Dynamical properties of random-field Ising model

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Extensive Monte Carlo simulations are performed on a two-dimensional random field Ising model. The purpose of the present work is to study the disorder-induced changes in the properties of disordered spin systems. The time evolution of the domain growth, the order parameter, and the spin-spin correlation functions are studied in the nonequilibrium regime. The dynamical evolution of the order parameter and the domain growth shows a power law scaling with disorder-dependent exponents. It is observed that for weak random fields, the two-dimensional random field Ising model possesses long-range order. Except for weak disorder, exchange interaction never wins over pinning interaction to establish long-range order in the system.

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I. INTRODUCTION

The random field Ising model (RFIM) belongs to a class of disordered spin models in which the disorder is coupled to the order parameter of the system. Much has been studied on various aspects of RFIM since Imry and Ma [1] introduced this model. It has experimental realizations in a diluted antiferromagnet [2]. Its Hamiltonian differs from that of a normal Ising model by the addition of a local random field term, which results in a drastic change of its behavior in both equilibrium and nonequilibrium situations. In one dimension ($d = 1$), the RFIM does not order at all [3]. Imry and Ma argued that the random fields assigned to spins change the lower critical dimension from $d_l = 1$ (pure case) to $d_l = 2$. Later a number of field theoretical calculations [4] suggested that $d_l = 3$. Finally, in 1987 came the exact results by Bricmont and Kupiainen [5], which showed that there is a ferromagnetic phase in three dimensions. In 1989 Aizenman and Wehr [6] provided us with a rigorous proof that there is no ferromagnetic phase in two-dimensional (2D) RFIM and thus $d_l = 2$. This means that the ground state is paramagnetic. However, in 1999 Frontera and Vives [7] had shown numerical signs of a transition in the 2D RFIM at $T = 0$ below a critical random field strength. They explained in their paper that the proof by Aizenman and Wehr cannot be misunderstood as a proof that ordered phase cannot exist. We mention in passing that Aizenman, in his recent seminars, claims that the 2D RFIM exhibits a phase transition in the disorder parameter [8]. Recently Spasojevic *et al*. [9] gave numerical evidence that the 2D nonequilibrium zero-temperature RFIM exhibits a critical behavior. The Hamiltonian for such a system is, in general, given by

$$H = -J \sum_{\langle ij \rangle} s_i s_j + \sum_i \eta_i s_i + H_{\mathtt{ext}} \sum_i s_i, \qquad (1)$$

where J is the coupling constant, conventionally set to unity in the present work. η_i is the quenched random field, and $H_{\mathtt{ext}}$ is the external magnetic field. In the present work the external magnetic field $H_{\mathtt{ext}}$ is set to zero.

The local static random fields give rise to many local minima of the free energy and the complexity of the random field free energy also gives rise to long relaxation times as the system lingers in a succession of local minima on its way to the lowest energy state. For the zero field Ising model in two or more dimensions below the critical temperature T_c, there form domains of moderate size in the early time regime, in which all the spins are either up or down and as time progresses the smallest of these domains shrink and vanish, closely followed by the next smallest, until eventually most of the spins on the lattice are pointing in the same direction. The reason for this behavior is that the domains of spins possess a surface energy, having a domain wall cost an energy which increases with the length of the wall, because the spins on either side of the wall are pointing in opposite directions. The system can therefore lower its energy by flipping the spins around the edge of a domain to make it smaller. Thus the domains "evaporate," leaving most of the spins either up or down.

However, the story is different for RFIM. In the RFIM, domains still form in the ferromagnetic regime, and there is still a surface energy associated with the domain walls, but it is no longer always possible to shrink the domains to reduce this energy. The random field acting on each spin in the RFIM means that it has a preferred direction. Furthermore, at some sites there will be a very large local field η_i pointing in one direction, say, up direction, which means that the corresponding spin will really want to point up, and it will cost the system a great deal of energy if it is pointing down. The acceptance ratio for flipping this spin contains a factor $\exp(-2\beta|\eta_i|)$, which is a very small number if $|\eta_i|$ is large. It is said that the domain wall is pinned by the local field; i.e., it is prevented from moving by an energy barrier produced by the large random field. If the domains eventually stop growing, the system will be in a disordered phase (although the domain size may be very large). This describes the $d = 2$ RFIM.

All these features can be visualized from some snapshots presented in Fig. 1. In the early time regime, domains begin to form and grow in size [Figs. 1(a)–1(d)]. This is seen in

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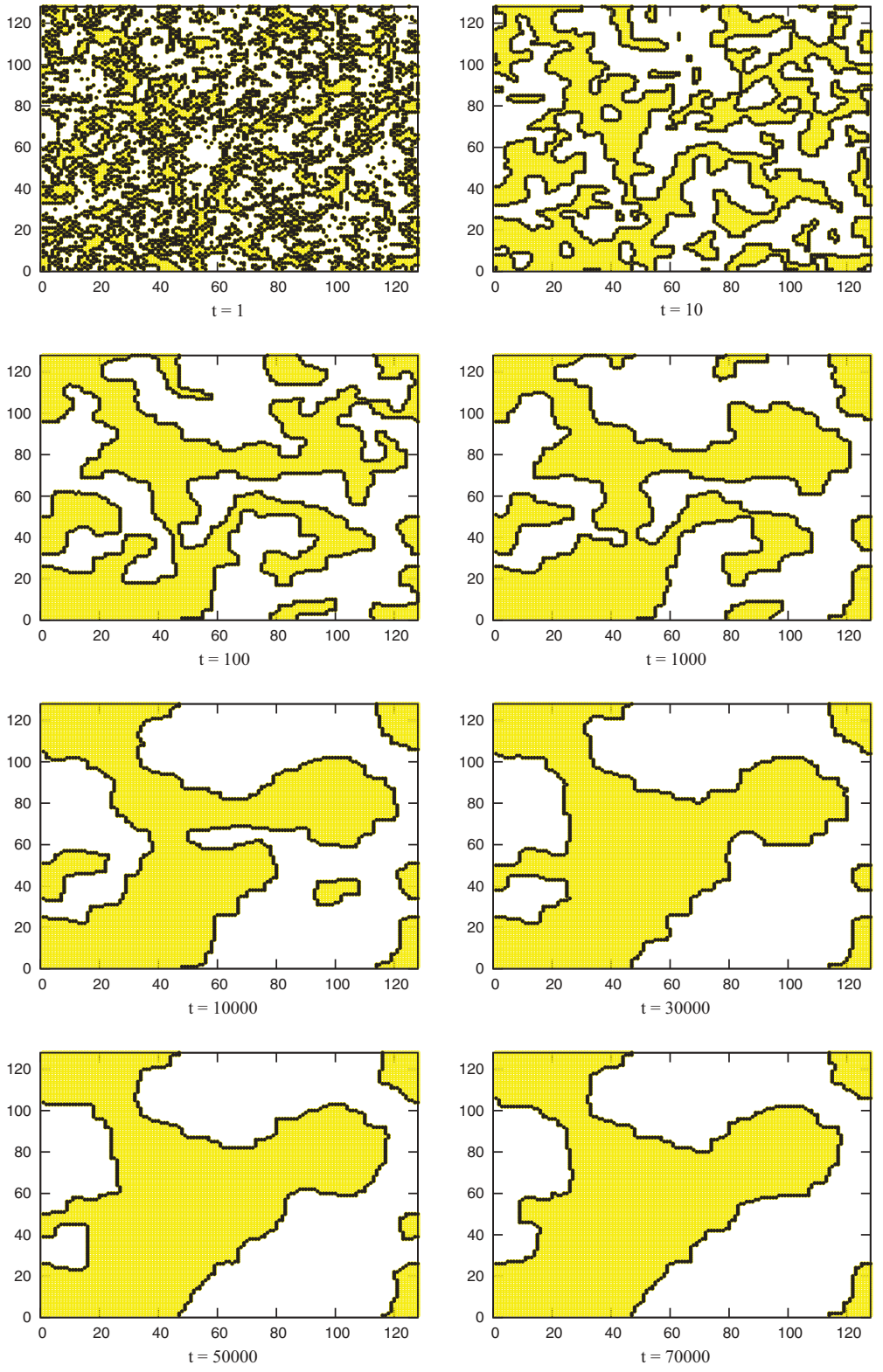

FIG. 1. (Color online) Time evolution of domains for a 128 × 128 system at $T = 0.50$. The strength of disorder is $\eta_0 = 1.0$. The yellow (gray) regions indicate domains of up spins.

a pure system too. As time flows, the domains stop growing and the domain wall is pinned [Figs. 1(e)–1(h)] even at a low temperature. The introduction of a local static random field term changes the behavior of the system completely in a later time regime and renders the system in a disordered phase even at a very low temperature.

These disordered spin systems have been an active area of research for quite some time now [10–16]. The purpose of the

present work is to study the disorder-induced changes in the properties of these systems, with emphasis on the dynamical evolution of the domain growth, the order parameter, and the spin-spin correlation functions in the nonequilibrium region.

In most of the studies on RFIM, the domain size (or the cluster size) was determined in terms of the fluctuations of the magnetization [17–21]. The fluctuations in magnetization are only a measure of domain size, not the actual domain size. In this work we have measured the actual domain size by the Hoshen-Kopelman algorithm [22]. The method of determining cluster size by the Hoshen-Kopelman algorithm allows us to make a more accurate study of its growth with time. Moreover, there are many ways in which the random fields could be chosen. In most of the studies, they were chosen to have a Gaussian distribution with some finite width σ, or to have values randomly $\pm h$ where h is a constant. However, the interesting properties of RFIM are believed to be independent of the exact choice of the distribution, which is a consequence of the phenomenon of universality [23]. In the present work, the random fields are chosen from a uniform distribution of varying strengths.

The rest of the paper is arranged as follows. Section II discusses the computational details and gives the definition of the thermodynamic quantities of our interest. Section III presents the results in detail. Finally in Sec. IV we summarize our results.

II. THE MODEL AND ITS SIMULATION

The first term of the Hamiltonian (1) is the usual exchange interaction term, while the second term represents the interaction of the random field with the spin. We call this term the pinning interaction term since this term is responsible for domain wall pinning. The simulations are performed on square lattice of sizes ranging from 32×32 to 512×512 and at a finite temperature $T = 0.50$, which is well below the critical temperature of a 2D zero field Ising model. The temperature is taken sufficiently low to reduce the thermal fluctuations. Properties of RFIM depend on the competition between the random fields and the ferromagnetic couplings with the thermal fluctuations serving only to renormalize the strengths of these couplings [24]. We have chosen the Metropolis algorithm [25] to simulate the system. The Metropolis algorithm is suitable here because the dynamics is local. Being a single spin flip dynamics, the Metropolis algorithm is believed to represent the natural way of evolution of a system, since the acceptance ratio is given by the Boltzmann probability. Periodic boundary conditions are used in the simulations.

The sizes of the domains (or clusters) are determined by Hoshen-Kopelman (HK) algorithm [22]. The general idea of the HK algorithm is that we scan through the lattice looking for up spins (or down spins). To each up spin (or down spin) we assign a label corresponding to the cluster to which the up spin (or the down spin) belongs. If the up spin (or the down spin) has zero neighbors of same sign, then we assign to it a cluster label we have not yet used (it is a new cluster). If the up spin (or the down spin) has one neighbor of same sign, then we assign to the current spin the same label as the previous spin (they are part of the same cluster). If the up spin (or the down spin) has more than one neighbor of same sign, then we choose the lowest-numbered cluster label of the up spins (or the down spins) to use the label for the current spin. Furthermore, if these neighboring spins have different labels, we must make a note that these different labels correspond to the same cluster. The HK algorithm is a very efficient cluster identification method for two-dimensional systems. The domain size corresponds to the number of spins enclosed by the boundary of a domain.

The time-dependent ensemble average of magnetization, i.e., the order parameter, has been defined as

$$M(\eta_0,t) = \left\langle \frac{1}{L^2} \left| \sum_i s_i \right| \right\rangle_t, \tag{2}$$

where L is the linear size of the system and s_i is the spin at the site i. The angular bracket $\langle \cdots \rangle_t$ indicates the ensemble average at time t.

The time-dependent ensemble average of the spin-spin correlation function is defined as

$$\psi_{ss}(\eta_0,l,t) = \langle \langle s_i s_{i+l} \rangle_l \rangle_t, \tag{3}$$

where l is the distance of separation between the spins. The angular bracket $\langle \cdots \rangle_l$ indicates the ensemble average over the distance of separation between the spins, while $\langle \cdots \rangle_t$ indicates the same over time t.

The thermodynamic quantities of our interest are averaged over 50 independent simulations to improve the accuracy and the quality of the results. The simulations start with a random spin configurations, characteristic of a high-temperature phase and then quenched to a low temperature.

III. RESULTS AND DISCUSSIONS

The dynamic evolution of the order parameter defined in Eq. (2) for different disorder strengths is shown in Fig. 2. It is seen from Fig. 2 that for weak disorder ($\eta_0 \leqslant 0.3$) the system reaches in a steady state after a certain time (here steady state means the fluctuations in order parameter is very small with time). For weak disorder, the system behaves more or less like a pure system. For larger disorder, the system takes longer time to reach in a steady state; i.e., the system relaxes, if at all,

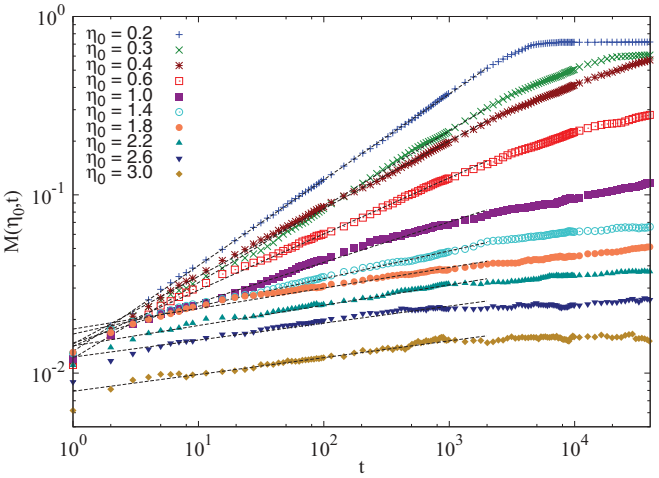

FIG. 2. (Color online) Plot of the time evolution of order parameter for $L = 256$. The dashed lines represent the best linear fits according to the scaling law defined in Eq. (4).

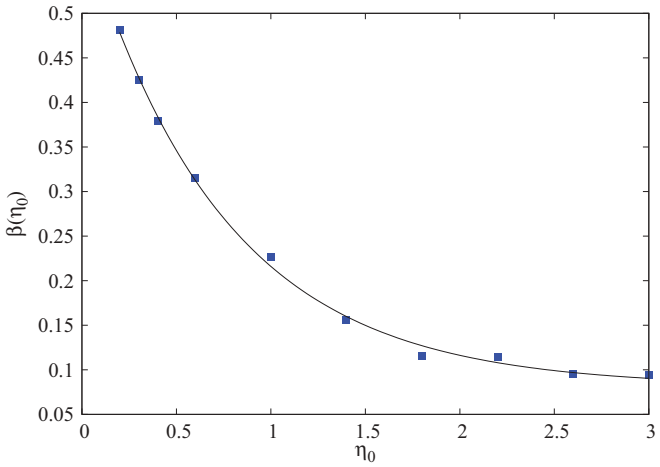

FIG. 3. (Color online) Variation of the exponent $\beta(\eta_0)$ with η_0. The solid line indicates the best exponential fit to the data points.

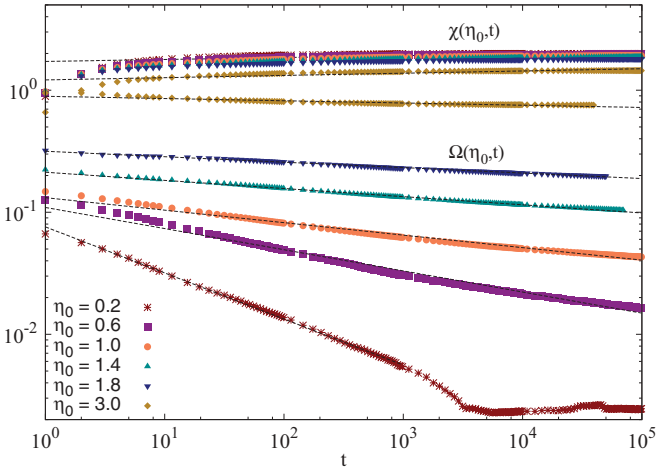

FIG. 4. (Color online) The time evolution of the exchange interaction $\chi(\eta_0,t)$ and the pinning interaction $\Omega(\eta_0,t)$ as defined in Eq. (5) and Eq. (6), respectively, for $L = 256$.

very slowly. The nature of the evolution of the order parameter strongly depends on the strength of the disorder. The early time behavior of the dynamic evolution of the order parameter can be characterized by the following power law behavior:

$$M(\eta_0,t) \sim t^{\beta(\eta_0)}, \tag{4}$$

where $\beta(\eta_0)$ is a disorder-strength-dependent exponent corresponding to the growth of the order parameter. The dependence of the power law exponent $\beta(\eta_0)$ on the disorder strength η_0 is shown in Fig. 3.

The exponent $\beta(\eta_0)$ falls off exponentially with the strength of the disorder η_0 as $\sim \exp(-\nu\eta_0)$ with $\nu = 1.35 \pm 0.07$. This is quite obvious. Two types of interactions, namely, the exchange interaction and the pining interaction, are present in the system. For larger disorder, spin flips are not favored due to the presence of the pinning interaction term, and consequently the domain walls get pinned, leaving the system in a disordered phase. As a result, the exponent $\beta(\eta_0)$ falls off sharply with η_0.

What happens is that there is always a competition between the exchange interaction and the pining interaction. Therefore it is interesting to observe the time evolution of the time-dependent ensemble average of the exchange interaction defined as

$$\chi(\eta_0,t) = \left\langle \sum_{\langle ij \rangle} s_i s_j \right\rangle_t \tag{5}$$

as well as the time evolution of the time-dependent ensemble average of the normalized pinning interaction defined as

$$\Omega(\eta_0,t) = \left\langle \frac{1}{\eta_0} \langle s_i \eta_i \rangle \right\rangle_t, \tag{6}$$

where the angular bracket $\langle \cdots \rangle_t$ indicates the ensemble average at time t. Figure 4 shows the dynamic evolution of the $\chi(\eta_0,t)$ and the $\Omega(\eta_0,t)$, respectively, and it reveals a striking feature. It is seen that the exchange interaction $\chi(\eta_0,t)$ remains almost same with time except at some initial time steps, whereas the pinning interaction $\Omega(\eta_0,t)$ decays more rapidly except for very large disorder strength. The system evolves with time in such a way that the pining interaction gets minimized. We would like to point out that this behavior is characteristic to the RFIM and was not observed earlier to the best of our knowledge. For weak disorder, the system achieves the steady state with the flow of time resulting in the saturation of the pining interaction and the exchange interaction as well. For disorder strengths lying in the intermediate range, the decay of pining interaction strongly depends on the strength of the disorder. Now, a natural question that arises is that how rapidly does the pining interaction $\Omega(\eta_0,t)$ decay or how slowly does the exchange interaction $\chi(\eta_0,t)$ increase with the strength of the random field η_0. According to the nature of the dynamic behavior seen in Fig. 4, the power law dependence of the $\chi(\eta_0,t)$ and the $\Omega(\eta_0,t)$ with time has been proposed as

$$\chi(\eta_0,t) \sim t^{\alpha_\chi(\eta_0)}, \quad \Omega(\eta_0,t) \sim t^{-\alpha_\Omega(\eta_0)}, \tag{7}$$

where $\alpha_\chi(\eta_0)$ and $\alpha_\Omega(\eta_0)$ are two characteristic exponents which determine the nature of growth of the exchange interaction and the pining interaction. The variation of these exponents with η_0 is shown in Fig. 5. The values of the exponent $\alpha_\chi(\eta_0)$ are very small, and its variation with the strength of the disorder η_0 is negligible, which indicates that $\alpha_\chi(\eta_0)$ is almost independent of η_0. On the other hand, the variation of the exponent $\alpha_\Omega(\eta_0)$ with η_0 is noticeable, and $\alpha_\Omega(\eta_0)$ decreases rapidly with η_0 and tends to saturate for very large values of η_0. These results confirm our earlier observation on the time evolution of the exchange interaction and the pinning interaction.

Next we focus our attention on the study of spin-spin correlation functions defined by Eq. (3). The spin-spin correlation functions $\psi_{ss}(\eta_0,l)$ for different strengths of disorder at different times t are shown in Fig. 6. For weak disorder, when the system is quenched from a high-temperature phase to a low-temperature one, long-range order is seen to develop at a later time regime. With decreasing strength of the random fields, the ferromagnetic couplings start to dominate over the random fields, the domains of parallel spins become larger, and the system is in a ferromagnetic regime. This observation is in agreement with Refs. [26,27], which shows that there is a critical field strength at which the correlation length

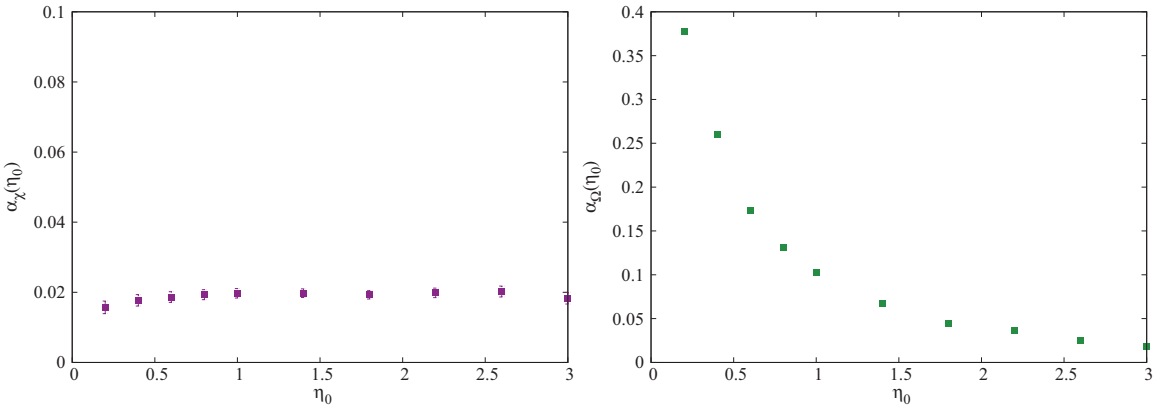

FIG. 5. (Color online) Variation of $\alpha_\chi(\eta_0)$ and $\alpha_\Omega(\eta_0)$ with η_0.

becomes divergent. This observation also supports the earlier findings [7] of a phase transition in the 2D RFIM at $T = 0$. Here the thermal fluctuations due to finite temperature serves only to renormalize the strengths of the random fields and the ferromagnetic couplings. For weak disorder, the pinning interaction decays faster with time and then saturates (see Fig. 4), which also implies presence of long-range order at later time regime. This is why we obtained a large value of the exponents $\beta(\eta_0)$ and $\alpha_\Omega(\eta_0)$ for weak disorder strengths. We would also like to point out here that with the increase in system size, it takes longer time in order for long-range order to be established. To check that the presence of long-range order for weak disorder is not a result of finite size effect, we have plotted the order parameter against time (the number of Monte Carlo steps) for various system sizes, which is shown in Fig. 7. It is evident from Fig. 7 that the number of Monte Carlo steps (t_x) required to achieve the long-range order for weak disorder increases with the increase in system size. The inset of Fig. 7 shows the plot of $\ln(t_x)$ against $\ln(L)$. It may also be noted that for weak disorder, the nature of spin-spin correlation functions changes from exponential decay to power law decay at late time stage, which suggests the possibility of existence of long-range order. This observation is in agreement with the recent observation of Aizenman [8].

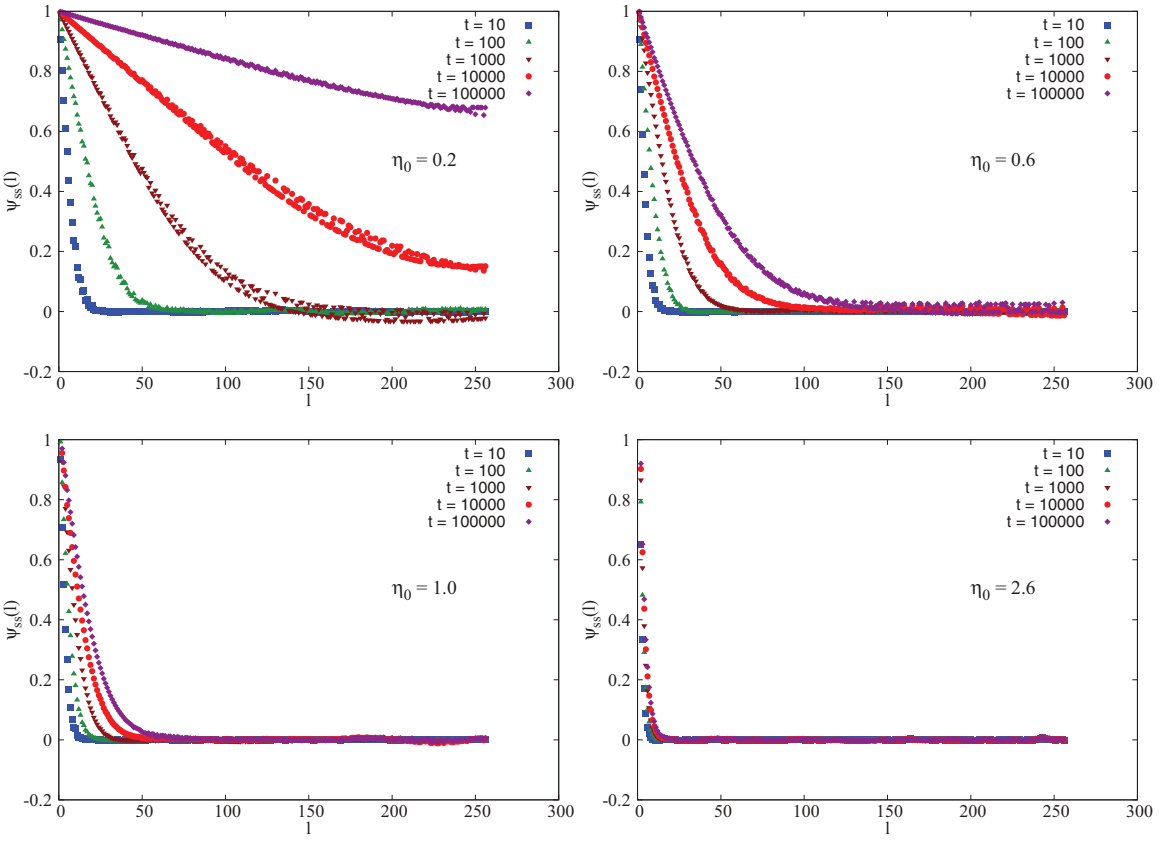

FIG. 6. (Color online) Plot of $\psi_{ss}(\eta_0,l)$ against l for different disorder strengths at different times for $L = 512$.

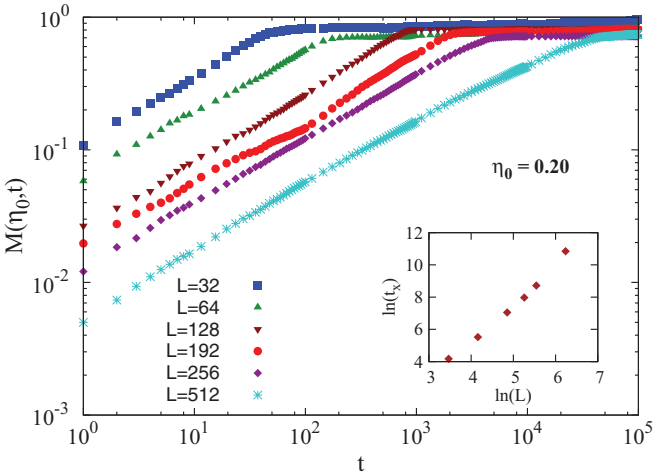

FIG. 7. (Color online) Plot of the order parameter against time for various system sizes at $\eta_0 = 0.20$.

For strengths of disorder lying in the intermediate range, no long-range order is seen to develop, and short-range order is being developed with time in the system. Except for weak random fields, exchange interaction never wins over pinning interaction to establish long-range order in the system. Due to this short-range order, there form small domains in the system initially, and as time elapses, these small domains "evaporate" to form a large domain. However, the presence of random fields prevents the system from growing into a single domain even at a low temperature. For very large disorder ($\geqslant$2.6), neither short-range nor long-range order prevails in the system, and the system remains in a complete disordered phase. There is always a competition between the ferromagnetic nearest neighbor interaction J (which favors ordering) and the random field strength η_0 (which favours disordering). In the limit of strong random fields, the direction of spins follows the direction of random fields.

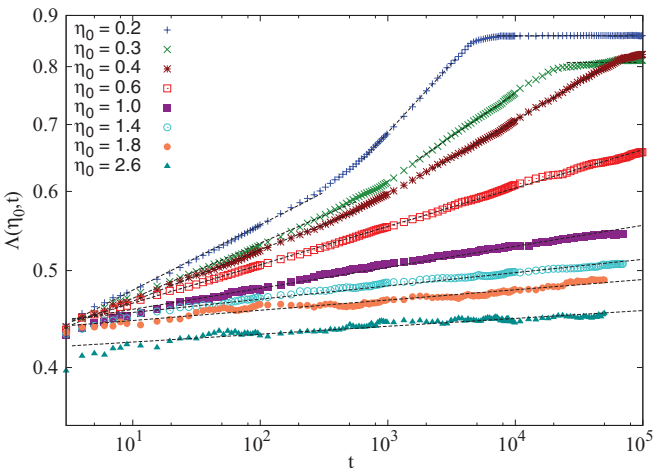

FIG. 8. (Color online) Plot of the growth of the largest domain of the system with time for $L = 256$. The dotted lines represent the best linear fit to the data points.

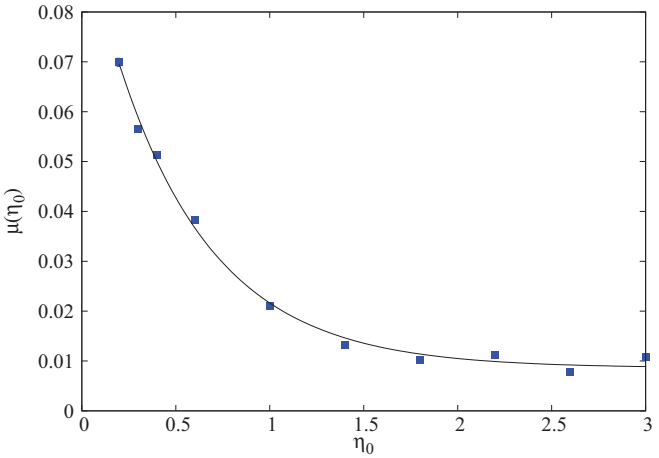

FIG. 9. (Color online) Plot of the exponents $\mu(\eta_0)$ with disorder strength η_0.

Now we present the results of the domain growth with time. Domain growth in quenched nonequilibrium systems is a widely studied topic [28]. Monte Carlo simulations have been carried out [29–32] to obtain numerically the time evolution of these domains, and to compare the results with theory [33–36]. We concentrate on the growth of the largest domain, and it is plotted in Fig. 8. The growth of the largest domain can be characterized by the following power law behavior :

For $\eta_0 \leqslant 0.3$

$$\begin{aligned} \Lambda(\eta_0,t) &= t^{\mu(\eta_0)} && t \ll t_{\times_1}, \\ \Lambda(\eta_0,t) &= t^{\nu(\eta_0)} && t_{\times_1} \ll t \ll t_{\times_2}, \\ \Lambda(\eta_0,t) &= \eta_0^{\sigma} && t \gg t_{\times_2}. \end{aligned} \tag{8}$$

For $\eta_0 > 0.3$

$$\Lambda(\eta_0,t) = t^{\mu(\eta_0)}, \tag{9}$$

where $\mu(\eta_0)$ is a disorder-strength-dependent exponent corresponding to the growth of the largest domain at the early time regime. Recently, Corberi *et al.* [37] also found that the domain growth shows a power law scaling with a disorder-dependent exponent in a preasymptotic regime. The variation of the exponent $\mu(\eta_0)$ with η_0 is shown in Fig. 9. The exponent $\mu(\eta_0)$ falls off exponentially with the strength of disorder η_0 as $\sim \exp(-\gamma\eta_0)$ with $\gamma = 1.92 \pm 0.14$. It is to be noted that although the order parameter exponent and the domain growth exponent fall off exponentially with the strength of disorder, the values of the exponents are different, and the fall of $\mu(\eta_0)$ is faster than that of $\beta(\eta_0)$.

IV. SUMMARY AND CONCLUSION

We conclude the paper with a summary of our results. This paper has attempted to consider some aspects of the nonequilibrium behavior of the 2D random-field Ising model numerically at a low temperature. As seen in the preceding sections, the RFIM exhibits a variety of behaviors depending on the strength of the random fields. The system relaxes with time in presence of two opposite kind of interactions, namely, the exchange interaction and the pinning interaction, and we have studied the dynamical evolution of these two interactions

separately. It is seen that the fall of pinning interaction depends on the strength of the random fields with a power law decay, and it decays faster for weak random fields. Therefore the dynamical evolution of the order parameter should also depend on the strength of the random fields with a power law growth, as has been observed. To get an insight of what is happening inside the system, we have calculated the dynamical spin-spin correlation functions. For weak disorder, the pinning interaction decays faster, and consequently the disordering effect reduces. As a result, the system is being correlated with time for weak disorder. Our numerical study suggests the possibility of presence of long-range order in the 2D 2*d* RFIM for weak disorder strengths. We are inclined to comment that the 2D RFIM exhibits a phase transition in disorder parameter even at a temperature $T > 0$. The transition is manifested by a change of nature of spin-spin correlation functions from an exponential decay at high disorder strengths to a power law decay at weak disorder strengths. The thermal fluctuations due to nonzero T plays the role only to renormalize the strengths of both the interactions, although it ceases to be of relevance at higher temperatures. Except for weak disorder, the exchange interaction never wins over the pinning interaction to establish long-range order in the system. The study of spin-spin correlation functions reveals that the 2D RFIM shows long-range order, short-range order, and no order at all, each of which occurs in a restricted range of random field strength. We have also measured the largest cluster size by using the Hoshen-Kopelman algorithm. The behaviors of the dynamical evolution of the largest cluster are consistent with our previous conclusions.

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