Nonlinear energy transfer in classical and quantum systems

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(Received 22 March 2012; revised manuscript received 17 July 2012; published 13 February 2013)

In this paper we investigate the effect of slowly-varying parameters on the energy transfer in a weakly coupled system. For definiteness, we consider a system of two nonlinear oscillators, in which the directly excited first oscillator with constant parameter is attached to the oscillator with slowly time-varying frequency. It is proved that the equations of the slow passage through resonance in this system are identical to the equations of nonlinear Landau-Zener (LZ) tunneling. Three types of dynamical behavior are distinguished, namely, quasilinear, moderately nonlinear, and strongly nonlinear ones. Quasilinear systems exhibit a gradual energy transfer from the excited to the attached oscillator, while moderately nonlinear systems are characterized by an abrupt transition from the energy localization on the excited oscillator to the localization on the attached oscillator. In strongly nonlinear systems, the transition from the energy localization to strong energy exchange between the oscillators is revealed. Explicit approximate solutions describing the transient processes in moderately and strongly nonlinear systems are suggested. Correctness of the constructed approximations is confirmed by numerical results. The results presented in this paper, in addition to providing an analytical framework for understanding the transient dynamics, suggest an approximate procedure for solving the nonlinear LZ problem with arbitrary initial conditions over a finite time-interval.

DOI: 10.1103/PhysRevE.87.022904 PACS number(s): 05.45.Xt, 02.30.Mv

I. INTRODUCTION

The concept of targeted energy transfer (TET) in nonlinear classical systems with constant coefficients was introduced earlier for a model of two weakly coupled nonlinear oscillators, one of which is initially excited [1]; motivated by this concept, a common consideration of classical and quantum energy transfers was suggested [2,3]. Classical and quantum transitions have interesting and nontrivial applications in diverse fields of applied mathematics, natural sciences, and engineering, and numerous studies have been devoted to the theory and applications of these phenomena, see, e.g., [4–11] and references therein. Most of the theoretical results concern the systems with constant parameters but over the last decade nonlinear quantum transitions under slow driving have been investigated using different methods [10–17].

The occurrence of irreversible energy transfer and an adequate analytical description of the transient processes in linear and quasilinear systems with time-dependent parameters and arbitrary initial conditions was presented in earlier works [7–10]. Additionally, it was shown that the equations of the slow dynamics in the passage through resonance in linear [7–9] and quasilinear [10] systems of two weakly coupled oscillators with slowly varying frequencies are equivalent to the well-known Landau-Zener (LZ) equations [12,13]. In this paper we extend earlier obtained results to the system with stronger nonlinearity and examine different types of the transient processes occurring in the systems with slowly varying parameters.

The paper is organized as follows. In the first part of the paper (Secs. II and III), the quasiresonant dynamics of a system of two weakly coupled nonlinear oscillators with *constant* parameters is analyzed.

Although the phase portraits for the conservative systems as well we for the systems with "frozen" time-dependent parameters were presented earlier [1–3,12–14,18–22], a more detailed analysis given in Secs. II and III enables us to distinguish three types of the dynamical behavior: quasilinear, moderately nonlinear, and strongly nonlinear ones.

It is important to note that earlier work on nonlinear tunneling [12,13] outlined a single parametric boundary associated with the change of the number of the stationary points and corresponding, in our terminology, to the boundary between quasilinear and moderately nonlinear systems. In Sec. III we derive the second parametric boundary corresponding to the boundary between moderately and strongly nonlinear systems. It is shown that this dynamical transition is closely related to the behavior of the so-called limiting phase trajectory (LPT) [20]. This implies that the dynamical analysis cannot be reduced to the study of the stationary points and their stability.

In Secs. IV and V we examine the dynamical behavior of the slowly time-dependent moderately and strongly nonlinear systems. We demonstrate an exact mathematical analogy between energy transfer in a classical system and nonlinear LZ tunneling. Section IV suggests an explicit asymptotic solution describing the transient processes in the moderately nonlinear system under adiabatically slow driving. The approach is based on the fact that at each stage of motion, both before and after tunneling, the energy of the system is localized on one of the oscillators, while the energy of the second one is small enough. This inherent property allows us to use an iterative procedure earlier developed for quasilinear systems [10]. The initial iteration expressed through the Fresnel integral formally explains the convergence of energy to a certain limiting value, while the successive iterations improve the accuracy of the approximate solution. In addition to the description of the transient processes, the explicit approximation allows the analytical calculation of the change of energy in tunneling. Numerical results show that even the main approximation provides an adequate representation of the transient processes.

Section V presents an approximate analysis of energy transfer in the strongly nonlinear system under slow driving.

Details of this analysis are provided in Appendix. A special case of the rapid irreversible energy transfer in the strongly nonlinear system with slow time-dependent parameters is also investigated.

II. STATIONARY MODEL

In this part of the paper, the dynamics of two weakly coupled nonlinear oscillators with *constant* parameters is examined in detail. Although the dynamics of the conservative systems was investigated earlier, e.g., in [18–22], here we analyze the dynamical transitions in more detail with the aim of highlighting different types of the nonlinear behavior manifested in both classical and quantum systems and finding the sets of the parameters corresponding to quasilinear, moderately nonlinear, and strongly nonlinear types of the dynamical behavior.

The classical model of two weakly coupled oscillators is presented in Fig. 1. We denote by m the equal masses of the oscillators; c_1 and c_2 denote the coefficients of linear stiffness of the corresponding oscillators, c denotes the coefficient of cubic nonlinearity, and c_{12} denotes the stiffness of the linear coupling.

The equations of motion are given by

$$\begin{aligned} &m\frac{d^2u_1}{dt^2}+c_1u_1+cu_1^3+c_{12}(u_1-u_2)=0,\\ &m\frac{d^2u_2}{dt^2}+c_2u_2+cu_2^3+c_{12}(u_2-u_1)=0. \end{aligned} \tag{2.1}$$

where u_1 and u_2 are the absolute displacements of the first and second oscillators, respectively (Fig. 1). The small parameter ε is defined through the relative stiffness of weak coupling: $c_{12}/c_1 = 2\varepsilon \ll 1$. Assuming weak nonlinearity and taking into account resonance properties of the system, we redefine the parameters as follows:

$$\begin{aligned} &c_1/m=\omega_0^2,\quad c_2/m=\omega_0^2(1+2\varepsilon g),\quad \tau_0=\omega_0 t,\\ &c/c_1=8\varepsilon\alpha,\quad c_{12}/c_r=2\varepsilon\lambda_r,\quad r=1,2, \end{aligned} \tag{2.2}$$

where a constant parameter g is chosen to ensure the desired dynamics of the originally symmetric ($c_1 = c_2$) system. It follows from (2.2) that $\lambda_1 = 1$, $\lambda_2 = (1 + 2\varepsilon g)^{-1}$.

Transforming Eqs. (2.1) to the dimensionless form and saving the notations $u_{1,2}$ and α for corresponding dimensionless quantities, we obtain the following system (the terms of order higher than ε are ignored):

$$\begin{aligned} &\frac{d^2u_1}{d\tau_0^2}+u_1+2\varepsilon(u_1-u_2)+8\varepsilon\alpha u_1^3=0,\\ &\frac{d^2u_2}{d\tau_0^2}+(1+2\varepsilon g)u_2+2\varepsilon(u_2-u_1)+8\varepsilon\alpha u_2^3=0. \end{aligned} \tag{2.3}$$

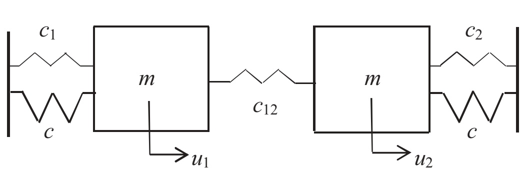

FIG. 1. Nonlinear weakly coupled oscillators.

System (2.3) is assumed to be initially at rest, with the initial unit impulse $v_1 = 1$ applied to the first oscillator. The corresponding initial conditions are given by

$$\begin{aligned} &\tau_0=0,\quad u_1=u_2=0;\\ &v_1=du_1/d\tau_0=1,\quad v_2=du_2/d\tau_0=0. \end{aligned} \tag{2.4}$$

If the initial impulse $v_2 = 1$ is applied to the second oscillator but the first one is initially at rest, then the initial conditions are written as

$$\begin{aligned} &\tau_0=0,\quad u_1=u_2=0;\\ &v_1=du_1/d\tau_0=0,\quad v_2=du_2/d\tau_0=1. \end{aligned} \tag{2.5}$$

Initial conditions (2.5) determine the limiting phase trajectory (LPT). The LPT concept for symmetric systems ($g = 0$) was introduced by one of the authors [20] and then used in various applications [21]. A more detailed discussion of the LPT properties in asymmetric systems ($g \neq 0$) is given below.

An asymptotic solution of system (2.3) for small ε is sought in the form of the multiple-scale expansion [23]. Changing to complex variables,

$$f_r=(v_r+iu_r)e^{-i\tau_0},\quad f_r^*=(v_r-iu_r)e^{i\tau_0},\quad r=1,2, \tag{2.6}$$

where the functions f_r and their time derivative are sought as

$$\begin{aligned} &f_r(\tau_0,\tau_1,\cdots)=f_{r0}(\tau_0,\tau_1,\cdots)+\varepsilon f_{r1}(\tau_0,\tau_1,\cdots)+O(\varepsilon^2),\\ &\frac{df_r}{d\tau_0}=\frac{\partial f_r}{\partial\tau_0}+\varepsilon\frac{\partial f_r}{\partial\tau_1}+O(\varepsilon^2),\quad \tau_1=\varepsilon\tau_0, \end{aligned} \tag{2.7}$$

and then reproducing the transformations of [20,21] we find that

$$f_{10}(\tau_1)=a(\tau_1)e^{i\tau_1},\quad f_{20}(\tau_1)=b(\tau_1)e^{-i\tau_1}, \tag{2.8}$$

where the envelopes $a(\tau_1)$ and $b(\tau_1)$ satisfy the equations

$$\begin{aligned} &\frac{da}{d\tau_1}+ib-3i\alpha|a|^2a=0,\\ &\frac{db}{d\tau_1}+ia-3i\alpha|b|^2b-2igb=0. \end{aligned} \tag{2.9}$$

It follows from (2.4)–(2.8) that the initial conditions take the form $a(0) = 1$, $b(0) = 0$ if the first oscillator is excited but the second one is initially at rest; in the opposite case, when the second oscillator is excited but the first one is initially at rest, the initial conditions are written as $a(0) = 0$, $b(0) = 1$. It is easy to prove that system (2.9) conserves the integral $|a|^2 + |b|^2 = 1$.

It is important to note that Eqs. (2.9) are identical to the equations of the two-state atomic tunneling [18], thereby confirming a mathematical analogy between quantum and classical transitions.

Once the solution $a(\tau_1)$, $b(\tau_1)$ is found, the leading-order approximations u_{r0} and $v_{r0} (r = 1,2)$ can be calculated by (2.6) and (2.8). As a result, we obtain

$$\begin{aligned} &u_{10}(\tau_0,\tau_1)=|a(\tau_1)|\sin[\tau_0+\gamma_1(\tau_1)],\\ &v_{10}(\tau_0,\tau_1)=|a(\tau_1)|\cos[\tau_0+\gamma_1(\tau_1)],\\ &u_{20}(\tau_0,\tau_1)=|b(\tau_1)|\sin[\tau_0+\gamma_2(\tau_1)],\\ &v_{20}(\tau_0,\tau_1)=|b(\tau_1)|\cos[\tau_0+\gamma_2(\tau_1)], \end{aligned} \tag{2.10}$$

where $\gamma_1(\tau_1) = \tau_1 + \arg a(\tau_1)$, $\gamma_2(\tau_1) = \tau_1 + \arg b(\tau_1)$. The well-known estimate of the discrepancy between the precise solution of Eq. (2.3) and its approximation (2.10) is of $O(\varepsilon)$ in the time interval $\tau_0 \sim O(1/\varepsilon)$ (see, e.g., [23,24]). However, a refined analysis has shown that the interval of convergence depends on the properties of approximate solutions (see, e.g., [24]). Moreover, relatively large values of ε used in particular problems do not necessarily imply that the derived analytical approximations will be poor at larger times (numerical examples can be found, e.g., in [6]). The accuracy of approximations should be additionally verified by numerical simulations.

The polar representation $a = \cos\theta e^{i\delta_1}$, $b = \sin\theta e^{i\delta_2}$, $\Delta = \delta_1 - \delta_2$, reduces system (2.9) to the real-valued equations

$$\frac{d\theta}{d\tau_1} = \sin\Delta,$$
$$\sin 2\theta \frac{d\Delta}{d\tau_1} = 2(\cos\Delta + 2k\sin 2\theta)\cos 2\theta - 2g\sin 2\theta, \quad (2.11)$$

where $k = 3\alpha/4$. The initial conditions for system (2.11) are chosen as $\theta = 0$, $\Delta = \pi/2$ (the first oscillator is excited but the second one is initially at rest) or $\theta = \pi/2$, $\Delta = \pi/2$ (the second oscillator is excited but the first one is at rest). Both conditions correspond to the LPTs of system (2.11).

It should be mentioned that system (2.11) is $2\pi -$ periodic in Δ, and the phase portrait in the interval $-\pi \leqslant \Delta \leqslant \pi$ is then reproduced in the interval $\pm\pi \leqslant \Delta \leqslant \pm 3\pi$, etc. Then, it is important to note that system (2.11) is integrable and conserves the integral of motion,

$$K = (\cos\Delta + k\sin 2\theta)\sin 2\theta + g\cos 2\theta, \quad (2.12)$$

whose properties are made in use in the dynamical analysis. The first step in the dynamical analysis is to define the steady states of system (2.11). The first condition of stationarity $d\theta/d\tau_1 = 0$ yields $\sin\Delta = 0$; this implies that all steady states lie on the vertical axes $\Delta_1 = 0$ and $\Delta_2 = \pi$ (Figs. 2–5).

The second condition, $d\Delta/d\tau_1 = 0$, implies that the corresponding values of θ are given by

$$(\pm 1 + 2k\sin 2\theta)\cot 2\theta = g, \quad (2.13)$$

where the signs "+" and "−" correspond to $\Delta = 0$ and $\Delta = \pi$, respectively.

If $k \leqslant 0.5$, then there exists a unique solution of Eq. (2.13) on each of the axes $\Delta = 0$ and $\Delta = \pi$. It was recently shown [10] that the solution of the nonlinear system is close to that of the linear system. The proximity of the solutions allows us to consider the systems with nonlinearity $0 \leqslant k \leqslant 0.5$ as *quasilinear.* Since the energy transfer in quasilinear systems was examined in our preceding work [10], in this paper we focus on the analysis of systems with $k > 0.5$.

As seen in Figs. 2–5, if $k > 0.5$, then there exists a certain g^* such that the system possesses three stationary points on the axis $\Delta = \pi$ for $|g| < |g^*|$ but it has a single point for $|g| \geqslant |g^*|$. Referring to Eq. (2.13), we denote $F(\theta) = (-1 + 2k\sin 2\theta)ctg2\theta$, $\theta_T = \arg\{\max F(\theta)\}$, $\theta \in [0,\pi/2]$. It is obvious that two stationary points coalesce if $F(\theta_T) = g^*$. The maximum condition $dF/d\theta = 0$ at $\theta = \theta_T$ shows that the point of coalescence of two stationary states θ_T and the critical parameter g^* are given by

$$\sin 2\theta_T = (2k)^{-1/3}, \quad g^* = \pm[(2k)^{2/3} - 1]^{3/2}. \quad (2.14)$$

Therefore, the inequalities

$$k > 0.5, \quad |g| < |g^*| \quad (2.15)$$

determine the domain of existence of four stationary states. Note that the parameter g^* coincides with the critical parameter γ_c reported without proof in [13] (in different notations).

As shown below, the system with four stationary states exhibits two different types of motion. First, we consider the system with $k = 0.65$. It is easy to deduce from (2.11) that the change of the sign of the parameter $g \to -g$ entails the following change of the solution: $\theta \to \pi - \theta$, $\Delta \to 2\pi - \Delta$. This allows the construction of the phase portraits only for $g \geqslant 0$ (Fig. 2). Bold lines in Fig. 2 depict the LPT of system (2.11); dotted lines correspond to the homoclinic separatrix. It is seen that with an increase of g the lower homoclinic loop vanishes through the coalescence of the stable and unstable states, and the number of the stationary states changes from four to two. The corresponding critical value $g^* = 0.083$ coincides with the theoretical value given by formula (2.14). It is easy to find that the number of the stationary points changes from two to four at $g^* = -0.083$.

Figure 2(a) illustrates the complete energy exchange between the symmetric oscillators ($g = 0$), i.e., the upper level $\theta = \pi/2$ is reached during the cycle of motion along the LPT starting at $\theta = 0$. Motion along the closed orbits within the domain encircled by the LPT obviously provides less extensive energy exchange than motion along the LPT. Vanishing of the homoclinic separatrix and the emergence of a new closed orbit of finite period [dotted line in Fig. 2(c)]

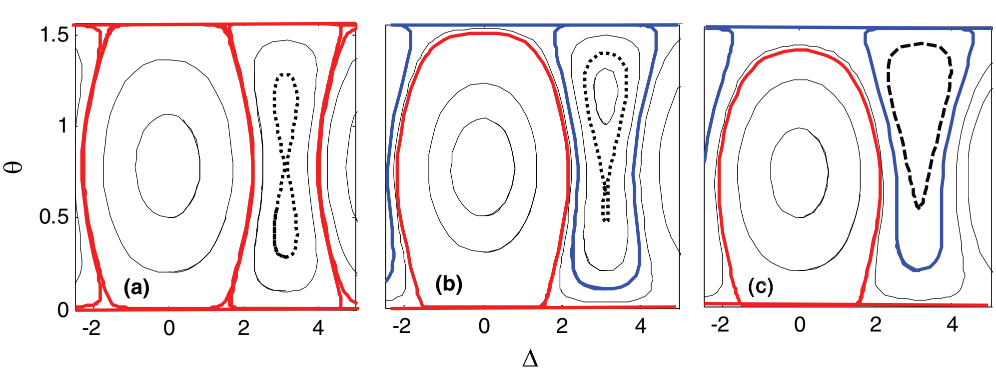

FIG. 2. (Color online) Phase portraits for $k = 0.65$. (a)–(c) $g = 0$, 0.075, and 0.083 correspondingly.

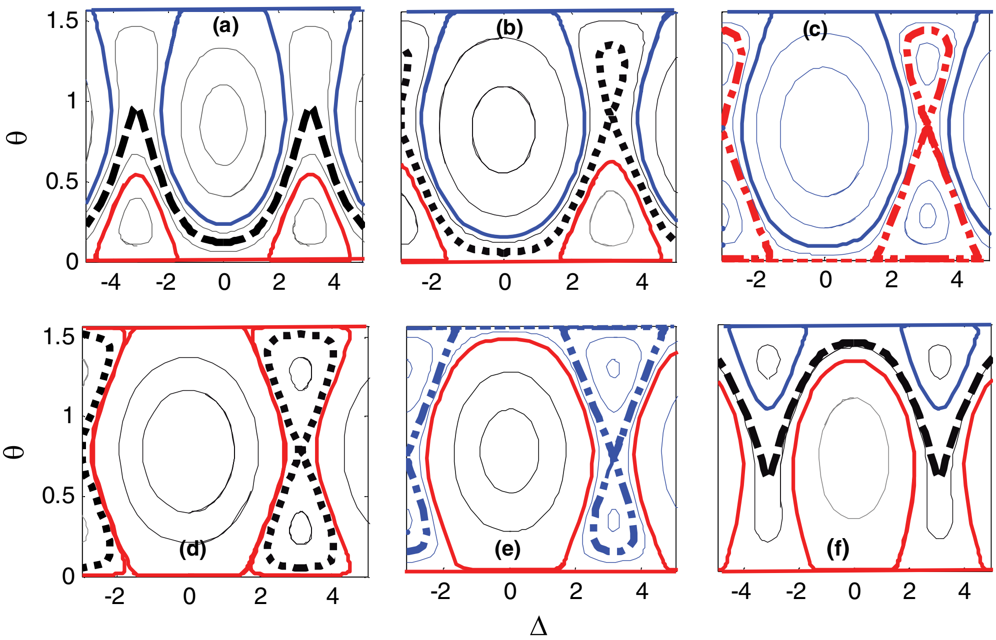

FIG. 3. (Color online) Phase portraits for $k = 0.9$. (a)–(f) $g = -0.33, -0.2, -0.0945, 0, 0.0945$, and 0.33, correspondingly.

characterizes *moderately nonlinear* systems. In these systems the localization of energy near the lower center changes to the localization near the upper center. This phenomenon, responsible for the occurrence of nonlinear tunneling in the slowly time-dependent systems [12,13], underlies the study of transient processes in Sec. IV.

From Fig. 3, it is seen that the system with $k = 0.9$ exhibits a more complicated dynamical behavior. For clarity, the phase portraits for both $g < 0$ and $g > 0$ are shown; bold lines correspond to the LPTs; dash-dotted lines in Figs. 3(c) and 3(e) depict the homoclinic separatrix coinciding with the LPT; edges of the dashed "beaks" in Figs. 3(a) and 3(f) lie at the points of annihilation of the stable and unstable states. The change from two to four fixed points at $g = -0.33$ [Fig. 3(a)] entails the emergence of a separatrix passing through the hyperbolic point that consists of the homoclinic and heteroclinic branches [Fig. 3(b)] [by a heteroclinic separatrix in Fig. 3(b) we understand a trajectory from the hyperbolic point on the axis $\Delta = \pi$ to the hyperbolic point on the axis $\Delta = -\pi$, which separates closed and unbounded orbits]. At $g = -0.0945$, the heteroclinic loop coincides with the LPT at $\theta = 0$ [Fig. 3(c)]. Further increase of g leads to the occurrence of the homoclinic separatrix [Fig. 3(d)] and then to the confluence of the separatrix with the LPT at $\theta = \pi/2$ for $g = 0.0945$ [Fig. 3(e)]; finally, the coalescence of the lower stable center with the hyperbolic point results in the degeneration of the separatrix and the change from four to two fixed points [Fig. 3(f)]. The numerical value $|g^*| = 0.33$ is coincident with the results of calculation by formula (2.14). Figure 3(d) illustrates complete energy exchange in the symmetric system ($g = 0$) moving along the LPT.

Figures 4 and 5 demonstrate the transformations of the phase portraits for $k = 1$, $g > 0$ and $k = 1.1$, $g > 0$, respectively. The phase portraits for $g < 0$ can be constructed by symmetry.

The system featuring the confluence of the separatrix with the LPT and the transition from energy localization to energy exchange is referred to as a *strongly nonlinear system*. The coalescence of the stable and unstable states entails the emergence of a new unlocked orbit and an

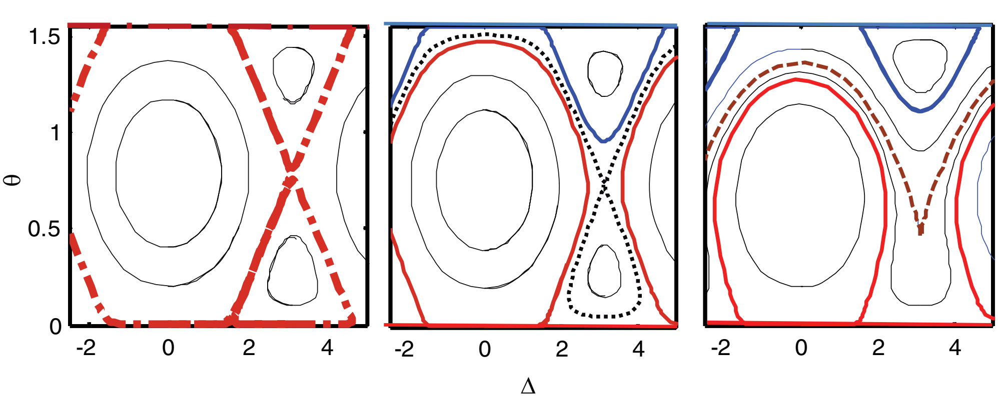

FIG. 4. (Color online) Phase portraits for $k = 1$ and $g = 0, 0.1, 0.45$ (from left to right).

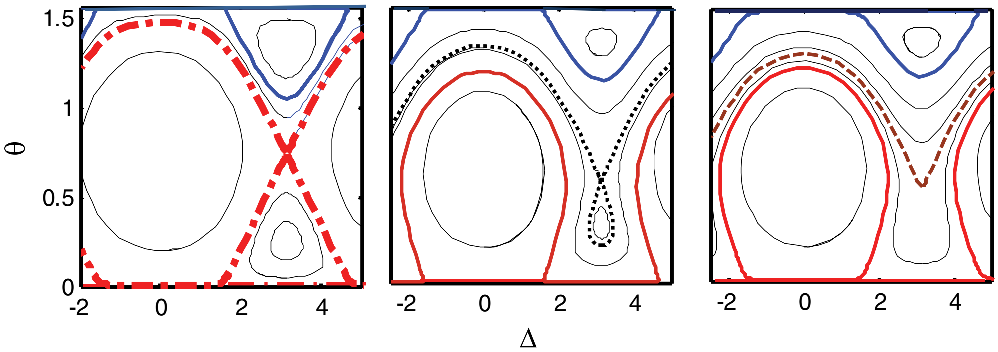

FIG. 5. (Color online) Phase portraits for $k = 1.1$ and $g = 0.1, 0.45, 0.575$ (from left to right).

associated transition from weak to strong energy exchange (Figs. 3–5).

III. CRITICAL PARAMETERS

The existence of the transition from energy localization near the stable state to energy exchange in the asymmetric system was first investigated in [22] but an analytical boundary between the dynamical domains was not derived. In this section, we find analytical conditions that ensure the transition from the energy localization on the excited oscillator to strong energy exchange.

As noted previously, the system may be considered as strongly nonlinear if its separatrix coincides with the LPT. We find the prerequisites for the existence of such a trajectory.

Consider the integral of motion (2.12). The initial condition $\theta = 0$ yields $K = g$ and, therefore,

$$(\cos\Delta + k\sin 2\theta)\sin 2\theta - 2g\sin^2\theta = 0 \tag{3.1}$$

on the LPT. It now follows from Eqs. (2.11) and (3.1) that

$$d\theta/d\tau_1 = V = \sin\Delta; \quad V = \pm[1 - (k\sin 2\theta - g\tan\theta)^2]^{1/2}. \tag{3.2}$$

Using equality (3.1) to exclude Δ, we replace system (2.11) by the second-order equation

$$\frac{d^2\theta}{d\tau_1^2} + \frac{dU}{d\theta} = 0, \tag{3.3}$$

with $\theta(0) = 0$, $V(0) = 1$. The potential $U(\theta)$ in (3.3) can be found from the energy conservation law $E = 1/2V^2 + U(\theta) = 1$. We thus obtain

$$U(\theta) = 1 - \tfrac{1}{2}V^2 = \tfrac{1}{2}[1 + (k\sin 2\theta - g\tan\theta)^2], \tag{3.4}$$

and, therefore, the maximum value $U(\theta) = 1$ is attained at $V = 0$. Potential $U(\theta)$ and phase portraits for system (3.1) with different coefficients k and g are shown in Fig. 6.

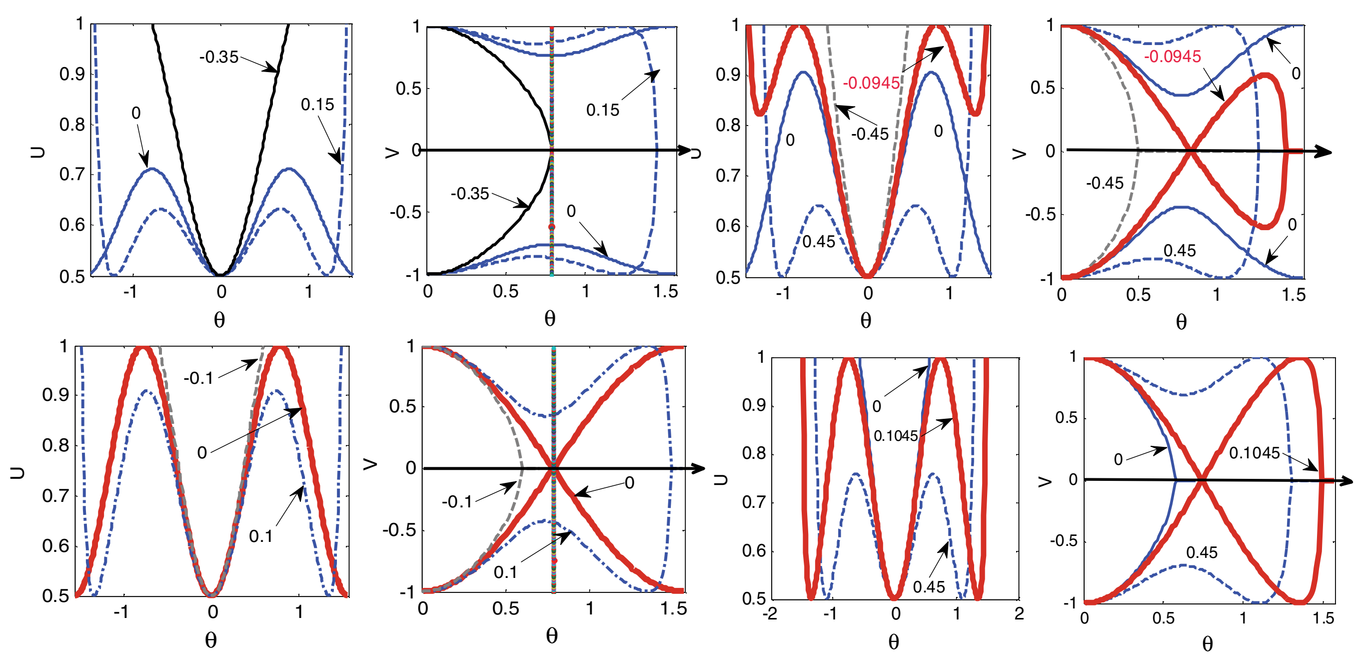

FIG. 6. (Color online) Potential $U(\theta)$ and phase portraits in the plane (θ, v) for system (3.1) at $k = 0.65$, $k = 0.9$ (from left to right in the upper row) and $k = 1$, $k = 1.1$ (from left to right in the bottom row). The value of detuning is indicated on each curve; bold lines depict critical potentials and corresponding separatrices confluent with the LPT at $\theta = 0$.

The sought separatrix may exist if and only if $dU/d\theta = 0$ at $\theta_h \in (0,\pi/2)$, as in this case there exists a potential barrier corresponding to the local maximum $U(\theta_h)$ and attained at $V = 0$; the latter condition is equivalent to $\Delta = 0$. This implies that $(\theta_h, 0)$ is a hyperbolic point. It now follows from (3.4) that the equality $dU/d\theta^* = 0$ is equivalent to

$$(k \sin 2\theta_h - g \tan\theta_h)(2k \cos 2\theta_h - g/\cos^2\theta_h) = 0. \quad (3.5)$$

It follows from (3.2) that

$$(k \sin 2\theta_h - g \tan\theta_h)^2 = 1 \text{ at } V = 0. \quad (3.6)$$

Using (3.5) and (3.6), we reduce Eq. (3.5) to a simple biquadratic equation for $\cos\theta_h$,

$$4k\cos^4\theta_h - 2k\cos^2\theta_h - g = 0, \quad \cos^2\theta_h = \frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{g}{4k}}. \quad (3.7)$$

If $g/k \ll 1/4$, then $\cos^2\theta_h \approx 1/2 + g/2k$, $\theta_h \approx \pi/4 - g/2k$. Using this approximation, we obtain from Eq. (3.6) a simple condition for the existence of the required separatrix:

$$|k - g_h| = 1. \quad (3.8)$$

It is easy to check that the theoretical threshold g_h closely agrees with the results of numerical calculations. In particular, for $k = 0.9$ we have $g_h = -0.1$, whereas the numerical threshold $g = -0.0945$ (Fig. 3); for $k = 1.1$ we have $g = g_h = 0.1$ (Fig. 5).

In the same way, one can prove that if the initial condition is taken at $\theta = \pi/2$, then the condition (3.8) is turned into the equality

$$|k + g_h| = 1. \quad (3.9)$$

If $g < 0$, then solution (3.7) exists provided $|g| \leqslant k/4$; in the limiting case $g = -k/4$ we have $\cos\theta_h = 1/2$, $\theta_h = \pi/3$, and condition (3.6) becomes

$$\frac{3\sqrt{3}}{4}k = 1, \quad k^* \approx 0.77. \quad (3.10)$$

The inequalities $k \geqslant k^*$, $|g| \leqslant k/4$ express *the necessary condition* for the existence of the separatrix coinciding with the LPT at $\theta = 0$. Additionally, one needs to calculate the argument θ_h from Eq. (3.7).

IV. TUNNELING IN MODERATELY NONLINEAR SYSTEMS

In this section, we analyze energy transfer in a system of two *moderately nonlinear* weakly coupled oscillators with slowly varying parameters. It is well known that energy transfer in a nonlinear system with adiabatically changed parameters is divided into two stages, with each characterized by adiabatic invariance but separated by an abrupt jump at a moment of tunneling. Breaking of adiabaticity under slow driving has been intensively studied over the last decades using various approximations [11–16]. However, an explicit analytical description of the transient processes in nonstationary nonlinear systems is prohibitively difficult, and the study of adiabatic tunneling was mainly focused on the calculation of the jump of the adiabatic invariant at tunneling. In this section we investigate slow transient processes before and after tunneling. The asymptotic analysis accounts for the fact that in the first interval of motion most of the energy is localized on the excited oscillator but the residual energy of the coupled oscillator is small enough. After tunneling, localization of energy takes place on the coupled oscillator but the energy of the initially excited oscillator becomes small. An introduction of a small parameter characterizing a relative energy level allows an explicit asymptotic solution.

As in Sec. II, we consider a system of two weakly coupled oscillators. We assume here that linear stiffness of the second oscillator is a slow function of time, namely $c_2(\tau_2) = c_1[1 + 2\varepsilon g(\tau_2)]$, where $g(\tau_2) = g_0 + g_1\tau_2$, $\tau_2 = \varepsilon\tau_1$. Reproducing the transformations (2.2)–(2.7), we reduce the equations for the slow envelopes to the form similar to (2.9) but involving the time-dependent detuning $g(\tau_2)$,

$$\begin{aligned} &\frac{da}{d\tau_1} + ib - 3i\alpha|a|^2 a = 0, \\ &\frac{db}{d\tau_1} + ia - 3ia|b|^2 b - 2ig(\tau_2)b = 0, \\ &a(0) = 1, \ \ b(0) = 0. \end{aligned} \quad (4.1)$$

Equations (4.1) are obviously identical to the nonlinear analog of the LZ equations of quantum tunneling [12,13]. It is easy to prove that the conservation law $|a|^2 + |b|^2 = 1$ holds true for system (4.1) despite its nonstationarity. The change of variables $a = \cos\theta e^{i\delta_1}$, $b = \sin\theta e^{i\delta_2}$, $\Delta = \delta_1 - \delta_1$ reduces system (4.1) to the equations similar to (2.11):

$$\begin{aligned} &\frac{d\theta}{d\tau_1} = \sin\Delta, \\ \sin 2\theta &\frac{d\Delta}{d\tau_1} = 2(\cos\Delta + 2k\sin 2\theta)\cos 2\theta - 2g(\tau_2)\sin 2\theta, \\ &\theta(0) = 0, \quad \Delta(0) = \pi/2. \end{aligned} \quad (4.2)$$

Figure 7 depicts the dynamical behavior of system (4.2) with $k = 0.65$, $g_0 = -0.5$, $\varepsilon g_1 = 0.001$ on the interval $0 \leqslant \tau_1 \leqslant 1000$.

The phase portrait of system (4.2) and the plots of $|a|^2$ and $|b|^2$ clearly demonstrate the occurrence of adiabatic tunneling at $T \approx 585$. The corresponding value of critical detuning $g = 0.085$ is close to the theoretical value $g^* \approx 0.083$. Note that the parameters of numerical simulations are chosen for illustrative purposes but the qualitative features of the results hold true for a wide range of parameters.

First we analyze the dynamical behavior in the interval $S_1: 0 \leqslant \tau_1 < T$ before tunneling. To this end, we employ the change of variables and rescaling of the parameters similar to that described in [15]. The change of variables,

$$\Psi = |b|e^{i\Delta} = |\sin\theta|e^{i\Delta}, \quad \Psi^* = |b|e^{-i\Delta} = |\sin\theta|e^{-i\Delta}, \quad (4.3)$$
$$|\Psi| = |b|,$$

reduces system (4.2) to a single complex-valued equation,

$$\frac{d\Psi}{d\tau_1} = 4i\left[\frac{1 - \frac{1}{2}(\Psi^2 + 2|\Psi|^2)}{\sqrt{1 - |\Psi|^2}} + 2\omega(\tau_2)\Psi - 8k\Psi|\Psi|^2\right], \quad (4.4)$$
$$\Psi(0) = 0,$$

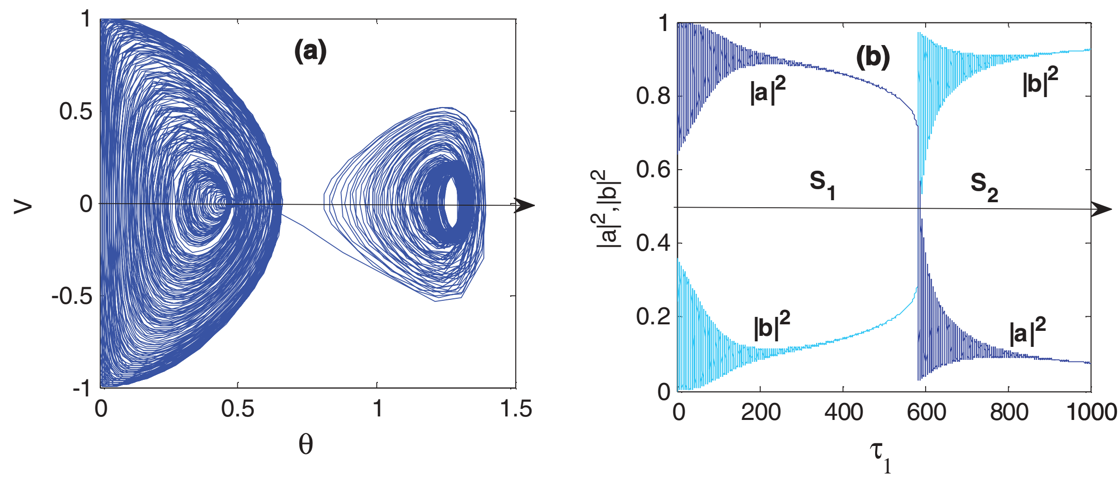

FIG. 7. (Color online) (a) Phase portrait in the plane $(\theta, V = d\theta/d\tau_1)$. (b) Plots of $|a|^2$ [blue (dark gray)] and $|b|^2$ [light blue (light gray)] in the interval $0 \leqslant \tau_1 \leqslant 1000$.

where $\omega(\tau_2) = 2k - g(\tau_2) = \omega_0 - g_1\tau_2$, $\omega_0 = 2k - g_0$. It is important to note that the frequency $\omega(\tau_2)$ directly depends on the coefficient k, thereby reflecting the effect of nonlinearity even in the linear approximation of Eq. (4.4).

Using the condition $|b| = |\Psi| \ll 1$ in S_1 [Fig. 7(b)], we define the variable $\psi = \varepsilon^{-1/2}\Psi$ and introduce a proper time scale $s = \varepsilon^{-1/2}\tau_1$. Then, direct estimation shows that the coefficients of Eq. (4.4) allow rescaling $8k = \varepsilon^{-1/2}\kappa$, $2\omega_0 = \varepsilon^{-1/2}w_0$, where the parameters κ and w_0 are of $O(1)$. Finally, we denote $\varepsilon^{3/2}g_1 = \beta^2/4$ keeping in mind that $\beta \ll 1$. When we substitute the variable ψ and the rescaled coefficients into Eq. (4.4) and ignore the terms of order higher than ε, we obtain the equation

$$\frac{d\psi}{ds} = 4i\left[w(s)\psi + 1 - \frac{1}{2}\varepsilon(\psi^2 + |\psi|^2) - \kappa\varepsilon\psi|\psi|^2\right], \quad \psi(0) = 0, \tag{4.5}$$

where $w(s) = w_0 - \beta^2 s/2$. Thus we get a quasilinear equation, and the earlier developed iteration procedure [10] can be used to construct an approximate solution. Following [10], the initial iteration $\psi_0(s)$ is chosen as a solution of the linear equation

$$\frac{d\psi_0}{ds} = 4i[w(s)\psi_0 + 1], \quad \psi_0(0) = 0,$$
$$\psi_0(s) = 4ie^{i\phi(s)}\Phi(s), \quad \Phi(s) = \int_0^s e^{-i\phi(z)}dz, \tag{4.6}$$
$$\phi(s) = w_0 s - (\beta s)^2.$$

Simple algebra shows that

$$\psi_0(s) = 4i\beta^{-1}e^{i(\phi(s)-\alpha^2)}F(s), \tag{4.7}$$

where $h(s) = \beta s - \alpha$, $\alpha = w_0/\beta$, and

$$F(s) = \int_{-\alpha}^{h(s)} e^{ih^2}dh = \{C[h(s)] + C(\alpha)\} + i\{S[h(s)] + S(\alpha)\}.$$

Here $F(s)$ is the complex-valued Fresnel integral, $C(h)$ and $S(h)$ are the cos- and sin-Fresnel integrals, respectively. If ψ_0 is found, the main approximation to the function $|b|$ is calculated as $|b_0| = \varepsilon^{1/2}|\psi_0|$. Note that the parameter w_0 depends on the coefficient of nonlinearity k, and thus, the behavior of the solution $\psi_0(s)$ is conditioned by the value of k.

As in [10], the first iteration ψ_1 should be found from the "linearized" equation

$$\frac{d\psi_1}{ds} = 4i\left\{[w(s) - \varepsilon\kappa|\psi_0(s)|^2]\psi_1 - e\left[\frac{1}{2}\psi_0^2(s) + |\psi_0(s)|^2\right] + 1\right\}, \quad \psi_1(0) = 0, \tag{4.8}$$

which yields the solution

$$\psi_1(s) = \psi_0(s) + 4ie^{i\varphi(s)}\Phi_1(s),$$
$$\varphi(s) = \phi(s) - \varepsilon\kappa\int_0^s|\psi_0(z)|^2dz,$$
$$\Phi_1(s) = \int_0^s e^{-i\varphi(z)}dz - \varepsilon\int_0^s e^{-i\varphi(z)}R(z)dz, \tag{4.9}$$
$$R(z) = \tfrac{1}{2}\psi_0^2(z) + |\psi_0(z)|^2.$$

Once the solution ψ_1 is derived, the first iteration to the function $|b|$ is calculated as $|b_1| = \varepsilon^{1/2}|\psi_1|$. The exact solution $|b(\tau_1)|^2$ and the iterations $|b_0(\tau_1)|^2$ and $|b_1(\tau_1)|^2$ are presented in Fig. 8.

Figure 8(a) clearly indicates that in the first half of the interval S_1 the maximum difference between the exact solution and the linear approximation is less than 15%. Increased divergence in the second part of S_1 is due to the different behavior of the exact and approximate solutions near the point of transition: While the exact solution undergoes a sudden increase at an instant of tunneling, the linear approximation tends to a certain limiting value.

The dynamical behavior in the interval $S_2 : \tau_1 > T$ is studied in a similar way. Given $|a| \ll 1$, the variables analogous to (4.3) are introduced:

$$Z = -|a|e^{i\Delta} = -|\cos\theta|e^{i\Delta},$$
$$Z^* = -|a|e^{-i\Delta} = -|\cos\theta|e^{-i\Delta}, \quad |Z| = |a|. \tag{4.10}$$

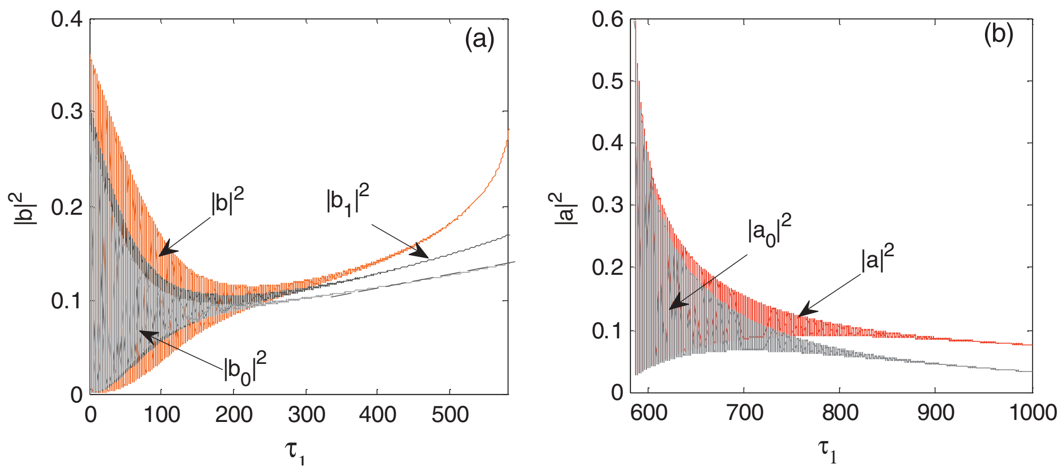

FIG. 8. (Color online) Exact and approximate solutions before and after tunneling: (a) Plots of the exact solution $|b|^2$ [red (dark gray) dashed lines] and the iterations $|b_0|^2$ (light gray) and $|b_1|^2$ (black) in the interval S_1; (b) plots of the exact solution $|a|^2$ [red (dark gray) dashed lines] and the iterations $|a_0|^2$ (light gray) in the interval S_2.

Transformations of the same sort that led to Eqs. (4.4) yield the following equation for Z:

$$\frac{dZ}{d\tau_1} = 4i\left[2\omega_1(\tau_2)Z - 1 + \frac{1}{2}(Z^2 + |Z|^2) - 8\kappa Z|Z|^2\right],\quad Z(T) = p_0, \tag{4.11}$$

with $\omega_1(\tau_2) = 2k + g(\tau_2) = \omega_{10} + g_1\tau_2$, $\omega_{10} = 2k + g_0$. As shown in Sec. II, the initial condition for Eq. (4.11) at $\tau_1 = T$ should be defined from the condition of coalescence of the stable and unstable points. This means that the quantity $\theta(T) = \theta_T$ [see (2.14)] determines the initial condition $|a(T)| = |\cos\theta_T| = p_0$.

Given $|Z| \ll 1$ in S_2, we introduce the transformations

$$Z = \varepsilon^{1/2}z,\quad \tau_1 = \varepsilon^{1/2}s,\quad 8k = \varepsilon^{-1/2}k,\quad 2\omega_{10} = e^{-1/2}w_{10},$$

and denote $\varepsilon^{3/2}g_1 = \beta^2/4, \varepsilon^{-1/2}T = s_0, \varepsilon^{1/2}p_0 = p_{10}$. Substituting the rescaled quantities into Eq. (4.11) and ignoring the higher-order terms, we obtain the quasilinear equation similar to (4.5),

$$\frac{dz}{ds} = 4i\left[w_1(s)z - 1 + \frac{1}{2}\varepsilon(z^2 + 2|z|^2) - \varepsilon kz|z|^2\right],\quad z(s_0) = p_{10}, \tag{4.12}$$

where $w_1(s) = w_{10} + \beta^2 s/2$ is an adiabatically increasing parameter. The initial iteration $z_0(s)$ is sought as a solution of the linear equation

$$\frac{dz_0}{ds} = 4i[w_1(s)z_0 - 1],\quad z_0(s_0) = p_{10},$$
$$z_0(s) = [p_1 - 4i\Theta(s)]e^{i\delta(s)},\quad \Theta(s) = \int_0^s e^{-i\delta(r)}dr. \tag{4.13}$$

where $\delta(r) = 4[w_{10}r + (\beta r)^2]$. Using the representation $\delta(r) = h_1^2(r) - \alpha_1^2$, $h_1(r) = 2\beta r + \alpha_1$, $\alpha_1 = w_{10}/\beta$, we rewrite $\Theta(s)$ as

$$\Theta(s) = e^{i\alpha_1^2}F_1(s)/2\beta,$$
$$F_1(s) = \int_{\alpha_1}^{h_1(s)} e^{-ih^2}dh = \{C[h_1(s)] - C(\alpha_1)\} - i\{S[h_1(s)] - S(\alpha_1)\}, \tag{4.14}$$

where $F_1(s)$ is the complex-valued Fresnel integral. Applying the inverse rescaling, we get the initial iteration to $|a|$ as $|\bar{a}_0| = \varepsilon^{1/2}|z_0|$. As seen in Fig. 8(b), the plot of $|a_0(\tau_1)|^2$ is close to $|a(\tau_1)|^2$.

Now we calculate the stationary state $|\bar{a}_0|$. If $s \to \infty$ and, therefore, $h_1(s) \to \infty$, then the cos- and sin-Fresnel integrals are approximated as [25]

$$C\,[h_1(s)] \to \frac{1}{2}\sqrt{\frac{\pi}{2}},\quad S[h_1(s)] \to \frac{1}{2}\sqrt{\frac{\pi}{2}}. \tag{4.15}$$

Since $\alpha_1 = w_{10}/\beta \gg 1$ for $\beta \ll 1$, the terms $C(\alpha_1)$ and $S(\alpha_1)$ in (4.14) allow the approximation

$$C(\alpha_1) \approx \frac{1}{2}\left(\sqrt{\frac{\pi}{2}} + \frac{\sin\alpha_1^2}{\alpha_1}\right) + O\left(\frac{1}{\alpha_1^2}\right),$$
$$S(\alpha_1) \approx \frac{1}{2}\left(\sqrt{\frac{\pi}{2}} - \frac{\cos\alpha_1^2}{\alpha_1}\right) + O\left(\frac{1}{\alpha_1^2}\right). \tag{4.16}$$

Combining (4.14)–(4.16), we obtain that $F_1(s) \to ie^{-i\alpha_1^2}/2\alpha_1$, $\Theta(s) \to i/4w_{10}$, $z_0(s) \to (p_{10} + 1/w_{10})e^{i\delta(s)}$, and therefore, $|z_0(s)| \to \bar{z}_0 = |p_1 + 1/w_{10}|$ as $s \to \infty$. Finally, we obtain

$$|\bar{a}_0| = \varepsilon^{1/2}\bar{z}_0 = |p_0 + 1/\omega_{10}|\,. \tag{4.17}$$

The resulting nonzero limiting value of $|\bar{a}_0|$ implies the existence of the residual energy depending on the initial condition p_0. Given $k = 0.65$, $g_0 = -0.5$, we obtain $|\bar{a}_0| = 0.445$. It follows from (4.17) that in the interval $S_2 : \tau_1 > T$ the quantity e_2 tends to the limiting value $\bar{e}_2 = |\bar{b}_0|^2/2 = (1 - |\bar{a}_0|^2)/2 = 0.401$ as $\tau_1 \to \infty$. The quantity $\bar{e}_2$ characterizes the amount of energy transferred from the excited oscillator with initial energy $e_{10} = 1/2$ to the coupled oscillator being initially at rest.

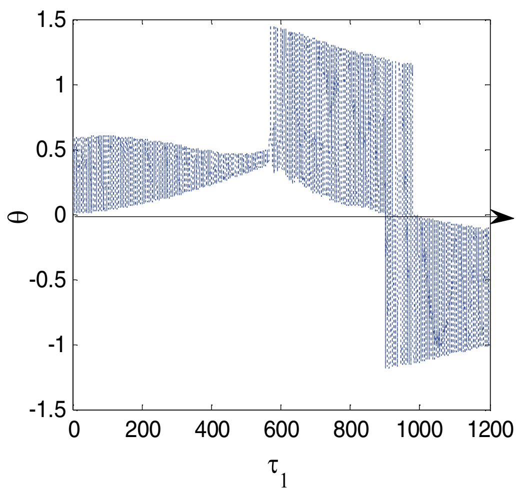

FIG. 9. (Color online) Transient processes in the interval $0 \leqslant \tau_1 \leqslant 1200$.

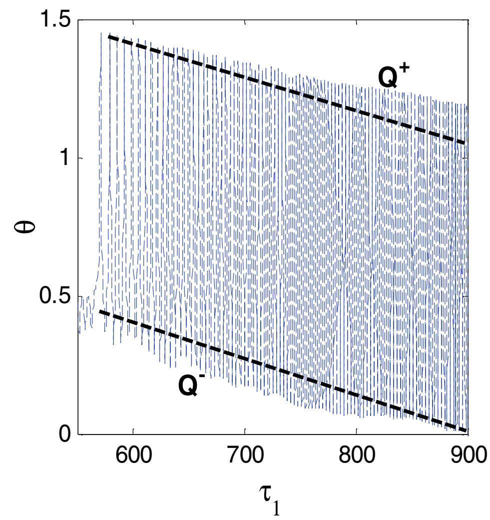

FIG. 10. (Color online) Plot of $\theta(\tau_1)$ in the interval S_2; black dashed lines depict the envelopes (5.2).

V. TUNNELING IN STRONGLY NONLINEAR SYSTEMS

A. Adiabatic transitions

The analysis of strongly nonlinear systems with constant detuning (Sec. II) demonstrates that the change of detuning g may result in a transition from weak to strong energy exchange (Figs. 3–5). In this section we extend this conclusion to strongly nonlinear systems with slowly varying detuning.

Figure 9 demonstrates the evolution of the coordinate θ for system (4.2) with the parameters $k = 0.9$, $g_0 = -0.25$, $\varepsilon g_1 = 0.001$. The phenomenon of energy localization is observed in the initial interval $S_1 : 0 \leqslant \tau_1 < T$. After tunneling, in the interval $S_2 : T \leqslant \tau_1 < T^*$, energy localization changes to intense exchange; in the interval $S_3 : \tau_1 > T^*$, a new transition to localization is observed. Note that tunneling occurs at the value of $T = 580$ that corresponds to $g = 0.33$; this value of g coincides with the result obtained for the stationary system (Fig. 3) as well as with the result of the calculation by formula (2.14).

The adiabatic convergence to the transition point at the first stage of motion can be examined using the techniques of Sec. IV. In the current section, we analyze the dynamical behavior at the second stage $S_2 : T < \tau_1 < T^*$, where variations of $\theta(\tau_1)$ are large enough, and the asymptotic approach of Sec. IV is inapplicable. We note that the local minima and maxima of θ lie on the slowly varying envelopes $Q^-(\tau_2)$ and $Q^+(\tau_2)$, respectively. The stroboscopic method (see Appendix) leads to the following nonlinear equations for the envelopes:

$$\begin{aligned} &2[\cos 2Q^- + k\sin 4Q^- - g(\tau_2)\sin 2Q^-]\frac{dQ^-}{d\tau_1} \\ &\quad = \varepsilon g_1\left(\cos 2Q^+ - \cos 2Q^-\right), \\ &2[-\cos 2Q^+ + k\sin 4Q^- - g(\tau_2)\sin 2Q^+]\frac{dQ^+}{d\tau_1} \\ &\quad = \varepsilon g_1(\cos 2Q^- - \cos 2Q^+), \end{aligned} \qquad (5.1)$$

with initial conditions $Q^- = \theta_0^-$, $Q^+ = \theta_0^+$ at $\tau_1 = T$. The point θ_0^- is defined as a point of coalescence of the stable and unstable states at the moment of tunneling; the point θ_0^+ may be calculated by Eq. (A4) in which $G_n = G_0 = g_0 + \varepsilon g_1 T$.

Given that the quantities $|Q^-|$ and $|\pi/2 - Q^+|$ are small enough (Fig. 10), the initial approximation can be chosen as $Q_0^- = 0$, $Q_0^+ = \pi/2$. Linearization of Eq. (5.1) near Q_0^-, Q_0^+ yields the first approximation:

$$Q_1^{\pm}(\tau_1) = \theta_0^{\pm} - \varepsilon g_1(\tau_1 - T). \qquad (5.2)$$

Figure 10 demonstrates a good agreement between the approximation (5.2) and the precise (numerical) solution in the interval S_2.

B. Rapid transitions

As noted previously, the adiabatic tunneling requires a very long time for reaching the transition point from the initial state. Here we demonstrate an alternative scenario of rapid tunneling, which may occur in the presence of the homoclinic separatrix coinciding with the LPT (Fig. 11).

Figures 11(a) and 11(c) show that the transition of the trajectory starting at $\theta = 0$ to the upper level near $\theta = \pi/2$ is due to the passage through the separatrix in a neighborhood of the saddle point. The time interval from the initial time instant to transition is small, and, therefore, in this interval the change of detuning in each cycle of oscillations as well as deviations of the bottom points from $\theta = 0$ can be ignored. By the same reason, deviations of the top points from $\theta = \pi/2$ remain negligible in the interval S_2 (in the first approximation).

We now evaluate extrema of $\theta(\tau_1)$ in each cycle of motion. One can derive from (2.11) that $V(\theta) = \pm[1 - (k\sin 2\theta - g\tan\theta)^2]^{1/2}$ on the lower branch of the LPT and $V(\theta) = \pm[1 - (k\sin 2\theta + g\cot\theta)^2]^{1/2}$ on the upper branch (the argument τ_2 is omitted). The extrema θ^* of the function $\theta(\tau_1)$ obey the condition $V(\theta) = 0$, yielding

$$\begin{aligned} &k\sin 2\theta - g\tan\theta = 1 \text{ on the lower branch of LPT,} \\ &k\sin 2\theta + g\cot\theta = 1 \text{ on the upper branch of LPT} \end{aligned} \qquad (5.3)$$

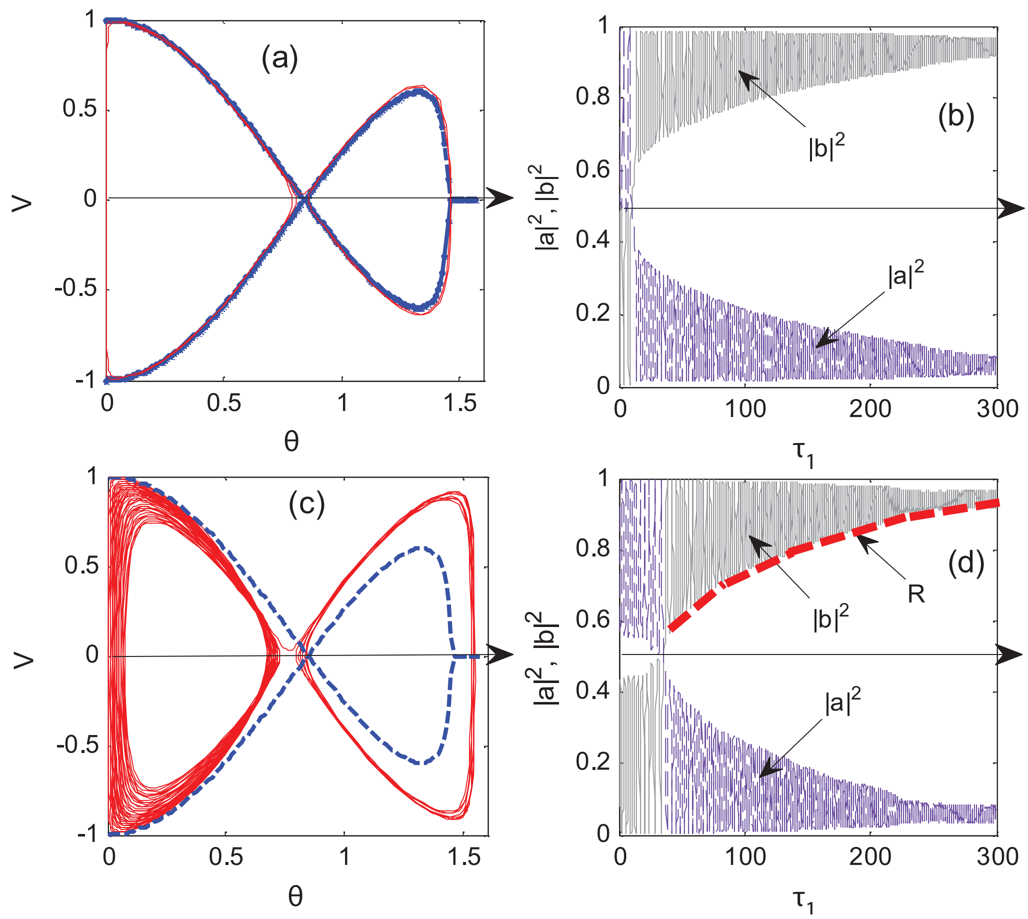

FIG. 11. (Color online) Phase portraits (left column) and plots of $|a|^2$ and $|b|^2$ in course of rapid tunneling with detuning $g(\tau_1) = -0.1 + 0.0015\tau_1$ (upper panel) and $g(\tau_1) = -0.15 + 0.0015\tau_1$ (lower panel). Bold (light gray) lines in the left column depict the separatrix of the system with critical detuning $g^* = -0.1$; bold red (black) dashed line R depicts the approximate envelope of $|b(\tau_1)|^2$ [see Eq. (5.3)].

Since $\theta^* = \pi/4 + q$, $|q| < 1$, expressions (5.3) can be transformed into the linear equations

$$\begin{aligned} &k - g(1 + 2q) = 1, \quad q = 1/2[(k - 1)/(g - 1)] \\ &\qquad \text{on the lower branch of LPT,} \\ &k + g(1 - 2q) = 1, \quad q = 1/2[1 - (1 - k)/g] \\ &\qquad \text{on the upper branch of LPT.} \end{aligned} \tag{5.4}$$

The insertion of the time-dependent detuning $g(\tau_2)$ into (5.4) provides an adiabatic approximation of the slowly varying envelopes of the function θ. Figure 11(d)) shows the approximation $R(\tau_1)$ of the lower envelope of the process $|b|^2 = \sin^2\theta$ in the system with parameters $k = 0.9$, $g_0 = -0.15$, $\varepsilon g_1 = 0.0015$. It is obvious that the theoretical approximation is close to the result of numerical calculations. The upper envelope is approximated as $\theta = \pi/2$, $|b| = 1$.

VI. CONCLUSIONS

In this paper we studied an effect of slowly varying parameters on the energy transfer between weakly coupled nonlinear oscillators. For definiteness, we considered a system of two nonlinear oscillators, in which the oscillator with constant parameters is connected with the slowly time-dependent oscillator. Motion is excited by an initial impulse applied to the time-independent oscillator.

The preliminary examination of the conservative system displayed three types of the energy transfer, namely, quasilinear, moderately nonlinearity, and strongly nonlinear ones. The specifics of the ensembles of trajectories corresponding to each type of motion were discussed. In particular, it was shown that the dynamical transitions are closely related to the behavior of the so-called limiting phase trajectory (LPT). The relationships between the parameters determining each type of the dynamical behavior were found.

In the second part of the paper we demonstrated the mathematical equivalence of the equations describing the slow passage through resonance in the classical system with slowly varying parameters with the equations of nonlinear LZ tunneling. It was noted that the well-known LZ tunneling is typical for moderately nonlinear systems, while strongly nonlinear systems can exhibit both the transition from the energy localization to intense energy exchange (with a large inductive period) as well as the rapid passage through the separatrix. An analytical description of the transient processes was achieved through a set of approximations. Correctness of the constructed approximations was confirmed by numerical simulations.

The revealed mathematical equivalence suggests a unified approach to the study of such physically different processes as energy transfer in classical oscillatory systems under slow driving and nonlinear quantum LZ tunneling. Furthermore, this analogy paves the way for a simple mechanical simulation of complicated quantum effects.

ACKNOWLEDGMENTS

The authors acknowledge support for this work received from the Russian Foundation for Basic Research through the RFBR Grant No. 10-01-00698. Also, the authors would like to thank the anonymous reviewers for their comments and advice.

APPENDIX

Here we derive Eqs. (5.1). Using the stroboscopic method [24], we approximate a smooth function $g(\tau_2)$ by its step approximation,

$$G(\tau_2) = \sum_{n=0}^{N} G_n \left[\eta\left(T_{n+1} - \tau_2\right) - \eta\left(T_n - \tau_2\right)\right], \quad \text{(A1)}$$

where $G_n = g(T_n)$, T_n corresponds to a time instant at which the function θ achieves its nth extremum θ_n; a corresponding value of the phase Δ is given by $\Delta_n = \pi n$. It now follows from (A1) that $G(\tau_2) = G_n$ in each interval $I_n : T_n \leqslant \tau_2 < T_{n+1}$. The step approximation allows one to replace Eqs. (4.2) by the system with constant detuning in each interval I_n, namely,

$$\frac{d\theta}{d\tau_1} = \sin\Delta, \quad \frac{d\Delta}{d\tau_1} = 2(\cos\Delta + 2k\sin 2\theta)\cot 2\theta - 2G_n. \quad \text{(A2)}$$

By analogy with (2.12), one can conclude that in each interval I_n system (A2) preserves the first integral of motion,

$$K_n = (\cos\Delta + k\sin 2\theta)\sin 2\theta + G_n\cos 2\theta. \quad \text{(A3)}$$

The parameter K_n on both ends of the interval I_n, namely at $\theta_n = \theta(T_n)$, $\Delta_n = \pi n$ and $\theta_{n+1} = \theta(T_{n+1})$, $\Delta_{n+1} = \pi(n+1)$, is expressed as

$$\begin{aligned} K_n &= [(-1)^n + k\sin 2\theta_n]\sin 2\theta_n + G_n\cos 2\theta_n = \Psi_n(\theta_n),\\ K_n &= [(-1)^{n+1} + k\sin 2\theta_{n+1}]\sin 2\theta_{n+1} + G_n\cos 2\theta_{n+1}\\ &= \Phi_n(\theta_{n+1}), \end{aligned} \quad \text{(A4)}$$

so that $\Psi_n(\theta_n) = \Phi_n(\theta_{n+1})$. Here and below the functions Ψ_n and Φ_n are defined at the left and right ends of the interval I_n, respectively. In the interval $I_{n+1} : T_{n+1} \leqslant \tau_2 < T_{n+2}$ system (A2) preserves the integral

$$K_{n+1} = (\cos\Delta + k\sin 2\theta)\sin 2\theta + G_{n+1}\cos 2\theta, \quad \text{(A5)}$$

such that

$$\begin{aligned} K_{n+1} &= [(-1)^{n+1} + k\sin 2\theta_{n+1}]\sin 2\theta_{n+1} + G_{n+1}\cos 2\theta_{n+1}\\ &= \Psi_{n+1}(\theta_{n+1}),\\ K_{n+1} &= [(-1)^n + k\sin 2\theta_{n+2}]\sin 2\theta_{n+2} + G_{n+1}\cos 2\theta_{n+2}\\ &= \Phi_{n+1}(\theta_{n+2}), \end{aligned} \quad \text{(A6)}$$

where $\Psi_{n+1}(\theta_{n+1}) = \Phi_{n+1}(\theta_{n+2})$ and θ_{n+1} and θ_{n+2} are the initial and end points of I_{n+1}, corresponding to $\Delta = \pi(n+1)$ and $\pi(n+2)$, respectively. The quantities K_{n+r} for any $r > 1$ may be expressed in a similar way. It now follows from (A4)–(A6) that

$$\begin{aligned} K_{n+1} - K_n &= \Phi_{n+1}(\theta_{n+2}) - \Psi_n(\theta_n)\\ &= \Psi_{n+1}(\theta_{n+1}) - \Phi_n(\theta_{n+1}). \end{aligned} \quad \text{(A7)}$$

Let $\theta_{n+2} - \theta_n = \delta\theta_n$, $G_{n+1} - G_n = \delta G_n$. Substituting (A4)–(A6) into (A7) and considering only the terms linear in $\delta\theta_n$, δG_n, we obtain the difference approximations,

$$\begin{aligned} &2[(-1)^n\cos 2\theta_n + k\sin 4\theta_n - G_n\sin 2\theta_n]\delta\theta_n\\ &\quad = (\cos 2\theta_{n+1} - \cos 2\theta_n)\delta G_n. \end{aligned} \quad \text{(A8)}$$

In the same way, a comparison of the integrals K_{n+2} and K_{n+1} in the intervals I_{n+2} and I_{n+1} yields the following difference equation:

$$\begin{aligned} &2[(-1)^{n+1}\cos 2\theta_{n+1} + k\sin 4\theta_{n+1} - G_{n+1}\sin 2\theta_{n+1}]\delta\theta_{n+1}\\ &\quad = (\cos 2\theta_{n+2} - \cos 2\theta_{n+1})\delta G_{n+1}. \end{aligned} \quad \text{(A9)}$$

For definiteness, we suppose that points θ_n and θ_{n+1} coincide with a local minimum and a consequent maximum of $\theta(\tau_1)$, respectively; that is, $n = 2k$. Since the local minima and maxima of $\theta(\tau_1)$ lie on the slowly varying envelopes $Q^-(\tau_2)$ and $Q^+(\tau_2)$, in the limit of small differences, Eqs. (A8) and (A9) can be replaced by the following system of differential equations for the slowly varying envelopes:

$$\begin{aligned} &2[\cos 2Q^- + k\sin 4Q^- - g(\tau_2)\sin 2Q^-]\frac{dQ^-}{d\tau_1}\\ &\quad = \varepsilon g_1(\cos 2Q^+ - \cos 2Q^-),\\ &2[-\cos 2Q^+ + k\sin 4Q^+ - g(\tau_2)\sin 2Q^+]\frac{dQ^+}{d\tau_1}\\ &\quad = \varepsilon g_1(\cos 2Q^- - \cos 2Q^+), \end{aligned} \quad \text{(A10)}$$

with initial conditions $Q^- = \theta_0^-$, $Q^+ = \theta_0^+$ at $\tau_1 = T$. The point θ_0^- is defined as a point of coalescence of the stable and unstable states at the moment of tunneling; the point θ_0^+ may be calculated by Eq. (A4) in which $G_n = G_0 = g_0 + \varepsilon g_1 T$.

[1] G. Kopidakis, S. Aubry, and G. P. Tsironis, Phys. Rev. Lett. **87**, 165501 (2001); S. Aubry, G. Kopidakis, A. M. Morgante, and G. P. Tsironis, Physica B **296**, 222 (2001).

[2] P. Maniadis, G. Kopidakis, and S. Aubry, Physica D **188**, 153 (2004).

[3] P. Maniadis and S. Aubry, Physica D **202**, 200 (2005).

[4] *Localization and Energy Transfer in Nonlinear Systems*, edited by L. Vazquez, R. MacKay, and M. P. Zorzano (World Scientific, Singapore, 2003).

[5] *Energy Localization and Transfer*, edited by T. Dauxois, A. Litvak-Hinenzon, R. MacKay, and A. Spanoudaki (World Scientific, Singapore, 2004).

[6] A. F. Vakakis, O. Gendelman, L. A. Bergman, D. M. McFarland, G. Kerschen, and Y. S. Lee, *Passive Nonlinear Targeted Energy Transfer in Mechanical and Structural Systems* (Springer-Verlag, Berlin, New York, 2008).

[7] L. I. Manevitch, Yu. A. Kosevich, M. Mane, G. M. Sigalov, L. A. Bergman, and A. F. Vakakis, Int. J. Non-Linear Mech. **46**, 247 (2011).

[8] Yu. A. Kosevich, L. I. Manevitch, and E. L. Manevitch, Phys.-Usp. **53**, 1281 (2010).

[9] A. Kovaleva, L. I. Manevitch, and Y. A. Kosevich, Phys. Rev. E **83**, 026602 (2011).

[10] A. Kovaleva and L. I. Manevitch, Phys. Rev. E **85**, 016202 (2012).

[11] R. Schilling, M. Vogelsberger, and D. A. Garanin, J. Phys. A: Math. Gen. **39** 13727 (2006); A. P. Itin, A. A. Vasiliev, G. Krishna, and S. Watanabe, Physica D **232**, 108 (2007); N. Sahakyan, H. Azizbekyan, H. Ishkhanyan, R. Sokhoyan, and A. Ishkhanyan, Laser Physics **20**, 291 (2010); R. Khomeriki, Eur. Phys. J. D **61**, 193 (2011).

[12] O. Zobay and B. M. Garraway, Phys. Rev. A **61**, 033603 (2000); D. Witthaut, E. M. Graefe, and H. J. Korsch, *ibid.* **73**, 063609 (2006); F. Trimborn, D. Witthaut, V. Kegel, and H. J. Korsch, New J. Phys. **12**, 05310 (2010).

[13] J. Liu, L. Fu, B. Y. Ou, S. G. Chen, D. I. Choi, B. Wu, and Q. Niu, Phys. Rev. A **66**, 023404 (2002).

[14] A. P. Itin and S. Watanabe, Phys. Rev. E **76**, 026218 (2007).

[15] A. P. Itin and P. Törmä, arXiv:0901.4778v1 (2010); Phys. Rev. A **79**, 055602 (2009).

[16] L. D. Landau, Phys. Z. Sowjetunion **2**, 46 (1932).

[17] C. Zener, Proc. R. Soc. London, Ser. A **137**, 696 (1932).

[18] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997); S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy, Phys. Rev. A **59**, 620 (1999).

[19] K. W. Mahmud, H. Perry, and W. P. Reinhardt, Phys. Rev. A **71**, 023615 (2005).

[20] L. I. Manevitch, Arch. Appl. Mech. **77**, 301 (2007).

[21] L. I. Manevitch and O. V. Gendelman, *Tractable Models of Solid Mechanics: Formulation, Analysis and Interpretation* (Springer-Verlag, Berlin, Heidelberg, 2011), Chap. 2; L. I. Manevitch, in *Vibro-Impact Dynamics of Ocean Systems and Related Problems*, Lecture Notes in Applied and Computational Mechanics Vol. 44, edited by R. A. Ibrahim, V. I. Babitsky, and M. Okuma (Springer, New York, 2009), pp. 149–160; L. I. Manevitch and V. V. Smirnov, in *Advanced Nonlinear Strategies for Vibration Mitigation and System Identification*, CISM International Centre for Mechanical Sciences Vol. 518, edited by A. F. Vakakis (Springer, New York, 2010), pp. 207–258.

[22] G. P. Tsironis, Phys. Lett. A **173**, 381 (1993); G. P. Tsironis, W. D. Deering, and M. I. Molina, Physica D **68**, 135 (1993).

[23] A. H. Nayfeh, *Perturbations Methods* (Wiley, New York, 2000).

[24] J. A. Sanders, F. Verhulst, and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, 2nd ed. (Springer, New York, 2007).

[25] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 6th ed. (Academic Press, San Diego, CA, 2000).