Power fluctuation theorem for a Brownian harmonic oscillator

J. I. Jiménez-Aquino* and R. M. Velasco†

Departamento de Física, Universidad Autónoma Metropolitana–Iztapalapa, 09340 México, D.F., Mexico

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In this paper we study the validity of the total power fluctuation theorem spent on a Brownian harmonic oscillator when the system is driven out of equilibrium through the drag of the potential minimum. The theorem is first proved for an ordinary harmonic oscillator in two cases: The first one considers the particle in a thermal bath under the action of Gaussian white noise, and in the second one the drift is provided by an additional external Gaussian colored noise satisfying the characteristics of an Ornstein-Uhlenbeck process. We go further, by considering a charged harmonic oscillator under the action of an electromagnetic field. The theorem is also proven as in the two cases given above. In both of those cases, we illustrate the theorem for a uniform motion of the trap potential minimum and show that in the presence of external colored noise, the theorem is only valid in the stationary state.

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I. INTRODUCTION

The fluctuation theorems (FT's) continue to be a hot topic in the statistical mechanics of small systems out of equilibrium. They have been studied from the experimental and theoretical points of view in a variety of systems ranging from physics to chemistry to biology [1–38]. A few years ago they were extended to quantum systems [25–32] as well. The study of FT's for thermodynamic quantities such as work, heat, and entropy production produces essential results for the understanding of nonequilibrium phenomena. Basically, the FT relates the probability densities $P(A), P(-A)$ in processes that drive the system out of equilibrium such that the ratio $P(A)/P(-A) = e^{A}$, where A represents the work, heat, entropy production, or some other physical quantity. The FT's for the aforementioned quantities have been proven for both the transient and stationary-state regimes. An important number of theoretical studies in this area rely upon the Langevin equations in which both the fluctuating and dissipative forces are originated from the same environment (commonly referred to as a thermal bath). When a balance between both forces is established by the fluctuation-dissipation relation, it enables the system to reach an equilibrium state. Otherwise, when the fluctuating and dissipative forces are originated from different environments, the system can reach a nonequilibrium steady state. This kind of fluctuating force is known as an external noise and leads to a breakdown of the fluctuation-dissipation relation [39–46]. The simplest generalization of the equilibrium state is a nonequilibrium steady state (NESS) that arises physically due to the lack of balance between dissipation and fluctuations caused by the driving of the external forces that maintain the system in a nonequilibrium situation. The simplest example is a Brownian particle in a fluid, confined into a harmonic potential and dragged with constant velocity. In addition to the potential force, the particle experiences friction and thermal forces due to the surrounding medium. To maintain the NESS, work on the particle has to be performed, which in turn is partly dissipated in the fluid as heat and partly stored as potential energy in the harmonic potential.

Over the past 5 years, the work and entropy production fluctuation theorems have also been explored and verified in the study of charged Brownian harmonic oscillators in the presence of electromagnetic fields [33–38]. It is interesting to note that the diffusion process in a plasma, studied as a Brownian motion problem across a magnetic field, was solved a long time ago [47,48] and assumed to arise from local fluctuations of the electric field that induce collisions between particles in a Brownian motion–like manner. In Ref. [47], a situation in which the density gradient occurs in a direction orthogonal to the magnetic field where the magnetic pressure is dominant was considered. The ion velocity is much less than the electron velocities, so the mean friction dynamics on an ion will be proportional to the ion velocity. After these pioneering works, the studies performed in this context have considered such fluctuations as a thermal noise where the fluctuation-dissipation relation holds [33–38,47,48]. In this case, a balance between both the fluctuating electric field and the friction force originated from the same environment allows the system to reach the equilibrium state.

Very recently, we have found that there exists another quantity closely related to the total work that also satisfies the fluctuation theorem. This is the *time derivative of the total work*, which we refer to as the *total power* spent on the system as it is driven out of equilibrium. It should be mentioned that the total work W_τ used in the present contribution is the same as that proposed in Ref. [8], which according to the energy conservation law, satisfies $\Delta U = W_\tau - Q$, where Q is the heat dissipated to the surrounding medium and ΔU is the potential energy of the Brownian particle that is dragged through water molecules; both the total work and heat are fluctuating quantities and path-dependent [8,9]. While in Ref. [8] it was shown that the conventional stationary-state fluctuation theorem and transient fluctuation theorem of the total work hold, another FT holds for the dissipated heat. However, in Ref. [9] a corrected fluctuation relation was shown to have a general validity for the dissipated heat. It

*ines@xanum.uam.mx
†rmvb@xanum.uam.mx

is our main interest in this contribution to study the validity of the total power fluctuation theorem (PFT), following similar ideas behind the proof of a corresponding theorem for the total work [6,8,33,37]. The total power $\mathcal{P}$ also satisfies $\mathcal{P} \equiv dW_\tau/d\tau = dQ/d\tau + d\Delta U/d\tau$, where $dQ/d\tau$ denotes the power dissipated in the form of heat to the surrounding medium and $d\Delta U/d\tau$ denotes the power stored in the form of potential energy by the particle. We first study the theorem for an ordinary Brownian harmonic oscillator in two cases: first, when the particle is embedded in a thermal bath of temperature T leading to the unavoidable presence of thermal fluctuations (internal noise), assumed to be a Gaussian white noise (GWN). In this case, the fluctuation-dissipation relation holds. The next case is when an additional external Gaussian Ornstein-Uhlenbeck (GOU) noise is considered. In both cases, we assume a uniform motion in the minimum of the harmonic trap responsible to drive the system out of equilibrium. In the case of the GWN only, the total PFT is shown to be valid in an exact way, whereas with the additional presence of the external GOU, the theorem is valid only in the stationary state. We also prove the PFT when the Brownian harmonic oscillator is electrically charged and in the presence of an electromagnetic field. In this problem, the magnetic field is considered a constant vector and the electric field a time-dependent vector. In addition, we first show the PFT when the system is embedded in a thermal bath of temperature T and the thermal noise comes from the local fluctuations of the electric field. In this case, the external time-dependent electric field will be responsible for the uniform motion of the minimum trap potential. In this case the exact validity of the total PFT is also shown. In the presence of an additional external GOU, the total PFT is valid only in the stationary state.

Our work is presented in the following way: In Sec. II we introduce the usual Langevin equation for a Brownian harmonic oscillator, whereas in Sec. III we prove the PFT assuming a uniform motion for the minimum trap potential. The theorem is proved first when the system is driven by a thermal GWN only. Then it is shown for an additional GOU noise. In Sec. IV we consider the problem in the presence of an electromagnetic field and Sec. V is devoted to the demonstration of the corresponding PFT. In addition, the case of thermal GWN is first studied, followed by that including an additional OU noise. Our concluding remarks are given in Sec. VI.

II. LANGEVIN EQUATION FOR A HARMONIC OSCILLATOR

Let us first consider a Brownian particle of mass m embedded in a thermal bath of temperature T such that the particle is subject to an external harmonic potential $U(\mathbf{r},\mathbf{r}^*) = (k/2)|\mathbf{r}-\mathbf{r}^*|^2$, with k a constant. Here $\mathbf{r}^*$ is the time-dependent position of the potential minimum that is dragged with arbitrary velocity $\mathbf{v}^*(t) = d\mathbf{r}^*/dt$. We will suppose that at time $t = 0$, the potential minimum is located at the origin of coordinates $\mathbf{r}^*(0) \equiv \mathbf{r}_0^* = 0$, whereas for $t > 0$ it moves arbitrarily with the velocity $\mathbf{v}^*(t)$. The Langevin equation for the harmonic oscillator is then

$$m\frac{d\mathbf{v}}{dt} = -\gamma\,\mathbf{v} - k(\mathbf{r}-\mathbf{r}^*) + \boldsymbol{\xi}(t), \tag{1}$$

where $\gamma > 0$ is the friction coefficient and $\boldsymbol{\xi}(t)$ a Gaussian noise with zero mean value $\langle \xi_i(t)\rangle = 0$ and correlation function

$$\langle \xi_i(t)\xi_j(t')\rangle = G_{ij}(t-t'), \quad i,j = x,y,z, \tag{2}$$

where $G_{ij}(t-t')$ is an arbitrary function. The problem will be studied in the overdamped approximation of the Langevin equation, which is then given by

$$\frac{d\mathbf{r}}{dt} = -a\,\mathbf{r} + a\,\mathbf{r}^* + \gamma^{-1}\boldsymbol{\xi}(t), \tag{3}$$

where $a = k/\gamma$ is a characteristic frequency in this problem. Equation (3) can be solved in terms of the deterministic trajectory $\langle\mathbf{r}\rangle$ and variable $\mathbf{R} = \mathbf{r} - \langle\mathbf{r}\rangle$; those variables satisfy the following differential equations:

$$\frac{d\langle\mathbf{r}\rangle}{dt} = -a\langle\mathbf{r}\rangle + a\mathbf{r}^*, \tag{4}$$

$$\frac{d\mathbf{R}}{dt} = -a\mathbf{R} + \gamma^{-1}\boldsymbol{\xi}(t). \tag{5}$$

In fact, the solution of Eq. (4) can be written, after integration by parts, as

$$\langle\mathbf{r}\rangle - \mathbf{r}^* = -\frac{\mathbf{v}^*(\tau)}{a} + \frac{1}{a}\int_0^\tau e^{-a(\tau-t)}\mathbf{A}^*(t)\,dt, \tag{6}$$

where we have assumed that $\langle\mathbf{r}(0)\rangle = 0$ and $\mathbf{A}^*(t) = d\mathbf{v}^*(t)/dt$ is the acceleration of the potential minimum. On the other hand, Eq. (5) has the following solution:

$$\mathbf{R}(\tau) = \mathbf{R}_0 e^{-a\tau} + \gamma^{-1}\int_0^\tau e^{-a(\tau-t)}\boldsymbol{\xi}(t)\,dt, \tag{7}$$

where $\mathbf{R}_0 = \mathbf{R}(0)$ is the initial condition. Such an initial condition is not fixed; instead it is given through a distribution. Due to the fact that the process $\mathbf{R}(t)$ is stationary, the initial distribution is a canonical one,

$$P(\mathbf{R}_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/2} e^{-|\mathbf{R}_0|^2/2\sigma^2}, \tag{8}$$

where $\sigma^2 = \langle R_{0i}R_{0j}\rangle$ is associated with the width of the distribution. This quantity must be evaluated each time we consider a process occurring in the system, and its value will be written when needed.

III. POWER AVERAGE AND VARIANCE

We begin our calculation using the definition of the total work [8,33,34,36–38] done on a system during a time interval τ given by

$$W_\tau = \int_0^\tau \mathbf{v}^*\cdot\mathbf{F}(\mathbf{r},\mathbf{r}^*)\,dt, \tag{9}$$

where $\mathbf{F}(\mathbf{r},\mathbf{r}^*) = -k(\mathbf{r}-\mathbf{r}^*)$ is the harmonic force. As stated in Ref. [8], this work is the total energy put into the system (the particle in the harmonic trap plus the water). As required by the energy conservation law, part of this total work goes into the fluid in the form of heat Q, while the other part goes into the potential energy ΔU of the particle in the harmonic potential such that $\Delta U = Q - W_\tau$, where $\Delta U = U(\mathbf{r}(t+\tau),t+\tau) - U(\mathbf{r}(t),t)$. According to Eq. (9), we thus define the total power spent on the system in the same time interval as

$$\mathcal{P} = \frac{dW_\tau}{d\tau} = \mathbf{v}^*(\tau) \cdot \mathbf{F}(\mathbf{r},\mathbf{r}^*). \tag{10}$$

It is now clear that $\mathcal{P} = dQ/d\tau + d\Delta U/d\tau$, where $dQ/d\tau$ would be the power dissipated in the form of heat to the surrounding medium and $d\Delta U/d\tau$ the power stored in the form of potential energy by the particle. We now proceed to calculate the statistics of the total power as the harmonic trap is moving in the time interval $[0,\tau]$. The average of the power as well as its variance can be further calculated if we use the variables $\langle \mathbf{r} \rangle$ and $\mathbf{R}$ and the solutions given in Eqs. (6) and (7). First we notice that the power defined in Eq. (10) in this case is written as

$$\mathcal{P} = -k[\mathbf{v}^* \cdot \mathbf{R} + \mathbf{v}^* \cdot (\langle \mathbf{r} \rangle - \mathbf{r}^*)]. \tag{11}$$

Since $\langle \mathbf{R} \rangle = 0$, the average value of the total power then reads

$$\langle \mathcal{P} \rangle = -k\mathbf{v}^*(\tau) \cdot [\langle \mathbf{r}(\tau) \rangle - \mathbf{r}^*(\tau)]. \tag{12}$$

The average of the total power needs the solutions we obtained in Eqs. (4) and (5). With the substitution of Eq. (6) into Eq. (12), we thus get the total power average as

$$\langle \mathcal{P} \rangle = \gamma\, \mathbf{v}^*(\tau) \cdot \mathbf{v}^*(\tau) - \gamma \int_0^\tau e^{-a(\tau - t)} \mathbf{v}^*(\tau) \cdot \mathbf{A}^*(t)\, dt. \tag{13}$$

Now, let us define the power fluctuation $\Delta\mathcal{P}(\tau) = \mathcal{P}(\tau) - \langle \mathcal{P}(\tau) \rangle$; its correlation is written as

$$\langle \Delta\mathcal{P}(\tau)\Delta\mathcal{P}(\tau') \rangle = k^2 \langle \mathbf{v}^*(\tau) \cdot \mathbf{R}(\tau) \mathbf{v}^*(\tau') \cdot \mathbf{R}(\tau') \rangle. \tag{14}$$

Now we express in a convenient way the quantities needed to calculate the correlation defined in Eq. (14), hence

$$\mathbf{v}^*(\tau) \cdot \mathbf{R}(\tau) = e^{-a\tau} \mathbf{R}_0 \cdot \mathbf{v}^*(\tau) + \gamma^{-1} \mathbf{C}(\tau) \cdot \mathbf{v}^*(\tau) \tag{15}$$

and

$$\mathbf{C}(\tau) = \int_0^\tau e^{-a(\tau - t)} \boldsymbol{\xi}(t)\, dt. \tag{16}$$

Upon substitution of Eq. (15) into Eq. (14) we get in terms of components that

$$\begin{aligned}\langle \Delta\mathcal{P}(\tau)\Delta\mathcal{P}(\tau') \rangle = k^2 [& e^{-2a\tau} v_i^*(\tau) v_j^*(\tau') \langle R_{0i} R_{0j} \rangle \\ & + \gamma^{-2} v_i^*(\tau) v_j^*(\tau') \langle C_i(\tau) C_j(\tau') \rangle]. \end{aligned} \tag{17}$$

The variance of the total power as required by its definition can be calculated as $V_{\mathcal{P}}(\tau) \equiv \langle \mathcal{P}(\tau)^2 \rangle - \langle \mathcal{P}(\tau) \rangle^2 = \langle \Delta\mathcal{P}(\tau)\Delta\mathcal{P}(\tau) \rangle$ and is shown to be

$$V_{\mathcal{P}}(\tau) = k^2 \langle \mathbf{v}^*(\tau) \cdot \mathbf{R}(\tau)\, \mathbf{v}^*(\tau) \cdot \mathbf{R}(\tau) \rangle. \tag{18}$$

A. Thermal Gaussian white noise

In this case we choose the function $G_{ij}(t - t') = 2\lambda\, \delta_{ij} \delta(t - t')$, corresponding to the Gaussian white noise, and thus

$$\langle \xi_i(t) \xi_j(t') \rangle = 2\lambda\, \delta_{ij}\, \delta(t - t'), \quad i,j = x,y,z, \tag{19}$$

where λ measures the noise intensity. According to the fluctuation-dissipation relation it is related to the friction coefficient by $\lambda = \gamma k_B T$ with k_B as Boltzmann's constant. To calculate the power variance in this case, we need the width in the distribution of initial conditions and the correlation function of the quantity given in Eq. (16). In this case, the initial conditions are distributed according to Eq. (8) with a width consistent with the thermal equilibrium between the particle and the thermal bath, then $\langle R_{0i} R_{0j} \rangle = 1/(k\beta)\delta_{ij}$ and $\beta = 1/k_B T$ as usual. On the other hand, the correlation function of $\mathbf{C}$ is a matter of a direct calculation and

$$\langle C_i(\tau) C_j(\tau') \rangle = \frac{\lambda}{a} \delta_{ij} [e^{-a|\tau - \tau'|} - e^{-a(\tau + \tau')}] \tag{20}$$

so that the correlation of the total power (14) reduces to

$$\langle \Delta\mathcal{P}(\tau)\Delta\mathcal{P}(\tau') \rangle = \frac{k}{\beta} \mathbf{v}^*(\tau) \cdot \mathbf{v}^*(\tau') e^{a|\tau - \tau'|}. \tag{21}$$

For equal times $\tau' = \tau$ we have

$$V_{\mathcal{P}} = \frac{k}{\beta} \mathbf{v}^*(\tau) \cdot \mathbf{v}^*(\tau) = \frac{a}{\beta} \gamma\, \mathbf{v}^*(\tau) \cdot \mathbf{v}^*(\tau). \tag{22}$$

It can be seen that in the case of a uniform motion of the minimum trap potential, which means that $\mathbf{A}^*(t) = 0$ and $\mathbf{v}^* \cdot \mathbf{v}^* \equiv v^2$ a constant, the average of the total power is $\langle \mathcal{P} \rangle = \gamma \mathbf{v}^* \cdot \mathbf{v}^* = \gamma v^2$. In this case, the variance and the total mean power will be related by

$$V_{\mathcal{P}} = \frac{a}{\beta} \langle \mathcal{P} \rangle. \tag{23}$$

To establish the total power fluctuation theorem we must notice that the total power as written in Eq. (11) is linear in the variable $\mathbf{R}$, which is described by a Gaussian distribution function. Accordingly, the power distribution is given as follows:

$$P(\mathcal{P}) = \frac{1}{\sqrt{2\pi V_{\mathcal{P}}}} e^{-(\mathcal{P} - \langle \mathcal{P} \rangle)^2 / 2V_{\mathcal{P}}}. \tag{24}$$

Now the relation between the power average and its variance Eq. (23) allows us to establish the total power fluctuation theorem, hence

$$\frac{P(\mathcal{P})}{P(-\mathcal{P})} = e^{2(\beta/a)\mathcal{P}}, \tag{25}$$

which corresponds to the conventional fluctuation theorem for the time derivative of the total work W_τ. In this case, the theorem is valid in its exact form for all times $\tau > 0$ [9].

B. External Gaussian Colored Noise

Here we will consider both the internal GWN denoted as $\boldsymbol{\zeta}(t)$ and the influence of an external noise $\boldsymbol{\eta}(t)$ that satisfies the properties of an Ornstein-Uhlenbeck noise (GOU). In this case the noise term of Eq. (1) is then $\boldsymbol{\xi}(t) = \boldsymbol{\eta}(t) + \boldsymbol{\zeta}(t)$, and the function $G_{ij}(t - t')$ leads specifically to

$$\langle \xi_i(t) \xi_j(t') \rangle = \frac{D}{\tau} \delta_{ij} e^{-|t - t'|/\tau} + 2\lambda\, \delta_{ij} \delta(t - t'), \tag{26}$$

where D and τ_c are the intensity and correlation time of the external noise, respectively, and $\lambda = \gamma k_B T$. We again consider a uniform motion for the potential minimum leading in this case to the same expression for the average of the total power, i.e., $\langle \mathcal{P} \rangle = \gamma \mathbf{v}^* \cdot \mathbf{v}^* = \gamma v^2$. To calculate the variance as given by Eq. (17), we need the correlation of the initial condition $\mathbf{R}_0$. In the absence of internal noise $\boldsymbol{\zeta}(t)$, the initial distribution function associated with Eq. (5) is given by the Gaussian distribution written in Eq. (8) [49] with $\sigma_0^2 = \langle R_{0i} R_{0j} \rangle = D/\gamma^2 a(1+a\tau_c)\delta_{ij}$. With the additional internal noise it can be shown that

$$\sigma_0^2 = \langle R_{0i} R_{0j} \rangle = \frac{D}{\gamma^2 a(1+a\tau_c)}\delta_{ij} + \frac{1}{k\beta}\delta_{ij}, \quad (27)$$

and therefore after integration of Eq. (16) we get the following expression:

$$\begin{aligned}\frac{k^2}{\gamma^2}\langle C_i(\tau)C_j(\tau')\rangle &= \frac{Da}{1-a^2\tau_c^2}[e^{-a|\tau-\tau'|} - e^{-a(\tau+\tau')}]\delta_{ij} \\ &\quad - \frac{Da^2\,\tau_c}{1-a^2\tau_c^2}\{e^{-a(\tau+\tau')}[1 - e^{-\frac{\tau}{\tau_1}} - e^{-\frac{\tau'}{\tau_1}}] \\ &\quad + e^{-\frac{|\tau-\tau'|}{\tau_c}}\}\delta_{ij} + \frac{\lambda}{a}\,\delta_{ij}[e^{-a|\tau-\tau'|} - e^{-a(\tau+\tau')}]. \quad (28)\end{aligned}$$

where $\tau_1 = \tau_c/(1-a\tau_c)$. For uniform dragging of the potential minimum and for equal times $\tau = \tau'$, Eq. (17) leads to

$$V_{\mathcal{P}} = \frac{a}{\beta}\gamma v^2 + \frac{Dv^2 a}{1+a\tau_c} + \frac{2Dv^2a^2\tau_c}{1-a^2\tau_c^2}\left(e^{-2a\tau} + e^{-\frac{\tau}{\tau_2}}\right), \quad (29)$$

where $\tau_2 = \tau_c/(1+a\tau_c)$. Notice that the variance calculated in Eq. (29) can be written in terms of the average power, yet such a relation contains a function of the time τ. This fact prevents the PFT from being valid for all times. When the system reaches the stationary state the variance reads

$$V_{\mathcal{P}_s} = \frac{a}{\beta}\gamma v^2 + \frac{Da}{1+a\tau_c}v^2, \quad (30)$$

which is related to the average of the total power by

$$V_{\mathcal{P}_s} = \left(\frac{a}{\beta} + \frac{D_e a}{\gamma}\right)\langle\mathcal{P}\rangle, \quad (31)$$

where $D_e = D/(1+a\tau_c)$ accounts for a renormalization of the noise intensity D by the factor $1+a\tau_c$. Hence in the stationary state, the total power Fluctuation theorem holds for both kinds of Gaussian noises, that is,

$$\frac{P(\mathcal{P})}{P(-\mathcal{P})} = e^{(2\beta\mathcal{P})/[a(1+D_e\beta/\gamma)]}. \quad (32)$$

Though the PFT is valid only in the stationary state, it is important that it is not necessary to correct it as in the case of heat [9]. In the GWN limiting case ($\tau_c = 0$) we have $D_e = D$. If we suppose that $D = \gamma k_B T = \gamma/\beta$ then $\gamma/aD_e = \beta/a$ and therefore $P(\mathcal{P})/P(-\mathcal{P}) = e^{4(\beta/a)\mathcal{P}}$, which is an expected result. Also, when the external noise is not present, $D = 0$ and we recover the power fluctuation theorem as written in Eq. (25).

IV. LANGEVIN EQUATION FOR A HARMONIC OSCILLATOR IN AN ELECTROMAGNETIC FIELD

Consider the above Brownian harmonic particle of mass m now with an electric charge q under the action of an electric field. Besides, a constant magnetic field pointing along the z axis is acting on the particle, so that $\mathbf{B} = (0,0,B)$. In this case, the Langevin equation associated with the charged particle can be split into two independent equations: One equation is parallel to magnetic field and the other is given on the x-y plane orthogonal to this field. Along the magnetic field the process is the same as the z component of Eq. (1), and therefore solved in the preceding sections. The influence of the magnetic field is then on the (x,y) plane orthogonal to this field, and it is the process in which we are interested now. The two-dimensional harmonic potential is defined as $\overline{U}(\mathbf{x},\mathbf{x}^*) = (k/2)|\mathbf{x}-\mathbf{x}^*|^2$, where $\mathbf{x} = (x,y)$ is the position vector for the charged particle, $\mathbf{x}^* \equiv \mathbf{x}^*(t) = (x^*,y^*)$ the position vector for the potential minimum, and $\mathbf{u} = (u_x,u_y)$ the velocity vector. As a first case, we prove the PFT, taking into account that the thermal fluctuations come from the local fluctuations of the electric field, and the external time-dependent electric field $\mathbf{E}_s(t)$ will account for the systematic electric field responsible for the dragging of the potential minimum. At time $t = 0$, the potential minimum is at the origin of coordinates, i.e., $\mathbf{x}^*(0) = \mathbf{x}_0^* = 0$, whereas for $t > 0$, it moves arbitrarily with velocity $\mathbf{u}^*(t)$, where $\mathbf{u}^*(t) = d\mathbf{x}^*/dt$ and $\mathbf{u}^*(t) = (u_x^*(t),u_y^*(t))$. The Langevin equation for the charged harmonic oscillator in this case can be written as

$$m\frac{d\mathbf{u}}{dt} = -\gamma\,\mathbf{u} + \frac{q}{c}\mathbf{u}\times\mathbf{B} - k\,\mathbf{x} + k\mathbf{x}^* + \boldsymbol{\mu}(t), \quad (33)$$

where $\mathbf{x}^* = (q/k)\mathbf{E}_s(t)$ is the position vector of the potential minimum, $\boldsymbol{\mu}(t)$ a Gaussian noise with zero mean value and correlation function

$$\langle\mu_i(t)\mu_j(t')\rangle = H_{ij}(t-t'), \quad i,j = x,y. \quad (34)$$

being $H_{ij}(t-t')$ an arbitrary function. In the overdamped approximation Eq. (33) reduces to

$$\frac{d\mathbf{x}}{dt} = -\Lambda\mathbf{x} + \Lambda\,\mathbf{x}^* + k^{-1}\Lambda\,\boldsymbol{\mu}(t), \quad (35)$$

where Λ is a 2×2 matrix given by

$$\Lambda = \begin{pmatrix} \widetilde{a} & \widetilde{\Omega} \\ -\widetilde{\Omega} & \widetilde{a} \end{pmatrix}, \quad (36)$$

and the parameters are defined as $\widetilde{a} = a/(1+C^2)$, $\widetilde{\Omega} = \widetilde{a}C$, and $C = qB/c\gamma$ and contain the influence of the magnetic field, also $a = k/\gamma$ as in the previous case.

To solve Eq. (35), we follow a similar procedure as we did in Sec. II and separate the deterministic behavior given by $\langle\mathbf{x}\rangle$ from the one corresponding to variable $\mathbf{X} = \mathbf{x} - \langle\mathbf{x}\rangle$, then

$$\frac{d\langle\mathbf{x}\rangle}{dt} = -\Lambda\,\langle\mathbf{x}\rangle + \Lambda\,\mathbf{x}^*, \quad (37)$$

$$\frac{d\mathbf{X}}{dt} = -\Lambda\,\mathbf{X} + k^{-1}\Lambda\,\boldsymbol{\mu}(t). \quad (38)$$

Their solutions are obtained in a direct way:

$$\langle\mathbf{x}\rangle - \mathbf{x}^*(t) = \Lambda^{-1}\mathbf{u}^*(t) + \Lambda^{-1}e^{-\Lambda t}\int_0^t e^{\Lambda t'}\mathbf{A}^*(t')\,dt', \quad (39)$$

$$\mathbf{X}(t) = e^{-\widetilde{a}t}\mathcal{R}^{-1}(t)\,\mathbf{X}_0 + k^{-1}\,\widetilde{\mathbf{C}}(t), \quad (40)$$

where Λ^{-1} is the inverse of matrix Λ and $\mathbf{A}^*(t) = d\mathbf{u}^*(t)/dt$ is the acceleration of the potential minimum, and also it is assumed the initial condition $\langle \mathbf{x}(0)\rangle = 0$. If we take into account that $\Lambda = \widetilde{a}\,\mathbb{I} + \mathbb{W}$, with $\mathbb{I}$ the unit matrix and $\mathbb{W}$ a real antisymmetric matrix, then $e^{\Lambda t} = e^{\widetilde{a}t}\mathcal{R}(t)$, $\mathcal{R}(t) = e^{\mathbb{W}t}$ being an orthogonal rotation matrix and $\mathcal{R}^{-1}(t) = e^{-\mathbb{W}t}$ its inverse:

$$\mathbb{W} = \begin{pmatrix} 0 & \widetilde{\Omega} \\ -\widetilde{\Omega} & 0 \end{pmatrix}, \quad \mathcal{R}(t) = \begin{pmatrix} \cos\widetilde{\Omega}\,t & \sin\widetilde{\Omega}\,t \\ -\sin\widetilde{\Omega}\,t & \cos\widetilde{\Omega}\,t \end{pmatrix}. \quad (41)$$

The vector $\widetilde{\mathbf{C}}(t)$ is defined as follows:

$$\widetilde{\mathbf{C}}(t) = \int_0^t e^{-\widetilde{a}(t-t')}\Lambda\,\mathcal{R}^{-1}(t')\boldsymbol{\mu}(t')\,dt'. \quad (42)$$

V. POWER AVERAGE AND VARIANCE IN THE PRESENCE OF AN ELECTROMAGNETIC FIELD

In a similar way as defined above, the total work $\mathcal{W}_\tau$ done on a system during a time τ is defined, in the two-dimensional case, by

$$\mathcal{W}_\tau = \int_0^\tau \mathbf{u}^*\cdot\mathbf{F}(\mathbf{x},\mathbf{x}^*)\,dt, \quad (43)$$

where $\mathbf{F}(\mathbf{x},\mathbf{x}^*) = -k(\mathbf{x}-\mathbf{x}^*)$ is the harmonic force. The total power dissipated by the system in the same time interval τ reads

$$\widetilde{\mathcal{P}} = \frac{d\mathcal{W}_\tau}{d\tau} = \mathbf{u}^*(\tau)\cdot\mathbf{F}(\mathbf{x},\mathbf{x}^*). \quad (44)$$

The statistics of the total power will be calculated using the results given in Eqs. (37) and (38). The average of the total power now is

$$\langle\widetilde{\mathcal{P}}\rangle = -k\mathbf{u}^*(\tau)\cdot[\langle\mathbf{x}(\tau)\rangle - \mathbf{x}^*(\tau)], \quad (45)$$

in which we substitute Eq. (39); after an integration by parts we obtain

$$\langle\widetilde{\mathcal{P}}\rangle = \gamma\,\mathbf{u}^*(\tau)\cdot\mathbf{u}^*(\tau) - k\Lambda^{-1}\int_0^\tau e^{-\widetilde{a}(\tau-t)}\mathbf{U}^*(\tau)\cdot\mathbf{A}^*(t)\,dt, \quad (46)$$

where $\mathbf{U}^*(\tau) = \mathcal{R}(\tau)\mathbf{u}^*(\tau)$ and $\widetilde{\mathbf{A}}^*(t) = \mathcal{R}(t)\mathbf{A}^*(t)$ is the acceleration of the potential minimum affected by the rotation matrix. The correlation of power fluctuations reads

$$\langle\Delta\mathcal{P}(\tau)\Delta\mathcal{P}(\tau')\rangle = k^2\langle\mathbf{X}(\tau)\cdot\mathbf{u}^*(\tau)\mathbf{X}(\tau')\cdot\mathbf{u}^*(\tau')\rangle. \quad (47)$$

Now the quantity

$$\mathbf{X}(t)\cdot\mathbf{u}^*(t) = e^{-\widetilde{a}t}\mathbf{u}^*(t)\cdot\mathbf{X}_0 + k^{-1}\mathbf{u}^*(t)\cdot\widetilde{\mathbf{C}}(t) \quad (48)$$

is required. In terms of the components, the correlation of the total power now reads

$$\langle\Delta\mathcal{P}(\tau)\Delta\mathcal{P}(\tau')\rangle = k^2 e^{-\widetilde{a}(\tau+\tau')}U_i^*(\tau)U_j^*(\tau')\langle X_{0i}X_{0j}\rangle + U_i^*(\tau)U_j^*(\tau')\langle\widetilde{C}_i(\tau)\widetilde{C}_j(\tau')\rangle, \quad (49)$$

where

$$\langle\widetilde{C}_i(\tau)\widetilde{C}_j(\tau')\rangle = e^{-\widetilde{a}(\tau+\tau')}\int_0^\tau\int_0^{\tau'} e^{\widetilde{a}(t+t')}\Lambda_{il}\Lambda_{jk}\mathcal{R}^{-1}_{lm}(t)\times\mathcal{R}^{-1}_{kn}(t')\langle\mu_m(t)\mu_n(t')\rangle\,dt\,dt'. \quad (50)$$

A. Thermal Gaussian white noise

Now we will consider that the correlation function for the noise $\boldsymbol{\mu}(t)$ satisfies

$$\langle\mu_i(t)\mu_j(t')\rangle = 2\lambda\delta_{ij}\,\delta(t-t'), \quad i,j = x,y, \quad (51)$$

where λ satisfies the fluctuation dissipation relation, so it is again $\lambda = \gamma k_B T$. Using the definitions given above, it can be shown that

$$\langle\widetilde{C}_i(\tau)\widetilde{C}_j(\tau')\rangle = \frac{k}{\beta}\,\delta_{ij}[e^{-\widetilde{a}|\tau-\tau'|} - e^{-\widetilde{a}(\tau+\tau')}]. \quad (52)$$

Due to the fact that the process $\mathbf{X}(t)$ is stationary, it has been shown that its initial distribution also satisfies the canonical distribution given in Eq. (8) with $\sigma^2 = 1/k\beta$, then $\langle X_{0i}X_{0j}\rangle = (1/k\beta)\,\delta_{ij}$. The power correlation (49) now reduces to

$$\langle\Delta\mathcal{P}(\tau)\Delta\mathcal{P}(\tau') = \frac{k}{\beta}\,\mathbf{u}^*(\tau)\cdot\mathbf{u}^*(\tau')\,e^{-a|\tau-\tau'|}. \quad (53)$$

For equal times $\tau = \tau'$ we obtain the power variance

$$V_{\widetilde{\mathcal{P}}} = \frac{k}{\beta}\mathbf{u}^*(\tau)\cdot\mathbf{u}^*(\tau) = \frac{a}{\beta}\gamma\mathbf{u}^*(\tau)\cdot\mathbf{u}^*(\tau), \quad (54)$$

which is quite similar to Eq. (22). Again, for a uniform motion $\mathbf{A}^* = 0$ and $\mathbf{u}^*\cdot\mathbf{u}^* \equiv u^2$ is a constant. In this case the variance of the total power and the power average are also related by

$$V_{\widetilde{\mathcal{P}}} = \frac{a}{\beta}\langle\widetilde{\mathcal{P}}\rangle. \quad (55)$$

Under these conditions, the PFT leads to

$$\frac{P(\widetilde{\mathcal{P}})}{P(-\widetilde{\mathcal{P}})} = e^{2(\beta/a)\widetilde{\mathcal{P}}}, \quad (56)$$

which is practically the same as that obtained in the absence of the magnetic field as shown in Eq. (25) if we take just $\mathbf{v}^* = (v_x,v_y) = \mathbf{u}^*$. Also, the PFT is valid in its exact way, i.e., for all times $\tau > 0$.

B. External Gaussian Ornstein-Uhlenbeck noise

In this case the external electric field will account for the following two contributions: $\mathbf{E}(t) = \mathbf{E}_s(t) + \mathbf{E}_f(t)$. Here $\mathbf{E}_s(t)$ represents a systematic electric field responsible for the dragging of the potential minimum and $\mathbf{E}_f(t)$ the external fluctuation with the properties of an OU noise. This fluctuating field is thus absorbed in the noise term $\boldsymbol{\mu}(t)$ such that $\boldsymbol{\mu}(t) = \widetilde{\boldsymbol{\eta}}(t) + \widetilde{\boldsymbol{\zeta}}(t)$, with $\widetilde{\boldsymbol{\eta}}(t) \equiv q\mathbf{E}_f(t)$ the external OU noise and $\widetilde{\boldsymbol{\zeta}}(t)$ the thermal noise such that

$$\langle\mu_i(t)\mu_j(t')\rangle = \frac{D}{\tau_c}\delta_{ij}\,e^{-\frac{|t-t'|}{\tau_c}} + 2\lambda\,\delta_{ij}\delta(t-t'). \quad (57)$$

Notice that the power variance depends on the noise characteristics through the correlation function $\langle\widetilde{C}_i(\tau)\widetilde{C}_j(\tau')\rangle$ and the initial conditions $\langle X_{0i}X_{0j}\rangle$. Again, we notice that the process $\mathbf{X}(t)$ in (40) is stationary, the initial conditions in the presence of the magnetic field and without the influence of the internal noise are distributed according to a Gaussian distribution as the one given in Eq. (8), where $\sigma^2 = \langle X_{0i}X_{0j}\rangle$ was calculated in Ref. [49], and it is given as

$$\langle X_{0i}X_{0j}\rangle = \frac{D(1+\widetilde{a}\tau_c)}{\gamma^2\widetilde{a}(1+C^2)[(1+\widetilde{a}\tau_c)^2 + (\widetilde{\Omega}\tau_c)^2]}\delta_{ij}. \quad (58)$$

With the additional internal noise it can be shown that this initial correlation function reads

$$\langle X_{0i} X_{0j} \rangle = \frac{D(1+\widetilde{a}\tau_c)}{\gamma^2 \widetilde{a}(1+C^2)[(1+\widetilde{a}\tau_c)^2 + (\widetilde{\Omega}\tau_c)^2]} \delta_{ij} + \frac{1}{k\beta}\,\delta_{ij}. \tag{59}$$

When the potential minimum is dragged at constant velocity $\mathbf{u}^*(t)\cdot\mathbf{u}^*(t) = u^2$, the acceleration $\mathbf{A}^* = 0$ and thus the average power as given in Eq. (46) is simply $\langle\widetilde{\mathcal{P}}\rangle = \gamma u^2$. Now, it is a matter of a direct calculation to obtain the correlation functions required in Eq. (49). In fact, the external OU noise drives to $\langle \widetilde{C}_1(\tau)\widetilde{C}_2(\tau')\rangle = -\langle \widetilde{C}_2(\tau)\widetilde{C}_1(\tau')\rangle = 0$ and $\langle \widetilde{C}_1(\tau)\widetilde{C}_1(\tau')\rangle = \langle \widetilde{C}_2(\tau)\widetilde{C}_2(\tau')\rangle$. Upon evaluation of this correlation function at equal times $\tau = \tau'$ we have

$$\langle \widetilde{C}_1(\tau)\widetilde{C}_1(\tau)\rangle = Da\left[\frac{(-1+\widetilde{a}\tau_c)}{\Lambda^-} e^{2\widetilde{a}\tau} - \frac{2\tau_c}{\Lambda^+\Lambda^-} G(\tau) e^{-\tau/\widetilde{\tau}_2} + \frac{(1+\widetilde{a}\tau_c)}{\Lambda^+}\right] + \frac{k}{\beta}(1 - e^{-2\widetilde{a}\tau}), \tag{60}$$

where $G(\tau) = [-1 + \widetilde{a}^2\tau_c^2(1+C^2)]\cos\widetilde{\Omega}\tau + 2\widetilde{\Omega}\tau_c \sin\widetilde{\Omega}\tau$, $\widetilde{\tau}_2 = \tau_c/(1+\widetilde{a}\tau_c)$, $\Lambda^+ = (1+\widetilde{a}\tau_c)^2 + (\widetilde{\Omega}\tau_c)^2$, and $\Lambda^- = (1-\widetilde{a}\tau_c)^2 + (\widetilde{\Omega}\tau_c)^2$. Taking into account Eqs. (59) and (52), the total power variance (49) for uniform motion then reads

$$\begin{aligned} V_{\widetilde{\mathcal{P}}} = {} & \frac{Du^2 a(1+\widetilde{a}\tau_c)}{\Lambda^+} + \frac{2Du^2 a[1 - \widetilde{a}^2\tau_c^2 + (\widetilde{\Omega}\tau_c)^2]}{\Lambda^+\Lambda^-} e^{-2\widetilde{a}\tau} \\ & + \frac{2Du^2a^2\tau_c[1-\widetilde{a}^2\tau_c^2(1+C^2)]}{(1+C^2)\Lambda^+\Lambda^-} e^{-\tau/\widetilde{\tau}_2}\cos\widetilde{\Omega}\tau \\ & - \frac{2Du^2a^2\tau_c^2\widetilde{\Omega}}{(1+C^2)\Lambda^+\Lambda^-} e^{-\tau/\widetilde{\tau}_2}\sin\widetilde{\Omega}\tau + \frac{a\gamma u^2}{\beta}(1-e^{-2a\tau}), \end{aligned} \tag{61}$$

which contains a complicated dependency on time τ and its coupling with the magnetic field. Nevertheless, the asymptotic behavior shows in a clear way a relation between the variance and the power average given by

$$V_{\widetilde{\mathcal{P}}_s} = \frac{a}{\beta}\gamma u^2 + \frac{Da(1+\widetilde{a}\tau_c)}{\Lambda^+} u^2 = \left(\frac{a}{\beta} + \frac{Da(1+\widetilde{a}\tau_c)}{\gamma\,\Lambda^+}\right)\langle\widetilde{\mathcal{P}}\rangle. \tag{62}$$

To check out the consistency of the result given by Eq. (61), it can be compared with the result obtained for zero magnetic field. In this case $\widetilde{\Omega} = 0, \widetilde{a} = a = k/\gamma$ and thus

$$V_{\widetilde{\mathcal{P}}} = \frac{a}{\beta}\gamma u^2 + \frac{Du^2 a}{1+a\tau_c} + \frac{2Du^2a^2\tau_c}{1-a^2\tau_c^2}(e^{-2a\tau} + e^{-\tau/\tau_2}), \tag{63}$$

with $\tau_2 = \tau_c/(1+a\tau_c)$. This variance is the same as Eq. (29) for $v^2 = u^2$ and therefore the SPFT (65) reduces to Eq. (32) as it should be.

As we can see, the power variance depends on the influence of the internal noise through the parameter a/β as well on the cooperative effect of the four parameters D, τ_c, $\widetilde{a}$, and $\widetilde{\Omega}$ where the magnetic field is strongly coupled to the noise correlation time through the factor $(\widetilde{\Omega}\tau_c)^2$ contained in Λ^+. Here we have two limiting cases, first when we consider a weak magnetic field $\widetilde{\Omega}\tau_c \ll \widetilde{a}\tau_c$, $\Lambda^+ \approx (1+\widetilde{a}\tau_c)^2$ and thus the variance simplifies $V_{\mathcal{P}_s} = \frac{a}{\beta}\gamma u^2 + \frac{Da}{(1+\widetilde{a}\tau_c)} u^2 = (\frac{a}{\beta} + \frac{aD_e}{\gamma})\langle\widetilde{\mathcal{P}}\rangle$. This result is consistent with the result obtained in Eq. (31) as expected, because in this limiting case, the weak magnetic field is practically decoupled of the noise correlation time τ_c. As a second case, we consider a strong magnetic field such that $\widetilde{\Omega}\tau_c \gg \widetilde{a}\tau_c$ and now the asymptotic behavior is then given as $V_{\widetilde{\mathcal{P}}_s} = \frac{a}{\beta}\gamma u^2 + \frac{Da(1+\widetilde{a}\tau_c)}{1+(\widetilde{\Omega}\tau_c)^2}u^2$ where the coupling between all parameters is present. In the general case given in Eq. (62) we can define an effective noise intensity as $D_e^m = D(1+\widetilde{a}\tau_c)/\Lambda^+$ and we obtain

$$V_{\widetilde{\mathcal{P}}_s} = \left(\frac{a}{\beta} + \frac{aD_e^m}{\gamma}\right)\langle\widetilde{\mathcal{P}}\rangle. \tag{64}$$

This relation between the power average and its variance, allows us to establish the total stationary state power fluctuation theorem (SPFT)

$$\frac{P(\widetilde{\mathcal{P}})}{P(-\widetilde{\mathcal{P}})} = e^{\frac{2\beta}{a}((1+D_e^m a/\gamma)^{-1})\widetilde{\mathcal{P}}}. \tag{65}$$

It should be noticed that the power fluctuation theorem as written in Eq. (65) contains the influence of the internal and external noises as well as the effect of the magnetic field. The internal noise allows the bath temperature to play a role through the fluctuation dissipation relation. In contrast, the external noise is an athermal contribution because it is independent of the bath characteristics. In both cases the effect of the magnetic field is present through the effective noise intensity D_e^m. When the internal noise is not taken into account, the dissipative contribution given through the friction is not balanced with the fluctuations, and in this case the system remains out of equilibrium by means of external agents, though it reaches an asymptotic behavior in which the power becomes constant. The corresponding work done on the system behaves linearly with time and it stays, as far as the external forcing is present.

VI. CONCLUDING REMARKS

The power fluctuation theorem presented in this paper has been established for an ordinary Brownian harmonic oscillator and when it is charged and under the action of an external electromagnetic field. In the ordinary case and for a uniform dragging of the potential minimum, the theorem has been proved in exact way when the system is embedded in a thermal bath of temperature T. In this case the validity of Eq. (23) is shown for the total power average and its variance, which remain constant in the time interval $[0,\tau]$, contrary to what happens with the rate of the work average studied in Ref. [8], being a constant only in the stationary state. If the system is also driven by an external GOU, a constant relation between the average and the variance is reached only in the stationary state as shown in Eq. (31), leading then to the validity of the total SPFT as established in Eq. (32). This constant relation is similar to that obtained for the rate of average work in the stationary state when the system is driven by a Gaussian white noise, as studied in Ref. [8]. In the presence of an electromagnetic field, the theorem also holds for a thermal GWN caused by the fluctuations of local electric fields. Here the external time-dependent electric field accounts for the uniform motion of the potential minimum. The relation between the average of the power and its variance satisfies the constant expression given by Eq. (55), which is exactly the same as that obtained in the absence of the magnetic field as given in Eq. (23) if $v^2 = u^2$. By considering the influence of an external GOU, the external electric field now accounts for two contributions, namely, a systematic force as before and a random contribution that satisfies an OU process. Again, the same as in the absence of the magnetic field, the total PFT is only valid in the asymptotic behavior. The bath temperature plays a role when the white internal noise is taken into account; however, if it is absent and just the external noise is applied, the system remains out of equilibrium and there is no balance mechanism as exists when the noise has an internal origin.

Lastly, we think that the experiment performed by Wang *et al*. [6] to demonstrate the total work fluctuation theorem for a colloidal particle surrounded by water molecules could be implemented in quite a similar way to demonstrate the total PFT, first in the absence of a magnetic field and for a thermal bath and then in the presence of this field. In the presence of additional external colored noise, our proposal suggests the validity of the total PFT only in the stationary state. In this case there is a possibility of performing similar experiments with colloidal particles (ordinary and charged).

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