Shape-dependent finite-size effect of the critical two-dimensional Ising model on a triangular lattice

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Using a bond-propagation algorithm, we study the finite-size behavior of the critical two-dimensional Ising model on a finite triangular lattice with free boundaries in five shapes: triangular, rhomboid, trapezoid, hexagonal, and rectangular. The critical free energy, internal energy, and specific heat are calculated. The accuracy of the free energy reaches 10^{-26}. Based on accurate data on several finite systems with linear size up to $N = 2000$, we extract the bulk, surface, and corner parts of the free energy, internal energy, and specific heat accurately. We confirm the conformal field theory prediction that the corner free energy is universal and find logarithmic corrections in higher-order terms in the critical free energy for the rhomboid, trapezoid, and hexagonal systems, which are absent for the triangular and rectangular systems. The logarithmic edge corrections due to edges parallel or perpendicular to the bond directions in the internal energy are found to be identical, while the logarithmic edge corrections due to corresponding edges in the specific heat are different. The corner internal energy and corner specific heat for angles $\pi/3$, $\pi/2$, and $2\pi/3$ are obtained, as well as higher-order corrections. Comparing with the corner internal energy and corner specific heat we previously found on a rectangle of the square lattice [Phys. Rev. E **86**, 041149 (2012)], we conclude that the corner internal energy and corner specific heat for the rectangular shape are not universal.

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I. INTRODUCTION

Finite-size-scaling theory, introduced by Fisher, finds extensive applications in the analysis of experimental, Monte Carlo, and transfer-matrix data, as well as in recent theoretical developments related to conformal invariance [1–4]. It has become of practical interest due to the recent progresses in fine processing technologies, which has enabled the fabrication of nanoscale materials with novel shapes [5–7]. Exact solutions have played a key role in determining the form of the finite-size scaling. Ferdinand and Fisher [8] pioneered the theory of the two-dimensional (2D) Ising model on a finite-size square lattice, which extended Onsager's exact solution [9] and stimulated the ideas of finite-size scaling. Since then, exact results of the model on finite-size lattices with various boundaries have been studied intensively [8–18]. Detailed knowledge has been obtained for the torus case [12], for the helical boundary condition [13], for the Brascamp-Kunz boundary condition [14,15], and for an infinitely long cylinder [16]. The exact solution on the triangular lattice has also been studied [17,18].

However, for the 2D Ising model the exact solution with free boundaries, i.e., free edges and sharp corners, is still lacking. As we know, the Bethe ansatz is a powerful technique for solving the 2D Ising model, but this technique does not allow the system to be placed on a rectangle with free boundary conditions on the top and bottom, since the expression for the boundary state in terms of the Bethe eigenvectors is unknown; this is a famous unsolved problem in statistical mechanics. Although there are Monte Carlo and transfer-matrix studies on this problem [19,20], the accuracy or the system sizes of the results are not sufficient to extract the finite-size corrections. Meanwhile, for 2D critical systems, a large amount of knowledge has been obtained by the application of the powerful techniques of integrability and conformal field theory (CFT) [3,4,21–23]. Cardy and Peschel predicted that the next subdominant contribution to the free energy on a square comes from the corners [3]; it is universal and related to the central charge c in the continuum limit. Kleban and Vassileva [21] extended the result to a rectangle. These results are consistent with the conjectured exact analytical formula for the Ising model on a square lattice [24].

Several years ago an efficient bond propagation (BP) algorithm was developed for computing the partition function of the Ising model with free edges and corners in two dimensions [25,26]. Making use of this algorithm, we recently determined numerically the exact partition function of the Ising model on a square lattice with a rectangular shape and free boundaries [27]. We not only confirmed the CFT predictions, but also found logarithmic corrections due to corners in the internal energy and specific heat.

In the present paper we apply the BP algorithm to study the Ising model on finite triangular lattices with free boundaries in five different shapes, focusing on how the shape affects the finite-size scaling. The five shapes are triangular, rhomboid, trapezoid, hexagonal, and rectangular, as shown in Fig. 1. Based on accurate data on a sequence of finite systems with linear size up to $N = 2000$, we extract the bulk, surface, and corner parts of the free energy, internal energy, and specific heat. We verify the conformal field theory prediction of the corner free energy and find logarithmic corrections in higher-order terms in the critical free energy for the rhomboid, trapezoid, and hexagonal systems, which are absent for the triangular and rectangular systems. The logarithmic edge corrections due to edges parallel or perpendicular to the bond direction in the internal energy are found to be identical, while the logarithmic edge corrections due to the corresponding

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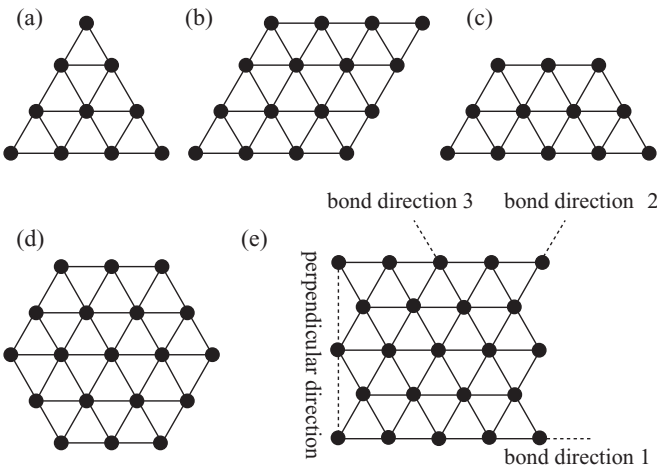

FIG. 1. (a) The triangle-shaped triangular lattice with $N = 4$. (b) The rhombus-shaped lattice with $N = 4$. (c) The trapezoid-shaped lattice with $N = 3$. (d) The hexagon-shaped lattice with $N = 3$. (e) The rectangle-shaped lattice with $N = 5$. Three bond directions and the perpendicular direction are shown (see the text).

edges in the specific heat are different. The corner internal energy and corner specific heat for angles $\pi/3$, $\pi/2$, and $2\pi/3$ are obtained, as well as higher-order corrections. Comparing with the previous found corner internal energy and corner specific heat on a rectangle of the square lattice [27], we conclude that the corner internal energy and corner specific heat for the rectangle shape are not universal.

Our paper is organized as follows: In Sec. II, we briefly describe the BP algorithm used here. We present in Sec. III our main results and discussion. We conclude in Sec. IV.

II. METHOD

The partition function of the Ising model on the 2D triangular lattice is

$$Z = \sum_{\{\sigma_i\}} \exp\left(\beta \sum_{\langle i,j\rangle} \sigma_i \sigma_j\right), \tag{1}$$

where the nearest neighbor couplings are dimensionless and β is the inverse temperature. This partition function for a finite triangular lattice with five different shapes and open boundaries is calculated with the BP algorithm [25] at the exact critical point $\beta_c = \frac{1}{4}\ln(3) = 0.274\,653\,072\,167\ldots$. Figure 1 shows the five shapes: triangle, rhombus, trapezoid, hexagon, and rectangle. The linear size N of a finite lattice is defined as the length of edges in the triangular, rhomboid, and hexagonal cases, of which the lengths of edges are equal. For the trapezoidal shape, the lengths of the three short edges are required to be equal and N is the length of the short edges. For the rectangle, N is defined as the length of the bottom edge, and the number of layers is also required to be N. However, the actual geometrical vertical length is $N\sqrt{3}/2$. According to the finite-size scaling [1], the system size should be the actual geometrical length, not the number of layers. Therefore the aspect ratio of the rectangle that we consider here is $\sqrt{3}/2$ rather than 1.

The BP algorithm for the Ising model on the triangular lattice has been described in detail in Ref. [26]. For the triangle-shaped and rhombus-shaped lattices, this algorithm can be applied directly. However, for the trapezoid-shaped, hexagon-shaped, and rectangle-shaped lattices, the inverse of the BP series reduction should be introduced, which corresponds to generating a new spin between two spins such that

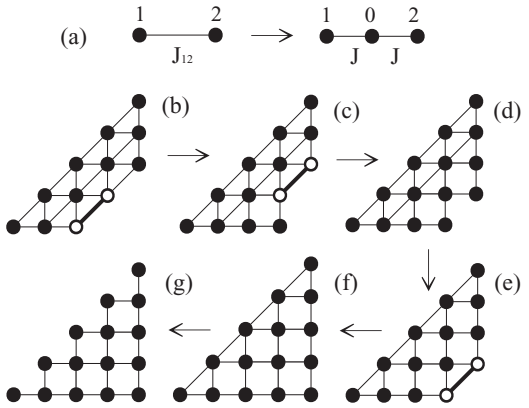

FIG. 2. (a) Inverse of the BP series reduction. See the formulation for this transformation in the text. (b)–(g) are schematics of the algorithm for the trapezoid-shaped lattice. From (b) to (c), the inverse of the BP series reduction is applied to the diagonal bond (thick line) at the bottom of the trapezoid. From (c) to (d), this operation is applied to another diagonal bond (thick line). From (d) to (e) bond-propagation operations are applied. From (e) to (f), the inverse of the BP series reduction is applied again. From (f) to (g) those diagonal bonds are eliminated by the usual BP procedure.

$$e^{J_{12}\sigma_1\sigma_2} \equiv \sum_{\sigma_0} e^{\delta F + J\sigma_0(\sigma_1+\sigma_2)}, \tag{2}$$

as illustrated in Fig. 2(a). It is convenient to use variables $j \equiv e^{-J}$, $j_{12} \equiv e^{-J_{12}}$, and $\delta f \equiv e^{\delta F}$. Then we get the solution $\delta f = j_{12}/2$ and $j = \sqrt{1/j_{12}^2 - \sqrt{1/j_{12}^4 - 1}}$.

In each step of the BP algorithm the transformation is exact. The numerical accuracy is limited only by the machine's precision, which is the round-off error 10^{-32} in the quadruple precision. The BP algorithm needs about N^3 steps to calculate the free energy of an $N \times N$ lattice (it is much faster than other numerical methods). Therefore the total error is approximately $N^{3/2} \times 10^{-32}$. This estimation has been verified in the following way: We compared the results obtained using double precision, in which there are 16 effective decimal digits, and those using quadruple precision. Because the latter results are much more accurate than the former, we can estimate the error in double-precision results by taking the quadruple results as the exact results. We thus found that the error is about $N^{3/2} \times 10^{-16}$. In our calculation, the largest size reached is $N = 2000$, and the round-off error is less than 10^{-26}.

The free energy density, internal energy per spin, and specific heat density are calculated at the critical point according to

$$f = \frac{\ln Z}{S}, \quad u = \frac{\partial f}{\partial \beta}, \quad c = \beta^2 \frac{\partial^2 f}{\partial \beta^2}, \tag{3}$$

respectively, where S is the number of spins on the lattice; this is $S = N(N+1)/2$, N^2, $N(3N-1)/2$, $3N^2 - 3N + 1$,

and $N^2-(N-1)/2$ for the triangular, rhomboid, trapezoid, hexagonal, and rectangular systems, respectively. One can alternatively define the free energy density according to the actual geometrical area, which differs from the present definition by a trivial constant.

With the BP algorithm, we obtain the free energy density f directly. The internal energy and specific heat are calculated by using a differentiation method:

$$u \approx -\frac{f(\beta_c+\Delta\beta)-f(\beta_c-\Delta\beta)}{2\Delta\beta},$$
$$c \approx \beta_c^2 \frac{f(\beta_c+\Delta\beta)+f(\beta_c-\Delta\beta)-2f(\beta_c)}{(\Delta\beta)^2}. \quad (4)$$

In our calculation, $\Delta\beta = 10^{-7}$ is used. The analysis of error in the calculations of u and c has been discussed in Ref. [27]. The final estimations of the accuracy of the free energy, internal energy, and specific heat are 10^{-26}, 10^{-11}, and 10^{-9}, respectively.

The calculations were carried out for 105, 104, 103, 85, and 127 systems, with linear size N varying from 30 to 2000, for the triangular, rhomboid, trapezoid, hexagonal, and rectangular triangular lattices, respectively.

III. RESULTS

A. Critical free energy density

By fitting the finite-size data, we find that the exact expansion of the critical free energy can be written in the following form, with k from 3 to 12 for the triangle and rectangle and from 3 to 9 for the other shapes:

$$f = f_\infty + f_{\rm surf}\frac{p(N)}{S} + \frac{f_{\rm corn}\ln N + f_2}{S} + \sum_{k=3}^{\infty}\frac{f_k + l_k \ln N}{S^{k/2}}, \quad (5)$$

where $p(N)$ is the perimeter, which is equal to $3N$, $4N$, $5N-1$, $6N$, and $(2+\sqrt{3})N$ for the triangle, rhombus, trapezoid, hexagon, and rectangle, respectively. In our fitting, the expansion is truncated to $k=12$ and $k=11$ for the triangle and rectangle,respectively. It is truncated to $k=9$ for the rhombus, trapezoid, and hexagon. This expansion is different from that for the triangular lattice with periodic boundary conditions [18], in which there are no surface or corner terms or logarithmic corrections, and only even k is presented.

The fitting method is the standard Levenberg-Marquardt method for nonlinear fits [28]. The standard deviation (SD) is defined by $\sigma = \sqrt{\sum_i (f_i - f_i^{(\rm fit)})^2/(n_d - n_f)}$ with f_i the numerical data, $f_i^{(\rm fit)}$ the value given by the fitting formula, n_d the number of data points used, and n_f the number of fitting parameters. For all cases, σ reaches 10^{-25}. The accuracy is seen from the fitted bulk free energy density f_∞: the worst fit among the five cases is for the rhombus, which yields $f_\infty = 0.879\,585\,386\,161\,571\,517\,093\,896\,2(7)$, and the best one is for the triangle, yielding $f_\infty = 0.879\,585\,386\,161\,571\,517\,093\,896\,05(3)$. These results coincide with the exact value $f_\infty = 0.879\,585\,386\,161\,571\,517\,093\,896\,028\,3\ldots$ [17] to more than 24 decimal places.

According to finite-size scaling, the surface correction term $f_{\rm surf}p(N)/S$ stems from free edges. As shown in Fig. 1(e), there are three bond directions for the triangular lattice. For the triangle, rhombus, trapezoid, and hexagon, every edge is parallel to a particular bond direction. The edge free energy per unit length on these edges should be equal. For the rectangle, there are two edges along one bond direction and two other edges perpendicular to that bond direction in a zigzag way. Therefore the edge free energy per unit length along the two different directions can be different in principle. To be clear, we denote the surface free energy per unit length (the straight length, not the total length of the zigzag line) along and perpendicular to the bond direction as $f_{\rm surf}^{\parallel}$ and $f_{\rm surf}^{\perp}$, respectively. The surface free energy is thus $f_{\rm surf}p(N) = f_{\rm surf}^{\parallel}p(N)$ for the triangle, rhombus, trapezoid, and hexagon, respectively. For the rectangle, it is $f_{\rm surf}P(N) = \bar{f}_{\rm surf}(2+\sqrt{3})N$, with $\bar{f}_{\rm surf} = (2f_{\rm surf}^{\parallel} + \sqrt{3}f_{\rm surf}^{\perp})/(2+\sqrt{3})$ the mean surface free energy per unit perimeter, considering that the straight length of the edges along the perpendicular direction is $\sqrt{3}N/2$, which is used to define the perimeter. The fitted values of $f_{\rm surf}$ are given in Table I. For the rectangular case, we obtain

$$f_{\rm surf}^{\parallel} = -0.103\,077\,638\,834\,090\,655\,334(2),$$
$$f_{\rm surf}^{\perp} = -0.116\,007\,497\,958\,656\,704\,304(2). \quad (6)$$

Cardy and Peschel showed in [3] that the presence of a corner of interior angle γ along a boundary of typical size N gives rise to a logarithmic correction to the critical free energy density,

$$f_{\rm corn}^{(\gamma)}\frac{\ln N}{S} = -\frac{c\gamma}{24\pi}\left[1-\left(\frac{\pi}{\gamma}\right)^2\right]\frac{\ln N}{S}, \quad (7)$$

TABLE I. The fitted edge and corner free energy and f_2 in Eq. (5).

Shape	$f_{\rm surf}$	$f_{\rm corn}$	f_2
Triangle	−0.103 077 638 834 090 655 334 3(2)	0.166 666 666 666 666 666 7(4)	0.006 804 832 446 685 952(4)
Rhombus	−0.103 077 638 834 090 655 334(2)	0.145 833 333 333 333 33(2)	0.183 972 833 468 758 7(1)
Trapezoid	−0.103 077 638 834 090 655 335(1)	0.145 833 333 333 333 35(2)	0.221 395 260 142 803 9(1)
Hexagon	−0.103 077 638 834 090 655 335(2)	0.104 166 666 666 666 68(3)	0.475 827 152 777 481 2(2)
Rectangle	−0.109 078 407 337 305 392 239(2)	0.125 000 000 000 000 001(2)	0.225 498 356 839 947 14(1)

where c is the conformal anomaly. The total corner free energy is

$$f_{\text{corn}} = \begin{cases} 3 f_{\text{corn}}^{(\pi/3)} = \frac{1}{6} = 0.166\,666\ldots & \text{for the triangle,} \\ 2\left(f_{\text{corn}}^{(\pi/3)} + f_{\text{corn}}^{(2\pi/3)}\right) = \frac{7}{48} = 0.145\,333\,3\ldots & \text{for the rhombus and trapezoid,} \\ 6 f_{\text{corn}}^{(2\pi/3)} = \frac{5}{48} = 0.104\,666\,6\ldots & \text{for the hexagon,} \\ 4 f_{\text{corn}}^{(\pi/2)} = \frac{1}{8} = 0.125 & \text{for the rectangle.} \end{cases}$$

The fitted corner free energies f_{corn} for the five shapes are listed in Table I, from which one can see that our results reproduce the CFT results very accurately.

In a previous study on the finite square lattice in a rectangular shape [27], we estimated the corner free energy $f_{\text{corn}} = 0.125 \pm 2.0^{-10}$ for rectangles with various aspect ratios. Here we find the same result $f_{\text{corn}} = 0.125 \pm 2.0 \times 10^{-18}$ on the rectangular triangular lattice. This indicates that the corner term of the free energy is independent of the microscopic properties of the lattice. Therefore we have proved the CFT prediction that the corner free energy is universal [3]. However, our calculations are for an aspect ratio fixed at $\rho = 2/\sqrt{3}$ in the present work. Thus our results are not sufficient to further verify Kleban and Vassileva's CFT predictions on the effect of the aspect ratio of the rectangle-shaped lattice [21].

For higher-order terms, we give fitted coefficients in Tables II and III. For the critical free energy of the triangle- and rectangle-shaped triangular lattices, there are no logarithmic corrections except for the corner term, i.e., all $l_k = 0$. For the triangle, the coefficient f_4 is determined to be zero since it is extremely small in our fits if included, and the fitting result changes little if we discard it. In contrast to the triangles and rectangles, for the other three shapes of triangular lattices, we find higher-order ($k \geqslant 5$) logarithmic corrections in addition to the corner term. Although these logarithmic corrections are very weak, with very small coefficients, our very accurate data indicate their existence.

TABLE II. The fitted parameters for order $k \geqslant 3$ in Eq. (5) for the critical free energy of the triangle- and rectangle-shaped triangular lattices. There are no logarithmic corrections except for the corner term, i.e., all $l_k = 0$.

Shape	Triangle	Rectangle
f_3	0.117 851 130 197 757 922 7(7)	−0.014 495 934 065 064 5(9)
f_4	0	0.018 068 370 551 1(1)
f_5	−0.002 455 231 879 142(5)	−0.012 371 010 66(1)
f_6	0.001 247 090 584 7(6)	0.016 864 148 9(1)
f_7	−0.001 625 545 87(4)	0.003 027 43(4)
f_8	0.001 247 103(1)	0.001 335(1)
f_9	−0.000 561 82(3)	−0.031 33(3)
f_{10}	−0.000 583 5(5)	0.0701(4)
f_{11}	0.001 968(5)	−0.161(2)
f_{12}	−0.002 26(2)	

B. Critical internal energy density

We fit the data on the critical internal energy with the following formula:

$$u = u_\infty + u_{\text{surf}} \frac{p(N) \ln N}{S} + u_{\text{corn}} \frac{\ln N}{S} + \sum_{k=1}^{\infty} \frac{u_k}{S^{k/2}}, \tag{8}$$

where $p(N)$ is again the perimeter. In our fits, the expansion is truncated to $k = 4$. Again, this formula is different from that for the triangular lattice with periodic boundary conditions [18], in which there are no logarithmic surface term, no corner term, and only odd k is presented. The bulk value u_∞ is known to be 2 [17]. Our fit of u_∞ is $2.0 \pm 1.0 \times 10^{-10}$ for the five shapes. The other fitted parameters are given in Table IV.

The leading correction is due to the edges (or surface), which has not been determined to our knowledge. We denote the surface internal energy per unit length of the edge along one bond direction by $u_{\text{surf}}^{\parallel}$, and that perpendicular to that direction by $u_{\text{surf}}^{\perp}$. For the triangle, rhombus, trapezoid, and hexagon, we have $u_{\text{surf}} = u_{\text{surf}}^{\parallel}$. For the rectangle we have $u_{\text{surf}} = (2u_{\text{surf}}^{\parallel} + \sqrt{3}u_{\text{surf}}^{\perp})/(2 + \sqrt{3})$, and

$$u_{\text{surf}}^{\parallel} = -0.551\,328\,89(2), \quad u_{\text{surf}}^{\perp} = -0.551\,328\,95(7). \tag{9}$$

This is an interesting result because it means that the surface internal energy per unit length is symmetric for an edge along or perpendicular to an arbitrary bond direction. Comparing this result with u_{surf} for the rectangle-shaped square lattice in our previous work [27], where $u_{\text{surf}} = 0.636\,619\,8(1)$, we find that the ratio is 1.154 700 5, which is very close to $2/\sqrt{3} = 1.154\,700\,538\ldots$. Because the exact value of u_{surf} for the square lattice is $2/\pi$ [11], we conjecture that the exact value of the surface internal energy for the triangular lattice is given by

$$u_{\text{surf}}^{\parallel} = u_{\text{surf}}^{\perp} = -\sqrt{3}/\pi. \tag{10}$$

Following the convention for the critical free energy, we write the coefficient of $(\ln N)/S$ as u_{corn}. We denote the corner correction by $u_{\text{corn}}^{(\gamma)}$, where γ is the angle of the corner. Under the assumption that u_{corn} is the sum of the corners, contributions, we have

$$u_{\text{corn}} = \begin{cases} 3u_{\text{corn}}^{(\pi/3)} & \text{for the triangle,} \\ 2\left(u_{\text{corn}}^{(\pi/3)} + u_{\text{corn}}^{(2\pi/3)}\right) & \text{for the rhombus and trapezoid,} \\ 6u_{\text{corn}}^{(2\pi/3)} & \text{for the hexagon,} \\ 4u_{\text{corn}}^{(\pi/2)} & \text{for the rectangle.} \end{cases}$$

TABLE III. The fitted parameters of Eq. (5) for the critical free energy. The third- and fourth-order logarithmic corrections l_3 and l_4 are zero.

Shape	Rhombus	Trapezoid	Hexagon
f_3	0.048 611 111 111 12(2)	−0.002 106 928 657 56(2)	−0.060 140 653 040 55(7)
f_4	−0.008 101 851 86(2)	0.013 632 817 06(2)	0.034 722 222 17(7)
f_5	0.003 811 44(6)	−0.006 861 68(6)	−0.018 097 6(4)
f_6	−0.002 15(1)	−0.008 94(1)0	0.0150(1)
f_7	−0.0008(5)	0.0316(6)	−0.018(7)
f_8	−0.026(4)	−0.113(5)	−0.56(9)
f_9	0.032(2)	0.097(3)	0.21(3)
l_5	0.003 463 39(1)	0.004 948 74(1)	0.009 997 97(6)
l_6	−0.003 461(2)	−0.008 767(3)	−0.008 63(2)
l_7	0.0024(1)	0.0117(2)	0.011(2)
l_8	0.006(2)	0.011(3)	0.17(5)
l_9	0.007(5)	0.070(8)	0.9(2)

From Table IV, we obtain the corner terms for the three angles:

$$\begin{aligned} u_{\rm corn}^{(\pi/3)} &= -0.551\,332(1), \\ u_{\rm corn}^{(\pi/2)} &= 0.036\,927(2), \\ u_{\rm corn}^{(2\pi/3)} &= 0.183\,781(3). \end{aligned} \tag{11}$$

In the previous study on the square lattice [27], we obtained the corner term $u_{\rm corn} = -0.4502(1)$ for rectangles with various aspect ratios. It is different from the present result $u_{\rm corn} = 0.147\,707(6)$ for the rectangle-shaped triangular lattice, which indicates that the corner term of the internal energy depends on the microscopic structure of the lattice, and thus is not universal.

The other parameters u_1, u_2, u_3, and u_4 are also estimated and listed in Table IV. We have tried other forms of formula to fit the critical internal energy. The coefficients of the possible logarithmic corrections $\ln S/S^{k/2}$ with $k \geqslant 3$ are extremely small. However, we cannot exclude these terms definitely based on the current data. More accurate data are needed to settle this issue.

C. Critical specific heat

The data for the critical specific heat are fitted using the following formula:

$$c = A_0 \ln N + c_0 + c_{\rm surf}\frac{p(N)\ln N}{S} + c_{\rm corn}\frac{\ln N}{S} + \sum_{k=1}^{\infty}\frac{c_k}{S^{k/2}}, \tag{12}$$

where $p(N)$ is the perimeter. In our fits, the expansion is truncated to $k = 4$. Compared with the expansion for the triangular lattice with periodic boundary conditions [18], there are additional logarithmic surface and corner terms.

The leading term $A_0 \ln N$ is known from the exact result [18], which reads $A_0 = \frac{3\sqrt{3}}{4\pi}(\ln 3)^2 \approx 0.499\,069\,378\,0\ldots$. Our fit yields $A_0 \approx 0.499\,069\,374(5)$. The other fitted parameters are listed in Table V.

The leading correction $p(N)\ln N/S$ is caused by the edges. We denote the surface specific heat per unit length of the edge along one bond direction by $c_{\rm surf}^{\parallel}$, and that perpendicular to this direction by $c_{\rm surf}^{\perp}$. For the triangle, rhombus, trapezoid, and hexagon, we have $c_{\rm surf} = c_{\rm surf}^{\parallel}$; for the rectangle, we set $c_{\rm surf} = (2c_{\rm surf}^{\parallel} + \sqrt{3}c_{\rm surf}^{\perp})/(2+\sqrt{3})$, and find

$$c_{\rm surf}^{\parallel} = 0.166\,354(2), \quad c_{\rm surf}^{\perp} = 0.105\,462(2). \tag{13}$$

Note that this term is absent in the torus case [8] and not mentioned in the long strip case [11], but exists in the cylindical case with Brascamp-Kunz boundary conditions [14,15].

Following the convention for the critical free energy, we write the coefficient of $(\ln N)/S$ as $c_{\rm corn}$. We denote the corner correction by $c_{\rm corn}^{(\gamma)}$ where γ is the angle of the corner. Again, under the assumption that the total correction is the sum of the corners, we have

$$c_{\rm corn} = \begin{cases} 3c_{\rm corn}^{(\pi/3)} & \text{for the triangle,} \\ 2\left(c_{\rm corn}^{(\pi/3)} + c_{\rm corn}^{(2\pi/3)}\right) & \text{for the rhombus and trapezoid,} \\ 6c_{\rm corn}^{(2\pi/3)} & \text{for the hexagon,} \\ 4c_{\rm corn}^{(\pi/2)} & \text{for the rectangle.} \end{cases}$$

TABLE IV. The fitted parameters of Eq. (8) for the critical internal energy per spin.

Shape	Triangle	Rhombus	Trapezoid	Hexagon	Rectangle
$u_{\rm surf}$	−0.551 328 91(2)	−0.551 328 867(8)	−0.551 328 88(1)	−0.551 328 91(2)	−0.551 329 0(7)
$u_{\rm corn}$	−1.653 995(3)	−0.735 055(7)	−0.735 06(1)	1.102 68(2)	0.147 707(6)
u_1	0.135 813 1(3)	−0.728 608 0(2)	−1.455 311 0(3)	−3.449 839 2(7)	−1.124 622 4(2)
u_2	−1.420 06(1)	−1.093 59(3)	−0.554 00(4)	1.710 14(4)	−0.954 85(2)
u_3	−0.702 91(8)	−0.2154(3)	0.0302(4)	−0.372(1)	0.1049(2)
u_4	−0.0243(4)	0.113(2)	0.044(4)	0.49(1)	−0.030(2)

TABLE V. The fitted parameters of Eq. (12) for the critical specific heat of the five shapes.

Shape	Triangle	Rhombus	Trapezoid	Hexagon	Rectangle
c_0	−0.804 242 37(1)	−0.605 103 31(1)	−0.519 950 11(1)	−0.267 339 5(1)	−0.573 388 895(9)
c_{surf}	0.166 355 8(3)	0.166 353 1(5)	0.166 352 0(5)	0.166 338(9)	0.138 094 1(8)
c_1	−0.187 862(5)	−0.036 692(3)	−0.103 91(2)	−0.2749(2)	−0.085 29(9)
c_{corn}	0.7484(1)	0.3864(2)	0.3857(3)	−0.343(6)	−0.1510(2)
c_2	0.6410(4)	0.6443(8)	0.615(1)	0.58(2)	0.6833(7)
c_3	0.339(2)	0.177(6)	0.04(1)	−0.09(3)	−0.057(5)
c_4	0.024(8)	−0.12(4)	−0.00(7)	1.8(3)	0.17(3)

From Table IV, we obtain the corner contributions of the three angles,

$$\begin{aligned} c_{\text{corn}}^{(\pi/3)} &= 0.249\,48(3), \\ c_{\text{corn}}^{(\pi/2)} &= -0.037\,75(5), \\ c_{\text{corn}}^{(2\pi/3)} &= -0.057(1). \end{aligned} \tag{14}$$

In the previous study on the square lattice [27], we obtained the corner term $c_{\text{corn}} = 0.368(1)$ for rectangles with various aspect ratios. It is different from the present result $c_{\text{corn}} = -0.1510(2)$ for the rectangle-shaped triangular lattice, which indicates that the corner term of the specific heat depends on the microscopic structure of the lattice, and thus is not universal.

We have also tried other forms of fitting formula to fit the critical specific heat data. The coefficients of the possible logarithmic corrections $\ln S/S^{k/2}$ with $k \geqslant 3$ are extremely small. However, we cannot exclude these terms definitely based on the current data. More accurate data are needed to settle this issue.

IV. CONCLUSION

Using the BP algorithm, we have studied the 2D critical Ising model on a triangular lattice with free boundaries. For five shapes, triangular, rhomboid, trapezoid, hexagonal, and rectangular, the critical free energy, internal energy, and specific heat have been calculated. We have proved the conformal field theory prediction about the corner free energy and have shown that the corner free energy, which is proportional to the central charge c, is indeed universal. For the edges parallel or perpendicular to the bond direction, the logarithmic edge corrections in the internal energy have been found to be almost identical, while these corrections in the free energy and in the specific heat have been found to be different. Comparing with the previous result on the square lattice in the shape of a rectangle, we have found that the corner internal energy u_{corn} and the corner specific heat c_{corn} for the rectangle are not universal, i.e., the coefficients in front of $\ln N/S$ in the expansion of the internal energy and the specific heat are different for the square and the triangular lattices.

We have also found that there exist logarithmic corrections in higher orders, say, there are terms $\ln N/S^{5/2}, \ln N/S^3, \ldots$ in the critical free energy for the rhombus, trapezoid, and hexagon. However, these terms are absent for the triangle and rectangle. This should be an interesting subject for further investigation.

The BP algorithm can be applied to study some different boundaries. The BP algorithm for a cylinder was also proposed in Ref. [26], so a lattice with periodic conditions in a particular direction can be studied. Moreover the BP algorithm can be applied to a lattice with a hole. The effects of a hole on the free energy, internal energy, and specific heat should be interesting. They must depend strongly on the position of the hole. For example, the change of free energy induced by a hole at the corner should be very different from that induced by a hole at the center. Further investigation is needed on these issues.

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