Exact relation with two-point correlation functions and phenomenological approach for compressible magnetohydrodynamic turbulence

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Compressible isothermal magnetohydrodynamic turbulence is analyzed under the assumption of statistical homogeneity and in the asymptotic limit of large kinetic and magnetic Reynolds numbers. Following Kolmogorov we derive an exact relation for some two-point correlation functions which generalizes the expression recently found for hydrodynamics. We show that the magnetic field brings new source and flux terms into the dynamics which may act on the inertial range similarly as a source or a sink for the mean energy transfer rate. The introduction of a uniform magnetic field simplifies significantly the exact relation for which a simple phenomenology may be given. A prediction for axisymmetric energy spectra is eventually proposed.

I. INTRODUCTION

Hydrodynamic turbulence, despite its ubiquitous nature, is extremely complex to be studied analytically. The degree of complexity gets considerably enhanced when the system consists of a magnetohydrodynamic (MHD) fluid. Yet several theoretical (analytical) works have been carried out in the framework of both the above cases when one assumes the incompressibility (density of the fluid is constant in space and time and the velocity field is a solenoidal vector) of the fluid. It was Kolmogorov [1] who for the first time derived in 1941 an exact relation for the third-order moment of the velocity structure functions in incompressible hydrodynamic turbulence—the famous 4/5 law (see also Ref. [2]). Later two important exact relations were derived respectively for the conduction of a passive scalar in a turbulent system [3] and incompressible MHD turbulence [4]. Note that exact relations can also be derived for quasigeostrophic flows [5] or dispersive MHD [6,7], whereas attempts are currently made to model axisymmetric turbulence [8].

In a recent Letter [9] we derived an exact relation for compressible hydrodynamic turbulence with an isothermal analytic closure $P = c_s^2 \rho$, where P, ρ, and c_s are, respectively, the fluid pressure, density, and sound speed (for our case it is supposed to be constant) defined at every point of the flow field. In the present article we generalize our previous work to the MHD case. Of course, the taking into consideration of compressibility of an MHD fluid leads us closer to the reality taking place in, for example, the solar wind because (i) it is a plasma (so contains charged species), (ii) Ulysses data analysis has shown clear evidence of the effect of compressibility [10], and (iii) Voyager data analysis at several astronomical units (AU) reveals the presence of both fluctuations and jumps with spectra steeper than Kolmogorov [11]. Moreover, turbulence inside the interstellar clouds, being highly compressible (supersonic), also demands a theoretical background in order to be properly understood [12–14]. A thorough theoretical work in compressible MHD turbulence is essential (suggested in [15]) above all because it will help to analyze and understand the results of direct numerical simulations (DNSs) done for compressible turbulence [16–20].

A number of numerical works have been carried out over the last two decades on compressible hydrodynamic turbulence. Initiated mainly around 1990 [21,22], the numerical simulations have become considerably sophisticated in the last ten years. Recently, Kritsuk *et al.* [18] studied three-dimensional (3D) isothermal compressible hydrodynamic turbulence using the piecewise parabolic method (PPM) for up to 2048^3 grid points where the rms Mach number is around 6. Their study consists of the Helmholtz decomposition of velocity and they obtain a velocity power spectrum around -2 (-1.95, -1.92, -2.02, respectively, for total velocity and solenoidal and compressible velocity components) using 1024^3 grid points, whereas they have a $-5/3$ spectrum for the density-weighted fluid velocity $\rho^{1/3}v$. Another important numerical work was accomplished by Federrath *et al.* [19] for an identical (to the previous one) system of turbulence using up to 1024^3 grid points by applying compressible or solenoidal forcing. For two different types of forcing, they obtained remarkably different power spectra (which questions the universality) corresponding to various quantities such as the total velocity, its longitudinal and transverse components, $\ln \rho$, $\rho^{1/2}v$, or $\rho^{1/3}v$. Total velocity gives a -1.86 and a -1.94 power spectra in the supersonic range for solenoidal and compressible forcings, respectively, which are close to the Burgers spectrum (in k^{-2}). The compensated spectra in the subsonic region cannot be affirmed to date to be free from a bottleneck effect and so it is hard to conclude anything for that part. However, a $-5/3$ spectra is obtained for $\rho^{1/3}v$ even in supersonic range using solenoidal forcing whereas a compressible forcing gives a -2.1 spectrum. Here too a plausible presence of the bottleneck effect prevents us from inferring for the subsonic region. Recently, Aluie *et al.* [23] provided empirical evidence of kinetic energy cascade (which is not an invariant in compressible turbulence) using subsonic and transonic simulations. Such cascading takes place far in the inertial zone due to the very fast decay of the pressure-dilatation term which dominates at large scales. They do not give, however, any insight into supersonic turbulence for which *a priori* the kinetic energy cascade may not be the dominant process. In the meantime, a theoretical work based on coarse-graining [24] shows that for the kinetic energy a range of scales independent of the dissipation and the forcing may exist in compressible turbulence.

Three-dimensional DNS of compressible MHD turbulence is more difficult to perform since it requires the computation of a heavier system in which several new regimes are possible. In this framework, for example, Cho and Lazarian [25] implemented a third-order accurate hybrid and essentially nonoscillatory numerical scheme to solve the isothermal, ideal MHD equations in a periodic box, and to obtain scaling relations they drove turbulence with a solenoidal forcing in Fourier space using 256^3 grid points. In their study, they executed a mode-separation method in order to study the individual scaling behavior of each of the present linear modes in a compressible MHD system, i.e., the Alfvén mode and slow and fast magnetosonic (MS) modes. They reported a $-5/3$ velocity fluctuation spectrum for each of the Alfvén and slow MS modes whereas a $-3/2$ spectrum for fast MS mode. The nature of spectra obtained is found to depend very weakly on the value of the plasma β parameter (the ratio of kinetic pressure to magnetic pressure). An approximately $\ell^{2/3}$ (ℓ is the length scale) scaling is also obtained for the second-order structure function. In [26] a Kolmogorov spectrum ($-5/3$) was reported for both the velocity (v) and density-weighted velocity ($\rho^{1/3}v$) fluctuations for subsonic MHD turbulence whereas only the second quantity follows a $-5/3$ spectrum for supersonic turbulence.

A relevant work to our present paper is that of Kritsuk *et al.* [27] in which the authors studied numerically supersonic turbulence in magnetized magnetic clouds. The work is realized with grid resolutions from 256^3 to 1024^3 using PPM, an isothermal closure, and a solenoidal forcing to drive the fluid. Interestingly, they observed the scale invariance of the total (kinetic + compressible) energy transfer rate in the inertial zone; we shall use this concept later in this article for constructing the different spectra. The most interesting and important aspect of their article is to obtain numerically the linear scaling in the 4/3 law of incompressible MHD turbulence to the weakly magnetized supersonic turbulence—the former Elsässer variables ($\mathbf{z}^{\pm}$) being replaced by the density-weighted Elsässer variables ($\rho^{1/3}\mathbf{z}^{\pm}$). The present article will give, with the help of an exact relation for compressible isothermal MHD turbulence, theoretical support for this generalization.

The rest of the paper is organized as follows. Section II introduces the compressible MHD equations and the correlation functions; the derivation of the exact relation which is the main result of this paper is given in Sec. III; Sec. IV is devoted to a discussion about the main result where in particular we look at the effect of a uniform magnetic field; finally a conclusion is drawn in Sec. V.

II. COMPRESSIBLE MHD EQUATIONS

For constituting an exact relation of compressible MHD turbulence of an isothermal plasma, we rewrite the basic equations (except the continuity equation which is kept unchanged) of ideal MHD in an unusual form [28] as follows:

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \,, \tag{1}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \boldsymbol{\nabla} \, \mathbf{v} = \mathbf{v_A} \cdot \boldsymbol{\nabla} \, \mathbf{v_A} - \frac{1}{\rho} \boldsymbol{\nabla} \left(P + \frac{1}{2} \rho \mathbf{v_A}^2 \right)$$
$$- \mathbf{v_A} (\boldsymbol{\nabla} \cdot \mathbf{v_A}) + \mathbf{d_K} + \mathbf{f_0} \,, \tag{2}$$

$$\partial_t \mathbf{v_A} + \mathbf{v} \cdot \boldsymbol{\nabla} \, \mathbf{v_A} = \mathbf{v_A} \cdot \boldsymbol{\nabla} \, \mathbf{v} - \frac{\mathbf{v_A}}{2} (\boldsymbol{\nabla} \cdot \mathbf{v}) + \mathbf{d_M} \,, \tag{3}$$

$$\mathbf{v_A} \cdot \boldsymbol{\nabla} \, \rho = -2\rho (\boldsymbol{\nabla} \cdot \mathbf{v_A}) \,, \tag{4}$$

with ρ the plasma density, $\mathbf{v}$ the fluid velocity field, $\mathbf{v_A} = \mathbf{b}/\sqrt{\mu_0 \rho}$ the Alfvén velocity, and P the pressure. The terms $\mathbf{d_K}$, $\mathbf{d_M}$, and $\mathbf{f_0}$ represent, respectively, the contribution of the kinetic viscosity, magnetic resistivity, and the external forcing. It can be identified that Eqs. (2), (3), and (4) represent the force equation, the Faraday's equation (with idealized Ohm's law injected), and the zero divergence of magnetic field, respectively. For the sake of simplicity in the present paper we use the isothermal closure (i.e., $P = C_s^2 \rho$). It is important to note that $\mathbf{v_A}$ as it is defined in our case does not correspond to the usual Alfvén velocity which does not depend on the density but just on the mean density. Moreover, the Alfvén mode is defined only when there is a mean external magnetic field $\mathbf{B_0}$ whereas our $\mathbf{v_A}$ is defined independently of the presence of $\mathbf{B_0}$.

As in the hydrodynamic case, our analysis will be carried out in the physical space in terms of two-point correlation functions and structure functions, where the unprimed quantities represent the properties at the point $\mathbf{x}$ and the primed quantities correspond to the point $\mathbf{x}'$ (with $\mathbf{x}' = \mathbf{x} + \mathbf{r}$). Our system is supposed to be statistically homogeneous and to undergo a completely developed turbulence. The analysis is general and contains no hypothesis of isotropy.

To establish an exact relation, we shall utilize the conservation of the total energy whose density is given as

$$E(\mathbf{x}) = \frac{\rho}{2} (\mathbf{v} \cdot \mathbf{v} + \mathbf{v_A} \cdot \mathbf{v_A}) + \rho e \,, \tag{5}$$

where e accounts for the compressible energy term due to the isothermal closure and is expressed as $C_s^2 \ln(\rho/\rho_0)$, where ρ_0 is the initial equilibrium density. We, however, introduce another quantity which we call compressible cross-helicity density and is written as

$$H(\mathbf{x}) = \rho \mathbf{v} \cdot \mathbf{v_A} \,. \tag{6}$$

Although this quantity is not conserved in compressible MHD, we need it (as we shall see later) to express the final flux term in a more indicative way (in connection with the incompressible exact relations [4]). We can write the expressions for $E'(\mathbf{x}')$ and $H'(\mathbf{x}')$ in an identical way as that of the above.

III. DERIVATION OF AN EXACT RELATION

Following the formalism of our previous paper [9], we define the relevant two-point correlation functions associated with the total energy and the compressible cross-helicity density as

$$R_E = \frac{\rho}{2} (\mathbf{v} \cdot \mathbf{v}' + \mathbf{v_A} \cdot \mathbf{v_A}') + \rho e' \,, \tag{7}$$

$$R_H = \frac{\rho}{2} (\mathbf{v} \cdot \mathbf{v_A}' + \mathbf{v_A} \cdot \mathbf{v}') \,, \tag{8}$$

and similarly for the primed quantities. Using the above definitions and introducing compressible Elsässer variables $\mathbf{z}^{\pm} \equiv \mathbf{v} \pm \mathbf{v_A}$, we obtain

$$(E + E') \pm (H + H') - (R_E \pm R_H + R_E' \pm R_H') - \delta\rho\delta e$$
$$= \tfrac{1}{2} \delta(\rho \mathbf{z}^{\pm}) \cdot \delta \mathbf{z}^{\pm} \,, \tag{9}$$

where for any variable ψ, $\delta\psi \equiv \psi(\mathbf{x}+\mathbf{r}) - \psi(\mathbf{x}) \equiv \psi' - \psi$. Under ensemble averaging the above relations get reduced to

$$\langle E \rangle \pm \langle H \rangle - \tfrac{1}{2}\langle R_E \pm R_H + R'_E \pm R'_H \rangle - \tfrac{1}{2}\langle \delta\rho\,\delta e \rangle$$
$$= \tfrac{1}{4}\langle \delta(\rho\mathbf{z}^\pm)\cdot\delta\mathbf{z}^\pm \rangle. \qquad (10)$$

Now we shall calculate

$$\partial_t \langle R_E \pm R_H + R'_E \pm R'_H \rangle, \qquad (11)$$

to see how the incompressible relations [4] can be generalized to the compressible pseudoenergies ($e^\pm = \rho\mathbf{z}^\pm \cdot \mathbf{z}^\pm$) which are no more conserved. By using Eqs. (1)–(4) and basic identities of vector calculus, we obtain after some calculation

$$\partial_t(\rho\mathbf{v}\cdot\mathbf{v}') = -\,\nabla\cdot[\rho(\mathbf{v}\cdot\mathbf{v}')\mathbf{v}] + \nabla\cdot[\rho\mathbf{v_A}(\mathbf{v}'\cdot\mathbf{v_A})] - (\mathbf{v}'\cdot\mathbf{v_A})[\nabla\cdot(\rho\mathbf{v_A})] - \nabla\cdot(P\mathbf{v}') - \nabla\cdot\left(\tfrac{1}{2}\rho\mathbf{v_A}^2\mathbf{v}'\right)$$
$$-\,\rho(\mathbf{v}'\cdot\mathbf{v_A})(\nabla\cdot\mathbf{v_A}) - \nabla'\cdot[\rho(\mathbf{v}\cdot\mathbf{v}')\mathbf{v}'] + \rho(\mathbf{v}\cdot\mathbf{v}')(\nabla'\cdot\mathbf{v}') + \nabla'\cdot[\rho(\mathbf{v}\cdot\mathbf{v_A}')\mathbf{v_A}'] - \rho(\mathbf{v}\cdot\mathbf{v_A}')(\nabla'\cdot\mathbf{v_A}')$$
$$-\,\frac{\rho}{\rho'}\nabla'\cdot(P'\mathbf{v}) - \frac{\rho}{\rho'}\nabla'\left(\tfrac{1}{2}\rho'\mathbf{v_A}'^2\mathbf{v}\right) - \rho(\mathbf{v}\cdot\mathbf{v_A}')(\nabla'\cdot\mathbf{v_A}') + d_1 + f_1, \qquad (12)$$

$$\partial_t(\rho\mathbf{v_A}\cdot\mathbf{v_A}') = -\,\nabla\cdot[\rho(\mathbf{v_A}'\cdot\mathbf{v_A})\mathbf{v}] + \nabla\cdot[\rho\mathbf{v_A}(\mathbf{v}\cdot\mathbf{v_A}')] - (\mathbf{v_A}'\cdot\mathbf{v})[\nabla\cdot(\rho\mathbf{v_A})] - \tfrac{1}{2}\rho(\mathbf{v_A}\cdot\mathbf{v_A}')(\nabla\cdot\mathbf{v}) - \tfrac{1}{2}\rho(\mathbf{v_A}\cdot\mathbf{v_A}')(\nabla'\cdot\mathbf{v}')$$
$$-\,\nabla\cdot[\rho(\mathbf{v_A}\cdot\mathbf{v_A}')\mathbf{v}'] + \rho(\mathbf{v_A}\cdot\mathbf{v_A}')(\nabla'\cdot\mathbf{v}') + \nabla'\cdot[\rho(\mathbf{v_A}\cdot\mathbf{v}')\mathbf{v_A}'] - \rho(\mathbf{v_A}\cdot\mathbf{v}')(\nabla'\cdot\mathbf{v_A}') + d_2, \qquad (13)$$

$$\partial_t(\rho\mathbf{v}\cdot\mathbf{v_A}') = -\,\nabla\cdot[\rho\mathbf{v}(\mathbf{v}\cdot\mathbf{v_A}')] + \nabla\cdot[\rho\mathbf{v_A}(\mathbf{v_A}\cdot\mathbf{v_A}')] - (\mathbf{v_A}'\cdot\mathbf{v_A})[\nabla\cdot(\rho\mathbf{v_A})] - \nabla\cdot(P\mathbf{v_A}') - \nabla\cdot\left(\tfrac{1}{2}\rho v_A^2\mathbf{v_A}'\right)$$
$$-\,\rho(\mathbf{v_A}'\cdot\mathbf{v_A})(\nabla\cdot\mathbf{v_A}) - \nabla\cdot[\rho(\mathbf{v}\cdot\mathbf{v_A}')\mathbf{v}'] + \rho(\mathbf{v}\cdot\mathbf{v_A}')(\nabla'\cdot\mathbf{v}') + \nabla'\cdot[\rho(\mathbf{v}\cdot\mathbf{v}')\mathbf{v_A}']$$
$$-\,\rho(\mathbf{v}\cdot\mathbf{v}')(\nabla'\cdot\mathbf{v_A}') - \tfrac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v_A}')(\nabla'\cdot\mathbf{v}') + d_3 + f_2, \qquad (14)$$

$$\partial_t(\rho\mathbf{v_A}\cdot\mathbf{v}') = -\,\nabla\cdot[\rho\mathbf{v}(\mathbf{v}'\cdot\mathbf{v_A})] + \nabla\cdot[\rho\mathbf{v_A}(\mathbf{v}'\cdot\mathbf{v})] - (\mathbf{v}\cdot\mathbf{v}')[\nabla\cdot(\rho\mathbf{v_A})] - \tfrac{1}{2}\rho(\mathbf{v}'\cdot\mathbf{v_A})(\nabla\cdot\mathbf{v})$$
$$-\,\nabla'\cdot[\rho(\mathbf{v_A}\cdot\mathbf{v}')\mathbf{v}'] + (\nabla'\cdot\mathbf{v}')(\rho\mathbf{v_A}\cdot\mathbf{v}') + \nabla'\cdot[\rho(\mathbf{v_A}\cdot\mathbf{v_A}')\mathbf{v_A}'] - \rho(\mathbf{v_A}\cdot\mathbf{v_A}')(\nabla'\cdot\mathbf{v_A}')$$
$$-\,\frac{\rho}{\rho'}\nabla'\cdot[P'\mathbf{v_A}] - \frac{\rho}{\rho'}\nabla'\cdot\left[\tfrac{1}{2}\rho' v_A'^2\mathbf{v_A}\right] - \rho(\mathbf{v_A}\cdot\mathbf{v_A}')(\nabla'\cdot\mathbf{v_A}') + d_4 + f_3, \qquad (15)$$

$$\partial_t(\rho e') = -\,\nabla'\cdot(\rho e'\mathbf{v}') - \nabla\cdot(\rho e'\mathbf{v}) - \nabla'\cdot(P\mathbf{v}') + \rho e'(\nabla'\cdot\mathbf{v}'). \qquad (16)$$

where d_i and f_i ($i = 1$ to 4) represent the dissipative and forcing contributions, respectively. Hence we have all the necessary elements for the dynamic equation of R_E and R_H. Similarly we can write (by symmetry) the expressions of R'_E and R'_H just by carefully interchanging the primed and the unprimed quantities. Now if we apply a statistical averaging operator on each term, after judicious rearrangement of all the terms we finally obtain

$$\partial_t\langle R_E \pm R_H + R'_E \pm R'_H \rangle = \nabla_r\cdot\left\langle\left[\tfrac{1}{2}\delta(\rho\mathbf{z}^\pm)\delta\mathbf{z}^\pm + \delta\rho\,\delta e\right]\delta\mathbf{z}^\mp + \overline{\delta}\left(e + \frac{v_A^2}{2}\right)\delta(\rho\mathbf{z}^\pm)\right\rangle - \frac{1}{4}\left\langle\frac{v_A'^2}{c_s^2}\nabla'\cdot(\rho\mathbf{z}^\pm e') + \frac{v_A^2}{c_s^2}\nabla\cdot(\rho'\mathbf{z}'^\pm e)\right\rangle$$

$$+\left\langle(\nabla\cdot\mathbf{v})\left[R'_E \pm R'_H - E' \mp H' \mp \frac{\overline{\delta}\rho}{2}(\mathbf{v_A}\cdot\mathbf{z}'^\pm) - \frac{P'}{2} + \frac{P'_M}{2}\right]\right\rangle$$

$$+\left\langle(\nabla'\cdot\mathbf{v}')\left[R_E \pm R_H - E \mp H \mp \frac{\overline{\delta}\rho}{2}(\mathbf{v_A}'\cdot\mathbf{z}^\pm) - \frac{P}{2} + \frac{P_M}{2}\right]\right\rangle$$

$$+\left\langle(\nabla\cdot\mathbf{v_A})\left[R_H \pm R_E - R'_H \mp R'_E \pm E' + H' - \overline{\delta}\rho(\mathbf{v_A}\cdot\mathbf{z}'^\pm) \pm \frac{5P'}{2} \pm \frac{P'_M}{2}\right]\right\rangle$$

$$+\left\langle(\nabla'\cdot\mathbf{v_A}')\left[R'_H \pm R'_E - R_H \mp R_E \pm E + H - \overline{\delta}\rho(\mathbf{v_A}'\cdot\mathbf{z}^\pm) \pm \frac{5P}{2} \pm \frac{P_M}{2}\right]\right\rangle + D_\pm + F_\pm, \qquad (17)$$

where $\overline{\delta}X \equiv (X + X')/2$ and $P_M = (1/2)\rho v_A^2$ is the magnetic pressure. $D_\pm$ and $F_\pm$ represent, respectively, the averaged dissipative and forcing terms. The above expression consists of two equations which represent the generalized form (to compressible MHD) of the coupled equations (4) and (6) of Ref. [4]. As it was already shown [28], for compressible flows the cross-helicity is no longer conserved and we cannot [by Eq. (6)] let the term $\partial_t\langle R_E \pm R_H + R'_E \pm R'_H \rangle$ vanish to the extent

of a stationary state where the average forcing term and the average resultant dissipative (kinetic + magnetic) terms cancel each other to ensure the conservation of total energy. Hence for the compressible case, these two equations individually do not lead to an exact relation comprising of a universal quantity. However, if we add both we find

$$\partial_t \langle R_E + R'_E \rangle = \frac{1}{2} \boldsymbol{\nabla}_r \cdot \left\langle \left[\frac{1}{2} \delta(\rho \mathbf{z}^-) \cdot \delta \mathbf{z}^- + \delta \rho \delta e \right] \delta \mathbf{z}^+ + \left[\frac{1}{2} \delta(\rho \mathbf{z}^+) \cdot \delta \mathbf{z}^+ + \delta \rho \delta e \right] \delta \mathbf{z}^- + \bar{\delta} \left(e + \frac{v_A^2}{2} \right) \delta(\rho \mathbf{z}^- + \rho \mathbf{z}^+) \right\rangle$$

$$- \frac{1}{8} \left\langle \frac{v_A'^2}{c_s^2} \boldsymbol{\nabla}' \cdot (\rho \mathbf{z}^+ e') + \frac{v_A^2}{c_s^2} \boldsymbol{\nabla} \cdot (\rho' \mathbf{z}'^+ e) + \frac{v_A'^2}{c_s^2} \boldsymbol{\nabla}' \cdot (\rho \mathbf{z}^- e') + \frac{v_A^2}{c_s^2} \boldsymbol{\nabla} \cdot (\rho' \mathbf{z}'^- e) \right\rangle$$

$$+ \left\langle (\boldsymbol{\nabla} \cdot \mathbf{v}) \left[R'_E - E' - \frac{\bar{\delta}\rho}{2} (\mathbf{v_A}' \cdot \mathbf{v_A}) + \frac{P'_M - P'}{2} \right] \right\rangle + \left\langle (\boldsymbol{\nabla}' \cdot \mathbf{v}') \left[R_E - E - \frac{\bar{\delta}\rho}{2} (\mathbf{v_A} \cdot \mathbf{v_A}') + \frac{P_M - P}{2} \right] \right\rangle$$

$$+ \langle (\boldsymbol{\nabla} \cdot \mathbf{v_A}) [R_H - R'_H + H' - \bar{\delta}\rho(\mathbf{v}' \cdot \mathbf{v_A})] \rangle + \langle (\boldsymbol{\nabla}' \cdot \mathbf{v_A}') [R'_H - R_H + H - \bar{\delta}\rho(\mathbf{v} \cdot \mathbf{v_A}')] \rangle + D + F \,, \quad (18)$$

where $D = (D_+ + D_-)/2$ and $F = (F_+ + F_-)/2$. Now to establish the final relation, we assume that in the limit of infinite (kinetic and magnetic) Reynolds numbers, the system attains a stationary state. This assumption, when applied to the relation (18), shows that the left-hand term of the above equation vanishes. Now if we concentrate far in the inertial zone where the dissipative terms are negligible with respect to the other terms, we are left with

$$-2\varepsilon = \frac{1}{2} \boldsymbol{\nabla}_r \cdot \left\langle \left[\frac{1}{2} \delta(\rho \mathbf{z}^-) \cdot \delta \mathbf{z}^- + \delta \rho \delta e \right] \delta \mathbf{z}^+ + \left[\frac{1}{2} \delta(\rho \mathbf{z}^+) \cdot \delta \mathbf{z}^+ + \delta \rho \delta e \right] \delta \mathbf{z}^- + \bar{\delta} \left(e + \frac{v_A^2}{2} \right) \delta(\rho \mathbf{z}^- + \rho \mathbf{z}^+) \right\rangle$$

$$- \frac{1}{8} \left\langle \frac{1}{\beta'} \boldsymbol{\nabla}' \cdot (\rho \mathbf{z}^+ e') + \frac{1}{\beta} \boldsymbol{\nabla} \cdot (\rho' \mathbf{z}'^+ e) + \frac{1}{\beta'} \boldsymbol{\nabla}' \cdot (\rho \mathbf{z}^- e') + \frac{1}{\beta} \boldsymbol{\nabla} \cdot (\rho' \mathbf{z}'^- e) \right\rangle$$

$$+ \left\langle (\boldsymbol{\nabla} \cdot \mathbf{v}) \left[R'_E - E' - \frac{\bar{\delta}\rho}{2} (\mathbf{v_A}' \cdot \mathbf{v_A}) + \frac{P'_M - P'}{2} \right] \right\rangle + \left\langle (\boldsymbol{\nabla}' \cdot \mathbf{v}') \left[R_E - E - \frac{\bar{\delta}\rho}{2} (\mathbf{v_A} \cdot \mathbf{v_A}') + \frac{P_M - P}{2} \right] \right\rangle$$

$$+ \langle (\boldsymbol{\nabla} \cdot \mathbf{v_A}) [R_H - R'_H + H' - \bar{\delta}\rho(\mathbf{v}' \cdot \mathbf{v_A})] \rangle + \langle (\boldsymbol{\nabla}' \cdot \mathbf{v_A}') [R'_H - R_H + H - \bar{\delta}\rho(\mathbf{v} \cdot \mathbf{v_A}')] \rangle \,, \quad (19)$$

where ε denotes the mean total energy injection rate (which is equal to the mean total energy dissipation rate). In the above equation the flux terms are deliberately written in terms of the compressible Elsässer variables $\mathbf{z}^\pm$ whereas the source terms are expressed in terms of the velocity and the compressible Alfvén velocity. This writing is expected to be useful in (i) attempting to generalize the Alfvén effect (introduced by Kraichnan) to describe the phenomenology of compressible MHD turbulence and (ii) understanding separately the contribution of the velocity field and the Alfvén term in the source term. It is interesting to note that in the second part of our flux, the β parameter (i.e., c_s^2/v_A^2) appears which can help us understand (or obtain) the different limits depending upon its value. The above equation, being considerably bulky, is laborious to handle. For quick references, we give below a schematic view of that equation:

$$-2\varepsilon = \left\langle \Phi_1 + \frac{1}{\beta} \Phi_2 + (\boldsymbol{\nabla} \cdot \mathbf{v}) S_1 + (\boldsymbol{\nabla} \cdot \mathbf{v_A}) S_2 \right\rangle. \quad (20)$$

IV. DISCUSSION

A. Relevant limits

Equation (19) is the main result of the paper. It is an exact relation for compressible isothermal MHD turbulence in terms of correlation functions. From this equation we can immediately investigate particular limits.

1. Incompressible MHD

In this limit, all the source terms vanish. Moreover, the compressible energy term becomes zero (evident from its definition as for the incompressible case $\rho = \rho_0$) and we get finally (normalizing the density to unity)

$$-2\varepsilon = \frac{1}{4} \boldsymbol{\nabla}_r \cdot \langle (\delta \mathbf{z}^-)^2 \delta \mathbf{z}^+ + (\delta \mathbf{z}^+)^2 \delta \mathbf{z}^- \rangle. \quad (21)$$

Note that the term $\boldsymbol{\nabla}_r \cdot \langle \bar{\delta}(v_A^2/2)\delta(\mathbf{z}^- + \mathbf{z}^+) \rangle$ can be shown to be zero in the incompressible case by expanding it and letting the $\boldsymbol{\nabla}_r$ enter inside the average operator in a judicious manner. A rewriting of the $\mathbf{z}^\pm$ in terms of $\mathbf{v}$ and $\mathbf{b}$ (assuming the constant density to be normalized to unity) helps us recognize the primitive form of the equations found in [4].

2. Compressible hydrodynamics

To verify the hydrodynamic limit, we put $\mathbf{v_A} = 0$ and so $\mathbf{z}^+ = \mathbf{z}^- = \mathbf{v}$; we are left with

$$-2\varepsilon = \boldsymbol{\nabla}_r \cdot \left\langle \left[\frac{1}{2} \delta(\rho \mathbf{v}) \cdot \delta \mathbf{v} + \delta \rho \delta e \right] \delta \mathbf{v} + \bar{\delta} e \delta(\rho \mathbf{v}) \right\rangle$$

$$+ \left\langle (\boldsymbol{\nabla} \cdot \mathbf{v}) \left[R'_E - E' - \frac{P'}{2} \right] \right\rangle$$

$$+ \left\langle (\boldsymbol{\nabla}' \cdot \mathbf{v}') \left[R_E - E - \frac{P}{2} \right] \right\rangle. \quad (22)$$

It is nothing but the exact relation which was derived in [9]. The only difference between the relation obtained above and that in the published paper is that here the pressure terms are written as the source terms whereas previously those were considered to contribute in flux terms.

3. High and low β plasmas

Without a problem we admit that in the limit where the β parameter of the plasma tends to infinity (very large value), i.e., the plasma becomes almost incompressible (although not entirely), the flux term Φ_2/β becomes negligible with respect to Φ_1 of Eq. (20). On the contrary for a very small β value, where the plasma can be assumed to be cold and magnetized (kinetic pressure negligible with respect to magnetic pressure), the term Φ_2/β dominates over Φ_1 term and at that situation the effective flux term becomes (after

rearrangement)

$$-\frac{1}{4}\left\langle \frac{1}{\beta'}\nabla'\cdot(\rho \mathbf{v}e') + \frac{1}{\beta}\nabla\cdot(\rho'\mathbf{v}'e)\right\rangle. \tag{23}$$

B. Presence of an external magnetic field

Relation (19) comprises the total magnetic field at each point of the flow field. This total field $\mathbf{b}$ at each point can be supposed (a realistic case) to have a fluctuating part (vary in space and time) $\tilde{\mathbf{b}}$ superimposed on a constant external magnetic field $\mathbf{B_0}$. In the following, we shall investigate the flux and the source terms under the said situation.

1. Flux contribution

The part of the total flux term which contains $\mathbf{B_0}$ can be expressed as (with $\mu_0 = 1$)

$$\langle\Phi_{\mathbf{B_0}}\rangle = \frac{\nabla_r}{2}\cdot\left\langle \delta\left(\frac{1}{\sqrt{\rho}}\right)\delta(\sqrt{\rho})[\mathbf{B_0}\times(\delta\mathbf{v}\times\mathbf{B_0})] + \delta(\sqrt{\rho})[\mathbf{B_0}\times(\delta\mathbf{v}\times\delta\tilde{\mathbf{v}}_\mathbf{A})] - \delta^2\left(\frac{1}{\sqrt{\rho}}\right)[\delta(\rho\mathbf{v})\cdot\mathbf{B_0}]\mathbf{B_0}\right.$$
$$\left. + \delta\left(\frac{1}{\sqrt{\rho}}\right)[\delta(\rho\tilde{\mathbf{v}}_\mathbf{A})\times(\delta\mathbf{v}\times\mathbf{B_0}) + \delta(\rho\mathbf{v})\times(\delta\tilde{\mathbf{v}}_\mathbf{A}\times\mathbf{B_0})] + B_0^2\bar{\delta}\left(\frac{1}{\rho}\right)\delta(\rho\mathbf{v}) + 2\left[\mathbf{B_0}\cdot\bar{\delta}\left(\frac{\tilde{\mathbf{v}}_\mathbf{A}}{\sqrt{\rho}}\right)\right]\delta(\rho\mathbf{v})\right\rangle$$
$$- \frac{1}{2}\left\langle \frac{B_0^2}{2\rho'^2}\rho\mathbf{v}\cdot\nabla'\rho' + \frac{\mathbf{B_0}\cdot\tilde{\mathbf{b}}}{\rho'^2}\rho\mathbf{v}\cdot\nabla'\rho' + \frac{B_0^2}{2\rho^2}\rho'\mathbf{v}'\cdot\nabla\rho + \frac{\mathbf{B_0}\cdot\tilde{\mathbf{b}}'}{\rho^2}\rho'\mathbf{v}'\cdot\nabla\rho\right\rangle, \tag{24}$$

where $\tilde{\mathbf{v}}_\mathbf{A} = \tilde{\mathbf{b}}/\sqrt{\rho}$. Now we assume the external field $\mathbf{B_0}$ to be very strong so that $B_0 \gg |\tilde{\mathbf{b}}|$ (and also $B_0 \gg |\delta\mathbf{v}|$). This situation is classical in astrophysics: for example, there are many evidences of magnetic loops in the turbulent solar corona which are interpreted as strong uniform magnetic fields on which small magnetic and velocity fluctuations are present [29,30]. We shall just consider the terms weighted by B_0^2. After some straightforward calculations, we obtain the resultant flux term (the magnetic terms without B_0 and with single power of B_0 are neglected) which writes at main order

$$\langle\Phi_{\mathbf{B_0}}\rangle = \frac{\nabla_r}{2}\cdot\left\langle \delta\left(\frac{1}{\sqrt{\rho}}\right)\delta(\sqrt{\rho})[\mathbf{B_0}\times(\delta\mathbf{v}\times\mathbf{B_0})]\right.$$
$$\left. - \left[\delta\left(\frac{1}{\sqrt{\rho}}\right)\right]^2[\delta(\rho\mathbf{v})\cdot\mathbf{B_0}]\mathbf{B_0}\right\rangle. \tag{25}$$

The above expression gives the modifying part of the flux in the presence of a strong constant magnetic field applied externally. One can easily understand that the modification is purely due to compressibility. [Note that the pure kinetic terms, not shown in Eq. (25), give also a contribution to the total flux.] It is also interesting to notice that in expression (25) the fluctuations are exclusively kinetic in nature (because of the absence of, e.g., a Hall-type term in the basic equations). One can easily verify that the $\delta\mathbf{v}$ of the first term and the $\delta(\rho\mathbf{v})$ of the second term of the above expression can be replaced by $\delta\mathbf{v}_\perp$ and $\delta(\rho\mathbf{v}_\parallel)$, respectively, where $\delta\mathbf{v}_\perp\perp\mathbf{B_0}$ and $\delta\mathbf{v}_\parallel\|\mathbf{B_0}$. The pure kinetic terms can, however, be omitted by assuming the external magnetic contribution to be dominant with respect to

the velocity and the density fluctuations and then expression (25) will represent the total flux contribution.

2. Source contribution

The source terms are also modified due to the effect of a strong external magnetic field. At main order (keeping only the terms in B_0^2), the terms of type $\langle(\nabla\cdot\mathbf{v})S_1\rangle$ get reduced to

$$\langle\Psi_\mathbf{v}\rangle = \frac{B_0^2}{2}\left\langle \delta(\nabla\cdot\mathbf{v})\delta\left(\frac{1}{\sqrt{\rho}}\right)\bar{\delta}(\sqrt{\rho}) - \bar{\delta}(\nabla\cdot\mathbf{v})\right\rangle, \tag{26}$$

and the source terms like $\langle(\nabla\cdot\mathbf{v_A})S_2\rangle$ write

$$\langle\Psi_{\mathbf{v_A}}\rangle$$
$$= \mathbf{B_0}\cdot\left\langle\nabla\left(\frac{1}{\sqrt{\rho}}\right)\left[(\mathbf{B_0}\cdot\mathbf{v}')\left\{\rho'\delta\left(\frac{1}{\sqrt{\rho}}\right)\right\} - (\mathbf{B_0}\cdot\mathbf{v})\frac{\delta\rho}{2\sqrt{\rho'}}\right]\right.$$
$$\left. - \nabla'\left(\frac{1}{\sqrt{\rho'}}\right)\left[(\mathbf{B_0}\cdot\mathbf{v})\left\{\rho\delta\left(\frac{1}{\sqrt{\rho}}\right)\right\} - (\mathbf{B_0}\cdot\mathbf{v}')\frac{\delta\rho}{2\sqrt{\rho}}\right]\right\rangle. \tag{27}$$

We note that the latter expression implies only parallel components of the velocity.

3. Reduced anisotropic law

Further simplifications are possible if we assume that the velocity field vector at each point of the flow field is (at the main order) perpendicular to the external magnetic field, i.e., if $v_\parallel = v'_\parallel = 0$, and therefore $\mathbf{v}\equiv\mathbf{v_\perp}$. In that case, $\langle\Psi_{\mathbf{v_A}}\rangle$ vanishes and so does the second term of $\langle\Phi_{B_z}\rangle$; then the corresponding

reduced exact relation can be simply written as

$$
-4\varepsilon = B_0^2 \nabla_{r_\perp} \cdot \left\langle \delta\left(\frac{1}{\sqrt{\rho}}\right) \delta(\sqrt{\rho}) \delta \mathbf{v}_\perp \right\rangle
$$
$$
- \frac{B_0^2}{2} \left\langle (\nabla_\perp \cdot \mathbf{v}_\perp)\left(1 + \sqrt{\frac{\rho}{\rho'}}\right) + (\nabla'_\perp \cdot \mathbf{v}'_\perp)\left(1 + \sqrt{\frac{\rho'}{\rho}}\right) \right\rangle,
$$
$$\text{(28)}$$

where $\nabla_\perp$ implies derivatives transverse to $\mathbf{B}_0$. Expression (28) is the second main result of this paper: it is a limiting case of the exact relation (19) for isothermal compressible MHD turbulence under the influence of a strong external uniform magnetic field (sub-Alfvénic turbulence regime). We may simplify the previous equation by assuming axisymmetry; the exact relation (28) can be written symbolically as (by using cylindrical coordinates)

$$
-4\varepsilon = \frac{1}{r_\perp} \partial_{r_\perp}(r_\perp \mathcal{F}_{r_\perp}) + \mathcal{S}, \qquad \text{(29)}
$$

where $\mathcal{F}_{r_\perp}$ is the radial component of the energy flux vector [first term in the right-hand side of Eq. (28)] and $\mathcal{S}$ is a sourceor sink term [last term in the right-hand side of Eq. (28)]. If we define an effective mean total energy injection rate as $\varepsilon_{\text{eff}} \equiv \varepsilon + \mathcal{S}/4$, a simple interpretation of expression (29) can be proposed as we see in Fig. 1; whereas for a direct cascade the energy flux vectors are oriented towards the axis of the cylinder, dilatation and compression are additional effects which act, respectively, in the opposite or in the same direction as the flux vectors (since terms like $1 + \sqrt{\rho'/\rho}$ are positive).
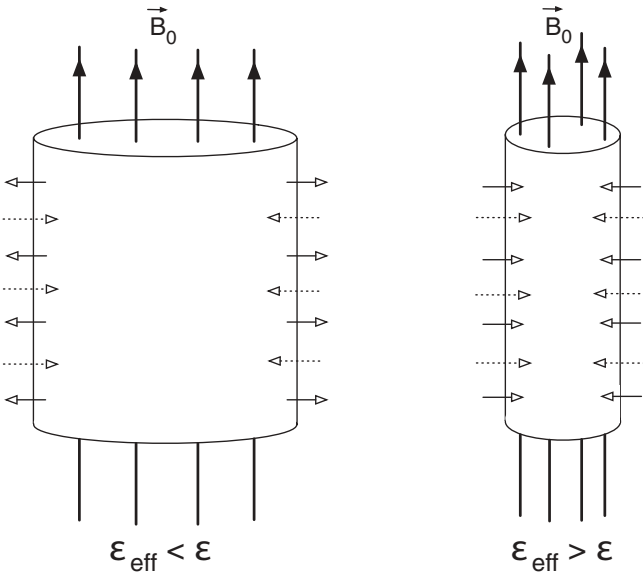

FIG. 1. Dilatation (left) and compression (right) phases in space correlation for axisymmetric MHD turbulence. In a direct cascade scenario the flux vectors (dotted arrows) are oriented towards the axis of the cylinder. Dilatation and compression (solid arrows) are additional effects which act, respectively, in the opposite or in the same direction as the flux vectors.

4. *Phenomenology and energy spectra*

In this section we shall make a prediction on compressible spectra using the derived theoretical relation (28); although simplified, it is considerably indicative of phenomenological insight. The existence of strong external magnetic field $\mathbf{B}_0$ renders the energy transfer in the parallel (along $\mathbf{B}_0$) direction negligible in comparison to the transverse one [31–36]. Then, our prediction will be made for transverse spectra (i.e., with a spectral dependence only in $k_\perp$). From the flux term of expression (28), we find dimensionally (keeping the source terms aside)

$$
\varepsilon \sim \frac{\rho_\ell v_{A\ell}^2 v_{\perp\ell}}{\ell_\perp} \sim \frac{\rho_\ell v_{A\ell}^2}{\ell_\perp/v_{\perp\ell}}. \qquad \text{(30)}
$$

The above expression corresponds to a compressible phenomenology in which the magnetic energy density, $\rho_\ell v_{A\ell}^2$, is transferred through the scales at the transfer time, $\tau_\ell \sim \ell_\perp/v_{\perp\ell}$, and the transfer rate ε. We may define the weighted variables

$$
\mathcal{U}_\ell \equiv \rho_\ell^{1/3} v_{\perp\ell}, \qquad \text{(31)}
$$

$$
\mathcal{B}_\ell \equiv \rho_\ell^{1/3} v_{A\ell} = \rho_\ell^{-1/6} B_0, \qquad \text{(32)}
$$

which leads to

$$
\varepsilon \sim \mathcal{B}_\ell^2 \mathcal{U}_\ell k_\perp \sim E^{\mathcal{B}}(k_\perp) k_\perp \sqrt{E^{\mathcal{U}}(k_\perp) k_\perp} \, k_\perp. \qquad \text{(33)}
$$

Whence the spectral prediction

$$
E^{\mathcal{B}}(k_\perp)\sqrt{E^{\mathcal{U}}(k_\perp)} \sim \varepsilon k_\perp^{-5/2}. \qquad \text{(34)}
$$

Taking into consideration the source terms will only modify ε for giving an effective mean total energy injection rate ε_{eff} (likewise in compressible hydrodynamics [9]). A compressional effect ($\nabla \cdot \mathbf{v} < 0$) will increase the effective energy flux rate whereas a dilatational effect ($\nabla \cdot \mathbf{v} > 0$) will reduce ε_{eff} as is evident from the source terms of expression (28). As stated before, a simple picture emerges (see Fig. 1) with the additional hypothesis of axisymmetry (symmetry in the plane perpendicular to $\mathbf{B}_0$). The flux vectors (dotted arrows) are always oriented towards the axis of the cylinder since a direct cascade is expected. Dilatation and compression act additionally (solid arrows); in the first case, the effect is similar to a decrease of the local mean total energy transfer rate whereas in the second case it is similar to an increase of the local mean total energy transfer rate. Then, the final prediction for compressible turbulence under a strong uniform magnetic field is

$$
E^{\mathcal{B}}(k_\perp)\sqrt{E^{\mathcal{U}}(k_\perp)} \sim \varepsilon_{\text{eff}}(k_\perp) k_\perp^{-5/2}, \qquad \text{(35)}
$$

with *a priori* a possible dependence of the effective mean total energy transfer rate on $k_\perp$. In that case, a power law steeper than $-5/2$ may be observed at large scales when compressible MHD turbulence becomes supersonic.

V. CONCLUSION

The present paper is a generalization of a recent work made for compressible hydrodynamics [9] in which it was possible to give a simple interpretation of the different terms of the exact relation. The introduction of both the Lorentz force and the induction equation renders the situation more difficult to

analyze but it is believed that the main physical characteristic of compressible MHD emerges when a uniform magnetic field is included. In that case, we have derived rigorously a reduced expression (28) which can be used easily to analyze either direct numerical simulations or *in situ* observations like those obtained in the solar wind. Since most of the natural plasmas are in interaction with a large-scale magnetic field, it is likely that the reduced form is the most interesting one.

Our work confirms the relevance of density-weighted Elsässer variables, $\rho^{1/3}\mathbf{z}^\pm$, for compressible MHD turbulence. However, unlike the hydrodynamic case, here one should not replace $\mathbf{z}^\pm$ by $\rho^{1/3}\mathbf{z}^\pm$ for recovering the incompressible scaling laws. Instead the appropriate variable should be $\rho^{1/3}(\mathbf{z}^{+2}\mathbf{z}^- + \mathbf{z}^{-2}\mathbf{z}^+)^{1/3}$ for obtaining a satisfactory energy scaling as suggested by relation (19). The said relation holds good for any type of compressible MHD turbulence (subsonic, transonic, supersonic, sub-Alfvénic or super-Alfvénic). Hence it can directly be used to analyze direct numerical simulations in compressible MHD turbulence at different Mach numbers or for plasmas with different β parameters in order to understand the underlying phenomenology. To approach the astrophysical turbulence more efficiently we have to take into account the gravitational force and also a polytropic closure although they would increase enormously the complexity of the analysis.

It was mentioned in the past that constant flux solutions are in general not relevant for compressible turbulence because the coupling to a sonic field provides a supplementary sink which will modify the inertial index [37–39]. A modification of the solenoidal energy spectrum (for hydrodynamic turbulence) was proposed by introducing a simple *ad hoc* model in which the main compressible parameter was the Mach number. Here, we may reach a similar conclusion with a rigorous analysis in terms of two-point correlation functions. We believe that our interpretation in terms of effective energy flux rate, source, and flux terms is the key to further understanding compressible turbulence.

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