat{\mathcal{L}}$, we obtain

$$e^{t\mathcal{Q}i\hat{\mathcal{L}}} = e^{it\hat{\mathcal{L}}}\Big(1+\sum_{n=1}^{\infty}(-1)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n \times i\hat{\mathcal{L}}^P(t_1)\cdots i\hat{\mathcal{L}}^P(t_n)\Big), \tag{B3}$$

where

$$\hat{\mathcal{L}}^P(t) \equiv e^{-i\hat{\mathcal{L}}t}\mathcal{P}\hat{\mathcal{L}}e^{i\hat{\mathcal{L}}t}. \tag{B4}$$

Multiplying $\hat{\mathcal{L}}^P(t)$ to $\hat{A}_n$, we obtain

$$\begin{aligned} i\hat{\mathcal{L}}^P(t)\hat{A}_m(-t') &= e^{-i\hat{\mathcal{L}}t}Pi\hat{\mathcal{L}}e^{i\hat{\mathcal{L}}t}\hat{A}_m(-t') \\ &= \hat{A}_n(-t)(i\hat{\mathcal{L}}\hat{A}_m(t-t'),\hat{A}^n) \\ &= \dot{\Xi}_m{}^n(t-t')\hat{A}_n(-t). \end{aligned} \tag{B5}$$

Then, for an operator $\hat{\mathcal{O}}$,

$$\begin{aligned} e^{t\mathcal{Q}i\hat{\mathcal{L}}}\hat{\mathcal{O}} &= e^{it\hat{\mathcal{L}}}\Big(1+\sum_{n=1}^{\infty}(-1)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n \times i\hat{\mathcal{L}}^P(t_1)\cdots i\hat{\mathcal{L}}^P(t_n)\Big)\hat{\mathcal{O}} \\ &= \hat{\mathcal{O}}(t)+\sum_{n=1}^{\infty}(-1)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\dot{\hat{\mathcal{O}}}_P(t_n)\dot{\Xi}(t_{n-1}-t_n) \\ &\quad\times\dot{\Xi}(t_{n-2}-t_{n-1})\cdots\dot{\Xi}(t_1-t_2)\hat{A}(t-t_1) \end{aligned} \tag{B6}$$

is satisfied. Here we defined

$$\dot{\hat{O}}_P(t) = (i\hat{\mathcal{L}}\hat{O}(t),\hat{A}^m). \tag{B7}$$

Performing the Laplace transform, we obtain

$$\int dt\, e^{-tz} e^{tQi\hat{\mathcal{L}}}\dot{\hat{O}} = \dot{\hat{O}}(z) - \dot{\hat{O}}_P(z)\frac{1}{1+\dot{\Xi}(z)}\hat{A}(z). \tag{B8}$$

Using this equation, we find the full expressions for the noise term and the memory function in the Fourier-Laplace space:

$$\hat{R}(z,\boldsymbol{k}) = i\hat{\mathcal{L}}\hat{A}(z,\boldsymbol{k}) - [i\Omega + \ddot{\Xi}(z,\boldsymbol{k})]\frac{1}{1+\dot{\Xi}(z,\boldsymbol{k})}\hat{A}(z,\boldsymbol{k}), \tag{B9}$$

$$\Phi(z,\boldsymbol{k}) = -[\ddot{\Xi}(z,\boldsymbol{k}) - i\Omega(\boldsymbol{k})\dot{\Xi}(z,\boldsymbol{k})]\frac{1}{1+\dot{\Xi}(z,\boldsymbol{k})}. \tag{B10}$$

APPENDIX C: PROJECTION OF δj ON $\{\delta\hat{e},\delta p,\delta\hat{n}\}$

Here we show that the projection of $\delta\hat{\boldsymbol{j}}$ on $\{\delta\hat{n},\delta\hat{e},\delta\hat{\boldsymbol{p}}\}$ gives Eq. (147). Namely, we will show

$$\mathcal{P}\left[\delta\hat{j}(t,\boldsymbol{x}) - \frac{n_0}{h_0}\delta\hat{\boldsymbol{p}}(t,\boldsymbol{x}) + \lambda\left(\frac{n_0T_0}{h_0}\right)^2\nabla\delta(\widehat{\beta\mu})(t,\boldsymbol{x})\right] = 0 \tag{C1}$$

to the first order in k. In the Fourier-Laplace space, Eq. (C1) becomes

$$\mathcal{P}\left[\delta\hat{\boldsymbol{j}}(z,\boldsymbol{k}) - \frac{n_0}{h_0}\delta\hat{\boldsymbol{p}}(z,\boldsymbol{k}) + i\boldsymbol{k}\lambda\left(\frac{n_0T_0}{h_0}\right)^2\delta(\widehat{\beta\mu})(z,\boldsymbol{k})\right] = 0. \tag{C2}$$

Here we decompose Eq. (C2) into the longitudinal and transverse components

$$\mathcal{P}\left[\delta\hat{j}_\parallel(z,\boldsymbol{k}) - \frac{n_0}{h_0}\delta\hat{p}_\parallel(z,\boldsymbol{k}) + ik\lambda\left(\frac{n_0T_0}{h_0}\right)^2\delta(\widehat{\beta\mu})(z,\boldsymbol{k})\right] = 0, \tag{C3}$$

$$\mathcal{P}\left[\delta\hat{\boldsymbol{j}}_\perp(z,\boldsymbol{k}) - \frac{n_0}{h_0}\delta\hat{\boldsymbol{p}}_\perp(z,\boldsymbol{k})\right] = 0. \tag{C4}$$

Let us show the equation for the transverse component (C4). Consider the projections $\mathcal{P}\delta\hat{\boldsymbol{j}}_\perp(z,\boldsymbol{k})$ and $\mathcal{P}\delta\hat{\boldsymbol{p}}_\perp(z,\boldsymbol{k})$, which are given as

$$\mathcal{P}\delta\hat{\boldsymbol{j}}_\perp(z,\boldsymbol{k}) = \Xi_{j_\perp}{}^{n}(z,\boldsymbol{k})\delta\hat{n}(0,\boldsymbol{k}) + \Xi_{j_\perp}{}^{e}(z,\boldsymbol{k})\delta\hat{e}(0,\boldsymbol{k}) + \Xi_{j_\perp}{}^{p_\perp}(z,\boldsymbol{k})\delta\hat{\boldsymbol{p}}_\perp(0,\boldsymbol{k}), \tag{C5}$$

$$\mathcal{P}\delta\hat{\boldsymbol{p}}_\perp(z,\boldsymbol{k}) = \Xi_{p_\perp}{}^{n}(z,\boldsymbol{k})\delta\hat{n}(0,\boldsymbol{k}) + \Xi_{p_\perp}{}^{e}(z,\boldsymbol{k})\delta\hat{e}(0,\boldsymbol{k}) + \Xi_{p_\perp}{}^{p_\perp}(z,\boldsymbol{k})\delta\hat{\boldsymbol{p}}_\perp(0,\boldsymbol{k}). \tag{C6}$$

We note that $\boldsymbol{k}$ expansions of $\Xi_{j_\perp}{}^{n}(z,\boldsymbol{k})$ and $\Xi_{j_\perp}{}^{e}(z,\boldsymbol{k})$ give only odd-order terms in $\boldsymbol{k}$ from the tensor structure. Then we can drop $\Xi_{j_\perp}{}^{n}(z,\boldsymbol{k})$ and $\Xi_{j_\perp}{}^{e}(z,\boldsymbol{k})$ because the odd terms are orthogonal to the transverse component. Then Eq. (C4) turns out to be

$$\left[\Xi_{j_\perp}{}^{p_\perp}(z,\boldsymbol{k}) - \frac{n_0}{h_0}\Xi_{p_\perp}{}^{p_\perp}(z,\boldsymbol{k})\right]\delta\hat{\boldsymbol{p}}_\perp(0,\boldsymbol{k}) = 0. \tag{C7}$$

Let us consider $\Xi_{j_\perp}{}^{p_\perp}(z,\boldsymbol{k})$ and $\Xi_{p_\perp}{}^{p_\perp}(z,\boldsymbol{k})$. For this task, we now use the equations of motion for $\{\delta\hat{n},\delta\hat{e},\delta\hat{\boldsymbol{p}},\delta\hat{\boldsymbol{j}}\}$ [Eqs. (103), (104), (112), and (119)]. From these equations we obtain the equations for the transverse components as

$$\begin{pmatrix} z+k^2\Gamma_\perp & 0 \\ z+\Gamma_j & -(h_0/n_0)\Gamma_j \end{pmatrix}\begin{pmatrix}\delta\hat{\boldsymbol{p}}_\perp(z,\boldsymbol{k}) \\ \delta\hat{\boldsymbol{j}}_\perp(z,\boldsymbol{k})\end{pmatrix} = \begin{pmatrix} \delta\hat{\boldsymbol{p}}_\perp(t=0,\boldsymbol{k}) \\ \delta\hat{\boldsymbol{p}}_\perp(t=0,\boldsymbol{k}) + n_0T_0g^{jj}\left(\delta\hat{\boldsymbol{j}}_\perp(t=0,\boldsymbol{k}) - \frac{n_0}{h_0}\delta\hat{\boldsymbol{p}}_\perp(t=0,\boldsymbol{k})\right)\end{pmatrix}. \tag{C8}$$

Therefore,

$$\delta\hat{\boldsymbol{p}}_\perp(z,\boldsymbol{k}) = \frac{1}{z+k^2\Gamma_\perp}\delta\hat{\boldsymbol{p}}_\perp(t=0,\boldsymbol{k}), \tag{C9}$$

$$\delta\hat{\boldsymbol{j}}_\perp(z,\boldsymbol{k}) = \frac{n_0/h_0}{z+k^2\Gamma_\perp}\delta\hat{\boldsymbol{p}}_\perp(t=0,\boldsymbol{k}) - \frac{n_0^2T_0g^{jj}}{h_0\Gamma_j} \times\left(\delta\hat{\boldsymbol{j}}_\perp(t=0,\boldsymbol{k}) - \frac{n_0}{h_0}\delta\hat{\boldsymbol{p}}_\perp(t=0,\boldsymbol{k})\right). \tag{C10}$$

If we notice that the conjugate variable for $\delta\hat{p}^i$ is

$$\hat{A}^{p^i} = \frac{1}{h_0T_0}\delta\hat{p}^i, \tag{C11}$$

in the space of $\{\delta\hat{e},\delta\hat{\boldsymbol{p}},\delta\hat{n}\}$, we find

$$\Xi_{j_\perp}{}^{p_\perp}(z,\boldsymbol{k}) = \frac{n_0/h_0}{z+k^2\Gamma_\perp}, \tag{C12}$$

$$\Xi_{p_\perp}{}^{p_\perp}(z,\boldsymbol{k}) = \frac{1}{z+k^2\Gamma_\perp}, \tag{C13}$$

where we used Eqs. (74) and (75). Finally, we arrive at

$$\left[\Xi_{j_\perp}{}^{p_\perp}(z,\boldsymbol{k}) - \frac{n_0}{h_0}\Xi_{p_\perp}{}^{p_\perp}(z,\boldsymbol{k})\right] = 0. \tag{C14}$$

We have shown the equation for the longitudinal component [Eq. (C4)]. Using a similar procedure, we can show the equation for the longitudinal component [Eq. (C3)].

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