# Proposal of a critical test of the Navier-Stokes-Fourier paradigm for compressible fluid continua

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A critical, albeit simple experimental and/or molecular-dynamic (MD) simulation test is proposed whose outcome would, in principle, establish the viability of the Navier-Stokes-Fourier (NSF) equations for compressible fluid continua. The latter equation set, despite its longevity as constituting the fundamental paradigm of continuum fluid mechanics, has recently been criticized on the basis of its failure to properly incorporate volume transport phenomena—as embodied in the proposed bivelocity paradigm [H. Brenner, Int. J. Eng. Sci. **54**, 67 (2012)]—into its formulation. Were the experimental or simulation results found to accord, even only qualitatively, with bivelocity predictions, the temperature distribution in a gas-filled, thermodynamically and mechanically isolated circular cylinder undergoing steady rigid-body rotation in an inertial reference frame would not be uniform; rather, the temperature would be higher at the cylinder wall than along the axis of rotation. This radial temperature nonuniformity contrasts with the uniformity of the temperature predicted by the NSF paradigm for these same circumstances. Easily attainable rates of rotation in centrifuges and readily available tools for measuring the expected temperature differences render experimental execution of the proposed scheme straightforward in principle. As such, measurement—via experiment or MD simulation—of, say, the temperature difference $\Delta T$ between the gas at the wall and along the axis of rotation would provide quantitative tests of both the NSF and bivelocity hydrodynamic models, whose respective solutions for the stated set of circumstances are derived in this paper. Independently of the correctness of the bivelocity model, any temperature difference observed during the proposed experiment or simulation, irrespective of magnitude, would preclude the possibility of the NSF paradigm being correct for fluid continua, except for incompressible flows.

## I. INTRODUCTION

Though it is commonly believed that the foundations of continuum fluid mechanics [1] (and transport processes in general [2]) rest on firm theoretical and experimental grounds, this belief is, in fact, unjustified when one includes situations where the fluid is compressible [3]. That is, the only experimental data that appear to unequivocally support the current model of continuum fluid mechanics, namely, the Navier-Stokes-Fourier (NSF) equations, are those pertaining to incompressible flows. Indeed, this fact is well known to gas kineticists [3] concerned, *inter alia*, with the flow of rarefied gases, in which field of study there currently exists no uncontested, generally applicable, macroscopic fluid-mechanical theory. This lack of a satisfactory hydrodynamic foundation is compounded by the fact that the traditional no-slip boundary condition generally imposed upon fluids at solid surfaces, whether NSF or bivelocity fluids, proves to be inapplicable when dealing with compressible gases [4].

The lack of a broadly applicable macroscopic fluid-mechanical theory embracing both incompressible and compressible fluids reflects a major gap in our current understanding of the foundations of hydrodynamics. It was in an attempt to close this gap that the original bivelocity hydrodynamic model [5] (later modified [6]) came into being. The current status of that model, based solidly upon the widely accepted macroscopic principles of linear irreversible thermodynamics (LIT) [7], was recently summarized [8] in a comprehensive review of the pertinent literature. (See also Refs. [9] and [10], which arrive at exactly these same bivelocity

equations [8], starting, however, from a molecular rather than macroscopic basis—the former involving the addition of a stochastic contribution to the collisional term in Boltzmann's gas-kinetic equation [11].) According to its macroscopic LIT-based derivation [8], bivelocity fluid mechanics is applicable to both liquids and gases. We focus here exclusively on gases owing to the greater abundance of high-quality experimental data pertinent to compressible gaseous flows, as well as because of the relatively large magnitude of compressibility effects in gases compared with those for liquids. And the larger the effect, the more readily is the modified theory likely to be accepted as pertinent to the issues at hand.

The subsequent bivelocity analysis developed herein predicts (when body forces such as gravity are absent or negligible) that the temperature of a gas undergoing steady, rigid-body rotation relative to an inertial reference frame in a circular cylinder possessing rigid, non-heat-conducting (i.e., insulated) walls will be *nonuniform*, with the temperature increasing radially with distance from the axis of rotation. This nonisothermal bivelocity prediction runs counter to NSF predictions of isothermalicity. That is, according to current beliefs, under these same conditions of thermodynamic isolation from its surroundings, the temperature will be uniform throughout the rotating fluid [12,13]. The issue thus focuses on whether rigid-body rotation occurring in an isolated system constitutes a thermodynamically reversible process, wherein dissipative processes (i.e., entropy production) are absent, such as is predicted to be the case for NSF fluids. Based upon statistical-mechanical arguments these issues are discussed at length by Landau and Lifshitz in their treatise on statistical physics [14].

Were temperature gradients discovered, either experimentally or through simulation, to exist within the rotating

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gas—such as will be seen to be the case for the bivelocity model—the gas would, by definition, presumably be in a state of thermodynamic disequilibrium. In terms of the possibility of observing nonuniformities in temperature, the proposed experiment thus serves to also test the prevailing, and more general, hypothesis [12–14] that isolated rigid-body fluid motions constitute states of thermodynamic equilibrium for *all* fluids. Even was the temperature distribution predicted by bivelocity theory to prove quantitatively wrong, experimental observation of *any* temperature nonuniformity, whatever its magnitude or direction, would suffice to discredit the currently accepted isothermalicity hypothesis ascribed to rigid-body fluid motions occurring in isolated systems. In turn, this would refute the claim of their thermodynamically reversible nature. Furthermore, this nonisothermal finding would also serve to undermine the prevailing entropy-based criticism [12,13] of the bivelocity model—according to which *any* fluid-mechanical model is, *ipso facto,* physically incorrect if its tenets predict the production of entropy during steady, isolated, rigid-body rotations.

In this present rotating fluid context the proposed experiment, establishing the temperature difference $\Delta T$ between the gas at the cylinder wall and along the axis of rotation, thus provides a simple, quantitative test of one of the fundamental tenets of fluid mechanics, applicable to all fluids, whether NSF, bivelocity, or otherwise. An approximate, albeit accurate, perturbation solution of the bivelocity equations for the stated set of circumstances is derived in subsequent sections of this paper.

Let $R$ denote the distance from the axis of rotation of the cylinder, $R_o$ the cylinder radius, and $T(R)$ the temperature field. A motivating objective in this paper is to calculate the magnitude of the expected temperature difference

$$\Delta T := T(R_o) - T(0) \qquad (1.1)$$

on the basis of belief in the principles of bivelocity theory, and hence to thereby suggest the feasibility of performing the proposed simulation or experiment over a wide range of operable test conditions.

## II. BIVELOCITY EQUATIONS

Relative to an observer fixed in an inertial reference frame (i.e., relative to the fixed stars) the conservation equations governing mass, momentum, and energy transport in all single-component fluids undergoing steady, time-independent flows in the absence of body forces are, respectively [8],

$$\nabla \cdot (\rho \mathbf{v}) = \mathbf{0}, \qquad (2.1)$$

$$\nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla \cdot \mathbf{P}, \qquad (2.2)$$

and

$$\nabla \cdot (\rho \mathbf{v} \hat{e}) = -\nabla \cdot \mathbf{j}_e, \qquad (2.3)$$

in which $\rho$ is the fluid's density and $\mathbf{v}$ the fluid's mass velocity. Furthermore,

$$\mathbf{P} = \mathbf{I}p - \mathbf{T} \qquad (2.4)$$

is the pressure tensor, in which $p$ is the pressure and $\mathbf{T}$ is the deviatoric or viscous stress (whose negative is equivalent

TABLE I. Bivelocity temperature differences for the noble gases when $C = 1$.

| Gas | $\hat{c}_p$ (kJ kg$^{-1}$ K$^{-1}$ $\equiv 10^{-3}$ m$^2$ s$^{-2}$ K$^{-1}$) | $\Delta T$ (°C) Eq. (6.4) | $\Delta T$ (°C) Numerical solution |
|---|---|---|---|
| Helium | 5.193 | 0.263 | 0.264 |
| Neon | 1.03 | 1.33 | 1.32 |
| Argon | 0.520 | 2.64 | 2.64 |
| Krypton | 0.248 | 5.53 | 5.53 |
| Xenon | 0.158 | 8.68 | 8.66 |

to the diffuse flux density of momentum). We assume $\mathbf{T}$ be symmetric and traceless (which is equivalent to supposing that bulk viscosity contributions to the stress are absent).

Appearing in the energy equation (2.3) is the fluid's specific (i.e., per unit mass) energy density

$$\hat{e} = \hat{u} + v^2/2, \qquad (2.5)$$

consisting of specific internal and kinetic energies. According to NSF theory the energy flux $\mathbf{j}_e$ is given by the constitutive expression

$$\mathbf{j}_e = \mathbf{j}_u + \mathbf{P} \cdot \mathbf{v}, \qquad (2.6a)$$

in which $\mathbf{j}_u$ is the diffuse internal energy flux ("heat" flux). On the other hand, according to bivelocity theory [8],

$$\mathbf{j}_e = \mathbf{j}_u + \mathbf{P} \cdot \mathbf{v}_v, \qquad (2.6b)$$

in which $\mathbf{v}_v$ is the fluid's volume velocity.

The volume velocity is related to the fluid's mass velocity by the expression

$$\mathbf{v}_v = \mathbf{v} + \mathbf{j}_v, \qquad (2.7)$$

wherein for gases the diffuse volume flux $\mathbf{j}_v$ is given constitutively by the expression [8]

$$\mathbf{j}_v = \frac{C}{\mathrm{Pr}} \upsilon \nabla \ln \rho. \qquad (2.8)$$

$C$ is a dimensionless, fluid-property-dependent constant [of $O(1)$ with respect to the Prandtl number], whose value is believed to be near unity for all gases (see Table I in Ref. [15]). The Prandtl number is defined as $\mathrm{Pr} = \upsilon/\alpha$ [2], in which $\upsilon = \eta/\rho$ is the kinematic viscosity, wherein $\eta$ is the shear viscosity. Moreover, $\alpha = k/\rho \hat{c}_p$ is the fluid's thermometric diffusivity [2], with $k$ and $\hat{c}_p$, respectively, the gas's thermal conductivity and isobaric specific heat.

Comparison of Eqs. (2.6a) and (2.6b) in light of (2.7) and (2.8) shows that the NSF equations correspond to circumstances wherein $\mathbf{j}_v = \mathbf{0}$, namely, where either $C = 0$ or where the fluid is incompressible, such that in the latter case $\rho$ is uniform throughout the fluid.

The rheological constitutive equations governing the respective viscous stresses for NSF and bivelocity fluids are [8]

$$\mathbf{T} = 2\eta \overline{\nabla \mathbf{v}} \qquad (2.9a)$$

and

$$\mathbf{T} = 2\eta \overline{\nabla \mathbf{v}_v}. \qquad (2.9b)$$

An overline above a dyadic denotes the dyadic's symmetric and traceless form, such that for any dyadic $\mathbf{D} \equiv D_{ij}$ we have in Cartesian tensor notation that $\bar{D}_{ij} = (1/2)(D_{ij} + D_{ji}) - (1/3)\delta_{ij}D_{kk}$, with $\delta_{ij}$ the Kronecker delta.

## III. RIGID-BODY ROTATION

The cylindrical container housing the gas rotates steadily at an angular velocity $\mathbf{\Omega} = \mathbf{i}_z\Omega$ about the $z$ axis of a circular cylindrical coordinate system $(R, \phi, z)$ affixed to the cylinder, whose origin $R = 0$ lies along the cylinder's symmetry axis, $-\infty < z < \infty$. In what follows, the trio $(\mathbf{i}_R, \mathbf{i}_\phi, \mathbf{i}_z)$ denotes an orthonormal right-handed set of unit vectors along each of the three coordinate axes.

### A. Symmetry considerations

By virtue of the anticipated radial symmetry of the respective solutions of the above equation sets for both NSF and bivelocity fluids (whose symmetries will be seen at the conclusion of our paper as having been verified *a posteriori*), the functional dependencies of the several hydrodynamic and other fields appearing therein are as follows:

$$\mathbf{v} = \mathbf{i}_\phi v(R), \quad p = p(R), \quad T = T(R),$$
$$\rho = \rho(R), \quad \hat{u} = \hat{u}(R). \tag{3.1}$$

Hence, as a result,

$$\mathbf{j}_v = \mathbf{i}_R j_v(R) \quad \text{and} \quad \mathbf{v}_v = \mathbf{i}_\phi v(R) + \mathbf{i}_R j_v(R), \tag{3.2}$$

in which

$$j_v = \frac{C}{\mathrm{Pr}}\eta \frac{1}{\rho}\frac{d\ln\rho}{dR} \equiv -\frac{C}{\mathrm{Pr}}\eta \frac{d}{dR}\left(\frac{1}{\rho}\right). \tag{3.3}$$

In what follows, $\eta$ will be regarded as a constant, independent of radial position.

### B. Continuity equation

As is readily shown, the continuity equation (2.1) is automatically satisfied as a consequence of the above symmetries, irrespective of the explicit dependencies of the fluid's density $\rho$ and mass velocity $v$ upon $R$.

## IV. MOMENTUM TRANSPORT AND VELOCITY CONSIDERATIONS

As shown in Appendix A, based upon the radial symmetry conditions set forth in Eqs. (3.1) and (3.2), the vector momentum equation (2.2) furnishes the following pair of scalar equations governing its respective azimuthal and radial $\phi$ and $R$ components:

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rv)\right] = 0 \tag{4.1}$$

and

$$\frac{4}{3}\eta \frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rj_v)\right] = \frac{dp}{dR} - \rho\frac{v^2}{R}. \tag{4.2}$$

### A. Velocity boundary conditions

With $\mathbf{n} = \mathbf{i}_R$ the outwardly directed unit normal vector at a point lying on the surface of the cylinder, the boundary condition $\mathbf{n} \cdot \mathbf{v} = 0$ at $R = R_o$ of no mass flow through its rigid walls is seen on the basis of $(3.1_1)$ to be automatically satisfied. Each point situated on the solid cylinder wall moves with the velocity $\mathbf{\Omega} \times \mathbf{R}_o = \mathbf{i}_\phi\Omega R_o$ (in which $\mathbf{R} = \mathbf{i}_R R$). The no-slip tangential velocity boundary condition imposed on the fluid in contact with the cylinder wall—whether regarded as being imposed on the fluid's mass velocity $\mathbf{v}$ in the form $(\mathbf{I} - \mathbf{nn}) \cdot \mathbf{v} = 0$ at $R = R_o$, or on the fluid's volume velocity $\mathbf{v}_v$ in the form $(\mathbf{I} - \mathbf{nn}) \cdot \mathbf{v}_v = 0$ at $R = R_o$—is seen to be physically satisfied in both instances by the single requirement that

$$v = \Omega R_o \quad \text{for} \quad R = R_o. \tag{4.3}$$

The general solution of the azimuthal momentum equation (4.1) is $v(R) = C_1 R - C_2 R^{-1}$. Consequently, the solution thereof that is free of singularities at the origin and satisfies the no-slip boundary condition (4.3) is represented by the rigid-body rotation velocity field

$$v = \Omega R \quad (\forall\, 0 \leqslant R \leqslant R_o). \tag{4.4}$$

Substitution of (4.4) into (4.2) furnishes the radial momentum equation:

$$\frac{4}{3}\eta \frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rj_v)\right] = \frac{dp}{dR} - \rho\Omega^2 R. \tag{4.5}$$

### B. Torque required to maintain the cylinder's rotation

With respect to an origin lying along the symmetry axis, the torque $\mathbf{N}$ exerted on a length $L$ of the cylinder wall that is required to maintain the gas's steady rotation is, by definition,

$$\mathbf{N} = -\int_{z=0}^{L}\int_{\phi=0}^{2\pi}\mathbf{R} \times (d\mathbf{S} \cdot \mathbf{P})|_{R=R_o}, \tag{4.6}$$

in which $d\mathbf{S} = \mathbf{i}_R dS$ is a directed element of surface area, wherein $dS|_{R_o} = R_o d\phi dz$. After some algebra (4.6) reduces to

$$\mathbf{N} = \mathbf{i}_z R_o^2 L \int_{\phi=0}^{2\pi} d\phi\, T_{R\phi}\big|_{R=R_o}.$$

From Appendix A we have for the present case, where the radial symmetries (3.1) and (3.2) prevail, that for both NSF and bivelocity fluids the shear stress is given by the expression

$$T_{R\phi} = \eta R \frac{d}{dR}\left(\frac{v}{R}\right).$$

Use of Eq. (4.4) in the latter shows that $T_{R\phi} = 0$ for all $R$. Hence, $\mathbf{N} = \mathbf{0}$, irrespective of the presence or absence of a diffuse volume flux $j_v$ in the governing fluid-mechanical equations. Consequently, independently of whether the NSF or bivelocity equations (if either) is regarded as being the applicable hydrodynamic equation set resulting in the fluid's steady rigid-body rotation, no torque is required to maintain that fluid motion.

## V. BIVELOCITY ENERGY TRANSPORT

### A. Bivelocity energy equation

It is readily shown on the basis of the symmetry conditions (3.1) and (3.2) that the left-hand side of (2.3) is identically zero. As such, it follows that $\nabla \cdot \mathbf{j}_e = \mathbf{0}$ at each point of the fluid. In turn, with use of (2.6b) it is shown in Appendix A that in present circumstances the energy equation (2.3) becomes

$$j_u + pj_v - \frac{2}{3}\eta \frac{dj_v^2}{dR} = 0. \tag{5.1}$$

Furthermore, the constitutive equation for the diffuse bivelocity internal energy flux in the case of ideal gases is [8]

$$j_u = -k' \frac{dT}{dR}, \tag{5.2}$$

in which

$$k' = \frac{\hat{c}_v}{\hat{c}_p} k, \tag{5.3}$$

wherein $\hat{c}_v$ is the isochoric specific heat.

As stated earlier, the gas is assumed to obey the ideal-gas equation of state

$$p = \frac{\mathcal{R}}{M_w} \rho T, \tag{5.4}$$

in which $\mathcal{R}$ is the universal gas constant and $M_w$ the gas's molecular weight.

### B. Bivelocity pressure, temperature, density, and diffuse volume flux fields

The radial momentum equation (4.5), together with the energy equations (5.1) and (5.2), the ideal-gas equation of state (5.4), and the constitutive equation (3.3) for the diffuse volume flux constitute a set of four equations involving the four bivelocity fields $(p, T, \rho, j_v)$. These equations are to be solved simultaneously so as to obtain expressions for the functional dependences of the preceding four fields upon $R$. As discussed in Appendix A, all of the physically imposed boundary conditions demanded of these fields are already implicitly satisfied as a consequence of the differential equations themselves. However, in order to render the solutions of these equations unique one must also specify (i) the mass of gas confined in the cylinder (per unit cylinder length) or, equivalently, the mean density $\bar{\rho}$ of the confined gas,

$$\bar{\rho} = \frac{2}{R_o^2} \int_0^{R_o} \rho R \, dR, \tag{5.5}$$

and (ii) say, the gas's mean temperature

$$\bar{T} = \frac{2}{\bar{\rho} R_o^2} \int_0^{R_o} \rho T R \, dR. \tag{5.6}$$

Alternatively, instead of the latter, upon using the ideal-gas law (5.4) in connection with the term $\rho T$ appearing in the above integrand, and upon defining the mean pressure $\bar{p}$ as $\bar{p} = (\mathcal{R}/M_w)\bar{\rho}\bar{T}$, one could, instead, specify the mean pressure

$$\bar{p} = \frac{2}{R_o^2} \int_0^{R_o} p R \, dR \tag{5.7}$$

in place of the mean temperature.

## VI. PERTURBATION SOLUTION OF THE BIVELOCITY EQUATIONS

Solution of the bivelocity-based mass, momentum, and energy equations satisfying the set of conditions prevailing during the proposed experiment or simulation thereof requires specifying the values of the gas's physical properties [e.g., $\eta$, $\hat{c}_p$, $M_w$, $C$ (if known), etc.], as well as the values of the other parameters governing the problem (e.g., $\Omega$, $R_o$, $\bar{\rho}$, etc.). As these choices lie within the province of the experimentalist undertaking the proposed test we make no attempt here to provide complete solutions of the above four-equation set, i.e., solutions valid for *arbitrary* choices of the specified parameters. However, the perturbation solutions of the field equations that we subsequently derive in Appendix B prove sufficient towards providing accurate results for all feasible experimental conditions likely to be encountered. Alternatively, was that not to be the case, one would have to solve the pertinent equation set numerically using the experimentalist's choice of parametric values in order to compare theoretical predictions with experimental results.

According to calculations set forth in Appendix B based upon bivelocity theory, the gas's radial temperature distribution in the cylinder is given for all practical purposes by the surprisingly elementary expression

$$T(R) - T(0) = f(C) \frac{(\Omega R)^2}{2\hat{c}_p}, \tag{6.1}$$

where $f(C)$ is the dimensionless $O(1)$ quantity

$$f(C) = \left[ \frac{1}{\gamma} \left( \frac{1}{C} - 1 \right) + 1 \right]^{-1}, \tag{6.2}$$

in which

$$\gamma = \frac{\hat{c}_p}{\hat{c}_v} \tag{6.3}$$

is the specific heat ratio. For monatomic gases, $\gamma = 5/3$. Note that, independently of $\gamma$, $f(C) = 1$ when $C = 1$ and $f(C) = 0$ when $C = 0$.

Upon setting $R = R_o$ in (6.1), the sought-after bivelocity temperature difference (1.1) is found to be

$$\Delta T = f(C) \frac{(\Omega R_o)^2}{2\hat{c}_p}. \tag{6.4}$$

### A. $\Delta T$ values for the noble gases and for the case where $C = 1$

By way of example, consider the case of a cylinder of radius 10 cm rotating with an angular velocity of 5000 rpm (approximately 2800 g's). Using specific heat capacity data for the noble gases (for which $\hat{c}_p = 20.786/M_w$ kJ kg$^{-1}$ K$^{-1}$), Table I presents data showing the temperature difference anticipated for each of these gases based upon Eq. (6.4) together with the assumption that $C = 1$. Also shown for comparison are the exact $\Delta T$ values obtained by solving the radial momentum and energy equations numerically for the case $C = 1$ at a mean temperature of $\bar{T} = 300$ K and a mean pressure of $\bar{p} = 1$ atm. Obviously, the approximate formula (6.4) is quite satisfactory [16]. As measurements of temperature differences of the respective orders of magnitude shown in the table are routine, there appears to be no significant barrier to performing the proposed experimental test.

### B. Attributes of the bivelocity formula (6.4) for $\Delta T$

Surprisingly, for a specified gas the preceding formula is independent of the gas's mean pressure, temperature, and density, as well as of the density gradient. Nor is Eq. (6.4) limited to rarefied gases, despite such diluteness having been required in all previous applications of bivelocity theory [8] in order to assure a sensible magnitude of the diffuse volume contribution $\mathbf{j}_v$ over and above the value $\mathbf{j}_v = \mathbf{0}$ for the comparable NSF case. Equally striking is the fact that Eq. (6.4) is independent of any and all of the gas's transport properties, particularly its viscosity and thermal conductivity. Only geometric, kinematic, and thermodynamic variables appear in (6.4). Collectively, these facts are unexpected given that the constitutive equation (3.3) for the diffuse volume flux—which is the sole factor distinguishing bivelocity hydrodynamics from NSF hydrodynamics—depends upon both the viscosity and density gradient, neither of which contributes directly to the above $\Delta T$ formula.

Equation (6.4) makes it clear that the numerical value of the coefficient $C$ entering into Eq. (3.3) constitutes the sole nonequilibrium (i.e., kinetic) contribution to the above $\Delta T$ formula. As such, its numerical value is key to the proposed test. For example, in place of the value $C = 1$ used in preparing Table I, had one, instead, set $C = 0$, thus making $j_v = 0$ (as in the NSF case), one would have obtained $f(C) = 0$ and hence $\Delta T = 0$, as was to be expected under the circumstances. As such, bivelocity theory is critically dependent upon the value of the parameter $C$. In effect, for a specified choice of gas the proposed experimental test or simulation is tantamount to experimentally establishing the value of the constant $C$ for that particular gas. That is, ideally, the test should involve experiments performed at different angular velocities, different mean pressures (or mean densities), etc., in order to establish whether the experimental value obtained for $C$ is indeed independent of operating conditions.

The overall physical significance of the preceding findings in broad general terms, involving issues of the thermodynamic irreversibility of rigid-body fluid motions, is discussed below in Sec. VII C.

### C. NSF fields

In contrast with the bivelocity fields $(p, T, \rho, j_v)$ whose solutions are derived in the Appendices, the comparable NSF fields $(p, T, \rho)$, corresponding to setting $j_v = 0$ in the respective mass, momentum, and energy equations, as well as in the boundary conditions, are given for the present set of circumstances by the following expressions:

$$T(R) = \text{const} = T(0), \qquad (6.5)$$

$$p(R) = p(0) \exp\left[\frac{M_w R^2}{2 \mathcal{R} T(0)}\right], \qquad (6.6)$$

and

$$\rho(R) = \rho(0) \exp\left[\frac{M_w R^2}{2 \mathcal{R} T(0)}\right]. \qquad (6.7)$$

The several centerline values represented in the above by the argument (0) are related through the ideal-gas expression $p(0) = (\mathcal{R}/M_w)\rho(0)T(0)$. The main feature to be noted here

is embodied in the fact that, for compressible gaseous continua,

$$\Delta T = 0 \quad \text{for NSF gases}, \qquad (6.8)$$

in contrast with (6.4) for bivelocity gases.

## VII. DISCUSSION

### A. Energy conservation

Despite the presence of dissipative processes (reflected in the nonzero radial temperature gradient) arising from the gas's rotation, the amount of energy $E = \int_V \hat{e}\rho\, dV$ contained within the steadily rotating cylinder's gaseous domain $V$ remains constant for all time. This represents an obvious thermodynamic necessity if our analysis is to prove to be physically correct. Demonstration of this energy constancy follows from the fact that with $D/Dt = \partial/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla}$ the material derivative, and with use of the continuity equation $\partial\rho/\partial t + \boldsymbol{\nabla} \cdot (\rho\mathbf{v}) = \mathbf{0}$, one obtains the relation

$$dE/dt = \int_V \rho D\hat{e}/Dt\, dV = \int_V [\partial(\rho\hat{e})/\partial t + \boldsymbol{\nabla} \cdot (\rho\mathbf{v}\hat{e})]dV.$$

For the present steady-state case the partial time derivative contribution appearing in the latter integrand is identically zero. Moreover, as earlier noted in connection with the left-hand side of the energy equation (2.3), one has that $\boldsymbol{\nabla} \cdot (\rho\mathbf{v}\hat{e}) = \mathbf{0}$. Together these confirm the constancy, $dE/dt = 0$, of the amount of energy $E$ confined within the gas present in the cylinder.

### B. Azimuthal energy flow

Use of the symmetry properties set forth in Eqs. (3.1) and (3.2) enables the energy flux vector $\mathbf{j}_e$ to be recast into component form as

$$\mathbf{j}_e = \mathbf{i}_R j_e^R(R) + \mathbf{i}_\phi j_e^\phi(R), \qquad (7.1)$$

in which, beginning with Eqs. (A13) and (A17), we eventually find that

$$j_e^R(R) = 0 \quad \text{and} \quad j_e^\phi(R)$$
$$= \Omega\left(Rp - \frac{4}{3}\eta j_v\right), \quad (\forall\, 0 \leqslant R \leqslant R_o). \quad (7.2)$$

The first of these two relations confirms the intuitive expectation (based jointly upon the gas's thermodynamically isolated status and radial symmetry) that there exists no radial flow of energy. The second relation supports a controversial view, originally introduced by Müller [17] (see also Ref. [18]), regarding the apparent existence of an azimuthal energy or heat flux $j_e^\phi$ in rarefied gases undergoing rigid-body rotation, this despite the absence of an azimuthal temperature gradient $\partial T/\partial\phi$. However, whereas Müller and others regard this phenomenon as constituting a noncontinuum effect, from our perspective this azimuthal flux prediction is a strictly continuum concept, arising not from Knudsen number effects but rather from centrifugal effects, as witness the presence of the angular velocity multiplier $\Omega$ stemming therefrom.

### C. Entropy production in rotating bivelocity fluids

Critics [12,13] of bivelocity theory have argued, among other things, that bivelocity theory cannot be correct because it predicts the continuous generation of entropy in a fluid undergoing a steady, external force-free, rigid-body rotation relative to an inertial reference frame. Their remarks refer only to the case where the fluid is isothermal throughout, presumably by virtue of the cylinder's contact with a heat bath (jointly with the cylinder wall now being heat conducting). However, was the cylinder wall, instead, to remain non-heat-conducting, the temperature would still have to be uniform according to their arguments. This owes to the critics' belief that a steadily rotating fluid necessarily constitutes a state of thermodynamic equilibrium [14].

Viewed thermodynamically rather than fluid mechanically, the above-cited criticism of bivelocity hydrodynamics is based upon the implicit belief existing among many fluid mechanicians that an isolated fluid, when rotating steadily in an inertial coordinate system, necessarily exists in a state of thermodynamic equilibrium, the latter representing a state wherein no entropy is generated as a consequence of the rotation [19]. That belief, however, represents an unproved assertion rather than an established fact, since (to the author's knowledge) it has never been explicitly confirmed experimentally.

The generic formula governing the transport of entropy at a point in a fluid continuum is [8]

$$\rho \frac{D\hat{s}}{Dt} = -\boldsymbol{\nabla} \cdot \mathbf{j}_s + \pi_s, \tag{7.3}$$

in which $\hat{s}$ is the specific entropy. Moreover,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \tag{7.4}$$

denotes the material derivative. Also,

$$\mathbf{j}_s = \frac{\mathbf{q}}{T} \tag{7.5}$$

is the constitutive equation governing the diffusive flux of entropy [8], whereas $\pi_s$ is the temporal rate of production of entropy per unit volume of fluid. On the assumption that $\hat{s}$ is radially symmetric, one has in view of $(3.1_1)$ and (4.4) that $\mathbf{v} \cdot \boldsymbol{\nabla}\hat{s} = \mathbf{0}$. Furthermore, given the steady-state nature of the processes under consideration, one has at each point of the fluid that $\partial\hat{s}/\partial t = 0$. It therefore follows that

$$\frac{D\hat{s}}{Dt} = 0. \tag{7.6}$$

Consequently,

$$\pi_s = \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}}{T}\right). \tag{7.7}$$

With

$$\dot{S} = \int_V \pi_S dV, \tag{7.8}$$

the temporal rate at which entropy is being produced within the cylinder, one finds from (7.7), together with use of the divergence theorem, that

$$\dot{S} = \frac{1}{T(R_o)} \oint_{\partial V} d\mathbf{S} \cdot \mathbf{q}, \tag{7.9}$$

in which $d\mathbf{S} \equiv \mathbf{i}_R \, dS|_{R_o}$ is an element of surface area on the cylinder wall and $T(R_o)$ is the (uniform) temperature of the gas along the cylinder surface. As $dS|_{R_o} = L R_o d\phi$, where $L$ is the length of the cylinder, we have from the above that

$$\dot{S} = \frac{2\pi L R_o}{T(R_o)} q(R_o). \tag{7.10}$$

According to bivelocity theory [8], in the absence of body forces the constitutive equation for the entropic heat flux $\mathbf{q}$, valid for both gases and liquids, is

$$\mathbf{q} = -k\boldsymbol{\nabla}T + \alpha\beta T\boldsymbol{\nabla}p. \tag{7.11}$$

Here, $k$ is the thermal conductivity,

$$\alpha = k/\rho\hat{c}_p \tag{7.12}$$

is the thermometric diffusivity, and $\beta = (\partial \ln \rho/\partial p)_T$ is the coefficient of isothermal compressibility. For ideal gases one has that $\beta T = 1$, whence (7.11) becomes, in the present radially symmetric circumstances,

$$q = \frac{k}{\hat{c}_p}\left(\frac{1}{\rho}\frac{dp}{dR} - \hat{c}_p\frac{dT}{dR}\right). \tag{7.13}$$

Conversion of the nondimensional Eq. (B13) appearing in Appendix B to dimensional form gives

$$\frac{dT}{dR} = f(C)\frac{\Omega^2 R}{2\hat{c}_p} + O(\varepsilon^2). \tag{7.14}$$

Similarly, conversion of (B11) from dimensionless to dimensional form yields

$$\frac{dp}{dR} = \rho\Omega^2 R + O(\varepsilon^2), \tag{7.15}$$

where, from Eqs. (B5) and (B6), $\varepsilon$ is a small dimensionless number (an inverse Reynolds number) of the order of $O(10^{-6})$ for the class of problems envisioned in connection with our proposed test. Hence, to terms of dominant order,

$$q = \frac{k}{\hat{c}_p}\left[1 - \frac{1}{2}f(C)\right]\Omega^2 R + O(\varepsilon^2). \tag{7.16}$$

Introduce the latter into (7.10) while noting that the cylinder volume is $V = \pi R_o^2 L$ and, consequently, that the mass of gas contained within the cylinder is $M = \bar{\rho}V$, where $\bar{\rho}$ is the mean density, defined in Eq. (5.5). This gives

$$\dot{S} = \left[1 - \frac{1}{2}f(C)\right]\frac{\bar{\alpha}M\Omega^2}{T(R_o)} + O(\varepsilon^2), \tag{7.17}$$

in which $\bar{\alpha} = k/\bar{\rho}\hat{c}_p$ is the mean thermometric diffusivity. For the proposed set of circumstances one thus has that

$$\dot{S} = \frac{1 - (1/2)f(C)}{1 + f(C)(\Omega R_o)^2/4\hat{c}_p\bar{T}}\left(\frac{\bar{\alpha}M\Omega^2}{\bar{T}}\right) + O(\varepsilon^2). \tag{7.18}$$

Based upon the bivelocity differential equations governing mass, momentum, and energy transport it is proved [8] for the choice of phenomenological coefficients entering into the above calculation that the local entropy production rate defined by (7.7) satisfies the inequality $\pi_s \geqslant 0$ at each point of the fluid, irrespective of the choice of boundary conditions and independent of the nature of the physical problem being addressed. It thus follows from (7.8) that $\dot{S} \geqslant 0$ in general, and hence certainly in the context of the formula (7.18).

According to (7.6) the amount of entropy contained in the gas confined within the cylinder remains constant in time. As such, the entropy being generated is not accumulated within the gas itself, but rather eventually appears in the surroundings of the rotating cylinder. Thus, the entropy of the Universe is continually increasing as a result of the gas's rigid-body rotation, despite the fact that the fluid itself is not undergoing any changes in its steady-state status. Since the system is thermodynamically isolated from its surroundings, this leads to the conclusion that the Universe's entropy increase is, in a fundamental sense, attributable exclusively to the body's rotation relative to the fixed stars, irrespective of the specific processes occurring within the fluid. As a result, it would be wrong to think of the body as being "isolated" from its surroundings. Rather, because of the nature of centrifugal force, which according to Mach's principle arises from the interaction of a body with all of the other bodies in the Universe, the gas—being a body—cannot really be regarded as isolated from its surroundings. It is this Mach-based interaction that appears to explain the source of the system's unexpected behavior, both locally with respect to the anticipated nonuniform temperature distribution within the body, and globally with respect to the Universe's entropy increase occurring outside of the body.

Interestingly, even in circumstances where $C = 0$, thereby rendering $f(C) = 0$ in (7.18), one finds that entropy is nevertheless being produced at a rate

$$\dot{S} = \frac{\bar{\alpha} M \Omega^2}{\bar{T}}, \tag{7.19}$$

despite the fact that Eq. (7.14) shows the temperature of the gas to be uniform throughout. This surprising result can be traced to the pressure gradient contribution $\nabla p = \rho \Omega^2 R$ to the entropic heat flux (7.13) [and hence to the entropy flux $\mathbf{j}_s$ in Eqs. (7.5) and (7.11)]. Thus, whereas a gravitationally induced pressure gradient would not result in entropy generation in circumstances where the fluid was not rotating, a centrifugally induced pressure gradient gives rise to a rather different result. With respect to Mach's principle, the role of the pressure gradient, which acts parallel to the centrifugal force in the proposed experiment, can be likened to the similar role played by the pressure gradient in Newton's water-filled rotating bucket experiment—which effect acts parallel to the gravitational force, and wherein the departure of the water's free surface from the horizontal is implicitly attributed to the interaction of the masses of the water molecules with the rest of the mass in the Universe.

In any event, issues of entropy production pertaining to the bivelocity model are irrelevant in the context of deciding whether the experimental or simulation data obtained from the proposed test supports or denies the viability of the NSF equations for compressible fluid continua. Simply stated, irrespective of entropy considerations, if the temperature is found to be nonuniform, the NSF model cannot be correct, and conversely.

### D. Irreversible thermodynamics of rotating fluids

Owing to recognition of the existence of centrifugal forces it has been known since at least Newton's time that the laws of mechanics are not invariant under the rotation of a body relative to "empty space," or, more precisely according to Mach, relative to the "fixed stars." Despite this knowledge, it appears to have always been implicitly assumed that the fundamental principles of irreversible thermodynamics, and hence of fluid mechanics, were invariant to rigid-body rotations relative to the fixed stars [14]. Empirical observation of temperature inhomogeneities during the proposed test would refute this assumption. Indeed, our proposed test amounts, *inter alia*, to a test of the so-called "principle of material frame indifference" (PMFI) [17,18,20] for fluid continua, according to which the constitutive responses of such fluids to changes in their respective states are independent of the observer's frame of reference.

This validity of this principle has been challenged by many, the earliest of whom include Müller [17], Edelen and McLennan [21], and Soderholm [22] (see also Ref. [23]). However, these challenges were based upon assuming the validity of Burnett's [24] (Boltzmann equation-derived) constitutive equations for the heat flux and stress in nonisothermal gases undergoing steady, rigid-body rotations. However, the viability of Burnett's equations have themselves been challenged [25,26] on various grounds. Moreover, the Burnett equations, because of their presumed noncontinuum, Knudsen number dependence, are not viewed as being applicable to fluid continua. And it is only for fluid continua that the PMFI is regarded as being applicable. As such, the viability of this principle cannot, as yet, be regarded as fully disproved.

Our LIT-based bivelocity equations [8] bear a strikingly close appearance to those of Burnett [24]. However, in contrast with those of Burnett—and as with all other LIT-based constitutive equations [7]—our equations are regarded as describing the behavior only of *continua,* while not including *noncontinua*. This view is confirmed by the independent works of Dadzie [9] and Meng, *et al.* [10], each grounded upon its own molecularly based modification of Boltzmann's gas-kinetic equation [11].

Accordingly, was our bivelocity model found to accord with experiment, this would serve to confirm the failure of the PMFI. For was the cylinder to be at rest, i.e., not rotating relative to the fixed stars, no temperature gradients would presumably be observed, as is surely true in the case of the NSF equations.

### E. Mach number dependence of $\Delta T$

As is well known, the Mach number, $\mathrm{Ma} = v/c$ (with $c$ the velocity of sound), plays a key role in quantifying the effect of compressibility on flowing fluid continua. Given the comparable role played by the diffuse volume flux $\mathbf{j}_v$ in compressible fluid motions [8] it should not be surprising to find that a close relation exists between these two different measures of compressibility. In what follows we establish a correlation between the two in the context of our proposed experiment.

With $v_o = \Omega R_o$ the velocity of the gas in proximity to the cylinder wall and $\bar{c} = \sqrt{\gamma \not R \bar{T}/M_w}$ the velocity of sound at the mean temperature $\bar{T}$ of the confined gas, one has that $\mathrm{Ma} = \Omega R_o \sqrt{M_w/\gamma \not R \bar{T}}$. For monatomic gases this yields $\mathrm{Ma} = \sqrt{3(\Omega R_o)^2/2\hat{c}_p \bar{T}}$. Comparison with (6.4) gives

$$\frac{\Delta T}{\bar{T}} = \frac{1}{3} f(C)(\mathrm{Ma})^2 \tag{7.20}$$

for the relation between the dimensionless temperature difference $\Delta T / \bar{T}$ and the Mach number. For example, for the case Ma $= 1$, $C = 1$, and a mean temperature of $\bar{T} = 300$ K, this gives $\Delta T = 100$ °C. It is obviously possible to encounter significant temperature differences without having to resort to extreme operating conditions.

## VIII. DISCUSSION

### A. MD versus DSMC simulations

The proposed test of the NSF equations outlined in this paper encourages performing either an experiment or a molecular dynamic (MD) simulation. One might ask, why not also encourage a comparable direct simulation Monte Carlo (DSMC) test [27]? The basis for cautioning against its use in the present context hinges upon the fact that the reliability of conclusions to be drawn from such a simulation are only as reliable as the Boltzmann equation is itself a physically reliable realization of Newton's laws of motion. That is, was a problem found to exist with regard to the physical basis of Boltzmann's gas-kinetic model in regard to accurately mirroring the macroscopic physical consequences of Newton's mechanical laws applied to collections of molecules—such as is, in fact, avoided in the course of effecting MD simulations— a comparable problem would then ensue with respect to the physical basis of the DSMC conclusions derived therefrom.

In this latter context, recent work by Dadzie, Reese, and their collaborators [9,10,28–35], as well as that of other independent researchers [36,37], serving to molecularly mirror comparable macroscopic bivelocity developments based upon irreversible thermodynamic principles [8]—point to the fact that a fundamental problem exists with regard to the current statistical-mechanical modeling of compressible fluid continua. This owes to the failure of most existing molecular models to incorporate diffuse or dissipative (stochastic) volume transport phenomena within their scope. In turn, as an implied consequence thereof, a subsequent problem arises with respect to the lack of completedness of Boltzmann's equation. Other, purely macroscopic, nonmolecular analyses [38–40] support the need for including stochastic, volume-based contributions to existing molecular models.

In contrast with our cautionary attitude displayed towards DSMC simulations of Boltzmann's original collision model, we see no objection to applying DSMC techniques to the proposed test at hand when using, say, Dadzie's stochastically modified Boltzmann model equation [9] in place of Boltzmann's original equation. Indeed, such an undertaking would presumably offer independent data bearing on the issue of the validity of the NSF equations for compressible gaseous continua. For was such a DSMC simulation to predict a nonuniform temperature distribution this would introduce further evidence serving to discredit the NSF equations.

### B. Rigid-body rotations

Proceeding beyond strictly constitutive issues pertaining to the NSF and bivelocity models, our proposed test bears, more generally, upon the viability of the currently accepted, albeit *ad hoc*, notion that rigid-body rotations of fluid continua are necessarily equipollent with states of thermodynamic equilibrium [14]. This issue impacts, among other things, on a variety of important physical fields of research. In short, the present proposal, whose challenges we hope will be addressed by experimentalists, is expected to stimulate renewed interest in the foundations of (compressible) fluid mechanics—especially if temperature gradients in isolated gases undergoing rigid-body rotations are indeed found to exist.

### C. Rarefied gases

In this paper we have proposed performing a simple class of experiments and/or MD simulations for gaseous continua. However, no objective distinction exists between continua and noncontinua, e.g., rarefied gases. That is, there exists no definite value of the Knudsen number below which the gas is a continuum and above which it is a noncontinuum. Thus, the oft-cited value of Kn $= 0.01$ [41] at which the transition is regarded by many as occurring is somewhat arbitrary. That said, it would be useful to extend our proposed experiment and simulation to now include rarefied gases in addition to gaseous continua. While it is already well known [3] that the predictions of the NSF equations (in conjunction with the no-slip boundary condition) become increasingly inaccurate as the Knudsen number increases, it would be of interest to establish whether under the circumstances of the proposed experiment and simulation this deviation would also give rise to a radial temperature variation, comparable to that presently anticipated for the case of gaseous continua.

*Note added in proof.* Recently, a molecular dynamics simulation of the test described herein was effected [42]. The findings, as evidenced by the nonuniform radial temperature distributions observed, were consistent with the hypothesis that the NSF paradigm is indeed invalid for gaseous continua. Furthermore, these findings were not inconsistent with the hypothesis that the bivelocity paradigm rather than the NSF paradigm should be regarded as constituting the fundamental paradigm underlying continuum fluid mechanics.

## APPENDIX A: COMPONENT FORMS OF THE BIVELOCITY MOMENTUM AND ENERGY EQUATIONS

### 1. Momentum

Joint use of Eqs. (2.9b) and (2.7) furnishes the relation

$$\mathbf{T} = \mathbf{T}_v + \mathbf{T}_j, \tag{A1}$$

wherein

$$\mathbf{T}_v := 2\eta \overline{\boldsymbol{\nabla} \mathbf{v}} \quad \text{and} \quad \mathbf{T}_j := 2\eta \overline{\boldsymbol{\nabla} \mathbf{j}}_v. \tag{A2}$$

In conjunction with the radial symmetry conditions expressed by Eqs. (3.1) and (3.2), the above relations, when written out

explicitly, adopt the respective forms

$$\mathbf{T}_v = (\mathbf{i}_R\mathbf{i}_\phi + \mathbf{i}_\phi\mathbf{i}_R)\eta R\frac{d}{dR}\left(\frac{v}{R}\right) \tag{A3}$$

and

$$\mathbf{T}_j = \frac{4}{3}\eta\left(\mathbf{i}_R\mathbf{i}_R\frac{dj_v}{dR} + \mathbf{i}_\phi\mathbf{i}_\phi\frac{j_v}{R}\right). \tag{A4}$$

On the assumption that the shear viscosity may be regarded as constant throughout the flow domain, and hence independent of position, the respective divergences of the preceding deviatoric stresses are found to be

$$\nabla\cdot\mathbf{T}_v = \mathbf{i}_\phi\eta\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rv)\right] \tag{A5}$$

and

$$\nabla\cdot\mathbf{T}_j = \mathbf{i}_R\frac{4}{3}\eta\frac{1}{R}\left[\frac{d}{dR}\left(R\frac{dj_v}{dR}\right) - \frac{j_v}{R}\right]$$
$$\equiv \mathbf{i}_R\frac{4}{3}\eta\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rj_v)\right]. \tag{A6}$$

Together, these combine to give

$$\nabla\cdot\mathbf{T} = \mathbf{i}_\phi\eta\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rv)\right] + \mathbf{i}_R\frac{4}{3}\eta\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rj_v)\right]. \tag{A7}$$

Furthermore, with use of $(3.1_1)$ we have by symmetry that $\mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{i}_R v^2/R$. Similarly, from $(3.1_2)$, $\nabla p = \mathbf{i}_R dp/dR$. Introduction of these relations into the momentum equation (2.2), along with use of the expressions $\nabla\cdot\mathbf{P} = \nabla p - \nabla\cdot\mathbf{T}$ and (A7), followed by equating corresponding $\phi$ and $R$ components, furnishes the pair of scalar momentum equations set forth in Eqs. (4.1) and (4.2).

### 2. Energy

As a consequence of the radial symmetries (3.1) and (3.2) it follows that the left-hand side of the energy equation (2.3) is identically zero. In turn, this requires that

$$\nabla\cdot\mathbf{j}_e = 0. \tag{A8}$$

In present circumstances, namely, with the cylinder both rigid and non-heat-conducting, the energy flux boundary condition at the cylinder wall requires that

$$\mathbf{n}\cdot\mathbf{j}_e = 0 \quad \text{at} \quad R = R_o. \tag{A9}$$

From Eqs. (2.6b), (2.4), (2.7), and (A1) we find that

$$\mathbf{j}_e = \mathbf{q} + p\mathbf{v} - (\mathbf{T}_v + \mathbf{T}_j)\cdot(\mathbf{v} + \mathbf{j}_v), \tag{A10}$$

where, for simplicity, we have introduced the symbol $\mathbf{q} = \mathbf{j}_u + p\mathbf{j}$ (whose physical significance [8] need not concern us at this moment). Owing to the respective radially symmetric natures of $\mathbf{j}_u$, $p$ and $\mathbf{j}_v$ we have that $\mathbf{q} = \mathbf{i}_R q(R)$, in which

$$q = j_u + pj_v. \tag{A11}$$

Furthermore, for the rigid-body motion (4.4) it follows from Eq. (A3) that

$$\mathbf{T}_v = \mathbf{0}. \tag{A12}$$

By symmetry, together with use of the relations $\mathbf{v} = \mathbf{i}_\phi\Omega R$, (A4), and (A12), it follows from (A10) that

$$\mathbf{j}_e = \mathbf{i}_\phi\Omega R\left(p - \frac{4}{3}\eta\frac{j_v}{R}\right) + \mathbf{i}_R\left(q - \frac{2}{3}\eta\frac{dj_v^2}{dR}\right). \tag{A13}$$

Equation (A8) in conjunction with the above relation leads to the expression

$$\frac{d}{dR}\left[R\left(q - \frac{2}{3}\eta\frac{dj_v^2}{dR}\right)\right] = 0, \tag{A14}$$

while the boundary condition (A9) requires that

$$q - \frac{2}{3}\eta\frac{dj_v^2}{dR} = 0 \quad \text{at} \quad R = R_o. \tag{A15}$$

Integration of (A14) gives

$$q - \frac{2}{3}\eta\frac{dj_v^2}{dR} = \frac{K}{R}, \tag{A16}$$

where $K$ is a constant. Finiteness at the origin requires that $K = 0$, whence the energy equation is satisfied for all $R$ by the relation

$$q - \frac{2}{3}\eta\frac{dj_v^2}{dR} = 0, \quad (\forall\, 0 \leqslant R \leqslant R_o). \tag{A17}$$

Accordingly, the boundary condition (A15) is automatically satisfied. Insertion of (A11) into the above furnishes the energy equation set forth in Eq. (5.1).

### APPENDIX B: PERTURBATION SOLUTIONS OF THE BIVELOCITY EQUATIONS

This Appendix furnishes perturbation solutions of the quartet of bivelocity equations governing the fields $(p,T,\rho,j_v)$ for the rigid-body rotation of a gas occurring in a thermodynamically isolated circular cylinder undergoing steady rotation in an inertial reference frame. From (4.5) the momentum equation is

$$\frac{4}{3}\eta\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Rj_v)\right] = \frac{dp}{dR} - \rho\Omega^2 R, \tag{B1}$$

in which, from (3.3), the constitutive equation for the diffuse volume flux is

$$j_v = -\frac{C}{\text{Pr}}\eta\frac{d}{dR}\left(\frac{1}{\rho}\right). \tag{B2}$$

Furthermore, Eqs. (5.1) and (5.2) combine to furnish the energy equation

$$-k'\frac{dT}{dR} + pj_v - \frac{2}{3}\eta\frac{dj_v^2}{dR} = 0. \tag{B3}$$

The above trio of equations together with the ideal-gas equation of state (5.4) are to be solved for the four fields $(p,T,\rho,j)$ as functions of $R$. All required boundary conditions have already been satisfied.

### 1. Nondimensionalization

We begin solving the above quartet of equations by reformulating them in dimensionless form. With the constants $(\bar{p},\bar{\rho},\bar{T})$ denoting the mean pressure, density and temperature

as defined in Eqs. (5.5)–(5.7), we introduce the following dimensionless fields denoted by asterisks:

$$p = \bar{p} p^*, \quad \rho = \bar{\rho} \rho^*, \quad T = \bar{T} T^*, \quad \text{and} \quad j_v = (\bar{\upsilon}/R_o) j_v^*.$$
(B4)

In addition, we define a dimensionless radial distance by the expression $R = R_o R^*$. As the mean fields are related to one another through the dimensional expression $\bar{p} = (\cancel{R}/M_w) \bar{\rho} \bar{T}$, the corresponding nondimensional fields are related through the expression $p^* = \rho^* T^*$. In the above, $\bar{\upsilon} = \eta/\bar{\rho}$ is the kinematic viscosity at the mean temperature and pressure prevailing in the rotating gas.

Subject to *a posteriori* verification, the dimensionless fields defined in (B4) are all assumed to be of $O(1)$ with respect to the small dimensionless parameter

$$\varepsilon = \frac{\bar{\upsilon}}{\Omega R_o^2} \ll 1.$$
(B5)

The parameter $\varepsilon$ will necessarily be very small in all feasible experiments. For example, in the case of air at room temperature and pressure (for which $\bar{\upsilon} = 1.568 \times 10^{-5}$ m$^2$ s$^{-1}$) and for a rotation rate of $\Omega = 5000$ rpm and a cylinder radius of $R_0 = 10$ cm (as used in preparing Table I) one finds that $\varepsilon = 3.0 \times 10^{-6}$. As the gas's velocity at the cylinder wall is $v_o = \Omega R_o$, the dimensionless parameter $\varepsilon$ is seen to be an inverse Reynolds number,

$$\varepsilon = \frac{1}{\text{Re}},$$
(B6)

in which $\text{Re} = R_o v_o \bar{\rho}/\eta \gg 1$. Accordingly, all conceivable experiments aimed at testing bivelocity theory based upon our presently proposed protocol necessarily occur at large Reynolds numbers.

After considerable algebra and rearrangement, the following dimensionless forms of Eqs. (B1) and (B3) are obtained:

$$\frac{dp^*}{dR^*} = 2B\rho^* R^* + \varepsilon^2 \frac{8}{3} B \frac{d}{dR^*} \left[ \frac{1}{R^*} \frac{d}{dR^*} (R^* j_v^*) \right],$$
(B7)

$$\frac{dT^*}{dR^*} = \text{Pr} \frac{\cancel{R}}{M_w \hat{c}_v} \left( p^* j_v^* - \varepsilon^2 \frac{4}{3} B \frac{d}{dR^*} j_v^{*2} \right),$$
(B8)

whereas Eq. (B2) becomes

$$j_v^* = -\frac{C}{\text{Pr}} \frac{d}{dR^*} \left( \frac{1}{\rho^*} \right).$$
(B9)

Appearing in the above expressions is the dimensionless parameter

$$B = \frac{M_w (\Omega R_o)^2}{2 \cancel{R} \bar{T}}.$$
(B10)

In arriving at (B8) we have noted that $k' = \hat{c}_v \eta / \text{Pr}$.

As will be seen, for all feasible experimental configurations $B = O(\Delta T/\bar{T})$, which parameter we will regard as being of $O(1)$ with respect to the small parameter $\varepsilon$. We also note that the dimensionless ratio $\cancel{R}/M_w \hat{c}_v$ appearing in (B8) is of $O(1)$. [For example, for ideal monatomic gases one has that $\hat{c}_v = (3/2)(\cancel{R}/M_w)$ whence $\cancel{R}/M_w \hat{c}_v = 2/3$.] The parameter $B$ can be given a simple physical interpretation. From Eq. (4.3) the quantity $(\Omega R_o)^2/2 = v_o^2/2$ is seen to be the specific kinetic energy of a molecule of the gas in proximity to the rotating

wall. Furthermore, $M_w/\cancel{R} \equiv m/k_B$, where $m$ is the mass of a single molecule of the fluid and $k_B$ is Boltzmann's constant. Thus (B10) possesses the alternative representation

$$B = \frac{m v_o^2/2}{k_B \bar{T}}.$$

The numerator and denominator of the above expression are, respectively, seen to be the mean kinetic energy of a near-wall molecule and the mean thermal energy of a molecule of gas within the cylinder. As such $B$ represents a measure of the relative strengths of the fluid's kinetic and thermal energies.

### 2. Temperature field

Given the above-cited facts, Eqs. (B7) and (B8) become, to terms of dominant order,

$$\frac{dp^*}{dR^*} = 2B\rho^* R^* + O(\varepsilon^2)$$
(B11)

and

$$\frac{dT^*}{dR^*} = -C \frac{\cancel{R}}{M_w \hat{c}_v} \left[ p^* \frac{d}{dR^*} \left( \frac{1}{\rho^*} \right) \right] + O(\varepsilon^2).$$
(B12)

The term appearing in square brackets in the preceding expression can be reformulated as

$$p^* \frac{d}{dR^*} \left( \frac{1}{\rho^*} \right) = \frac{d}{dR^*} \left( \frac{p^*}{\rho^*} \right) - \frac{1}{\rho^*} \frac{dp^*}{dR^*} \equiv \frac{dT^*}{dR^*} - \frac{1}{\rho^*} \frac{dp^*}{dR^*}.$$

Consequently, with use of (B11), Eq. (B12) becomes

$$\frac{dT^*}{dR^*} = \frac{\chi}{1+\chi} 2BR^* + O(\varepsilon^2),$$
(B13)

in which the constant $\chi$ denotes the dimensionless parameter

$$\chi = C \frac{\cancel{R}}{M_w \hat{c}_v}.$$
(B14)

Integration of (B13), followed by subsequent conversion of the resulting expression to dimensional form, gives

$$T(R) - T(0) = \frac{1}{(\hat{c}_v/C) + (\cancel{R}/M_w)} \frac{(\Omega R)^2}{2} + \bar{T} O(\varepsilon^2).$$
(B15)

For the case of ideal gases one has from thermodynamics that $\hat{c}_p = \hat{c}_v + \cancel{R}/M_w$. Thus,

$$T(R) - T(0) = f(C) \frac{(\Omega R)^2}{2\hat{c}_p} + \bar{T} O(\varepsilon^2),$$
(B16)

where $f(C)$ is the dimensionless $O(1)$ quantity

$$f(C) = \left[ \frac{1}{\gamma} \left( \frac{1}{C} - 1 \right) + 1 \right]^{-1}$$
(B17)

in which

$$\gamma = \frac{\hat{c}_p}{\hat{c}_v}$$
(B18)

is the specific heat ratio. Note that $f(C) = 1$ when $C = 1$ and $f(C) = 0$ when $C = 0$.

Upon setting $R = R_o$ in (B16), the sought-after temperature difference (1.1) is found to be

$$\Delta T = f(C)\frac{(\Omega R_o)^2}{2\hat{c}_p} + \bar{T}O(\varepsilon^2). \qquad \text{(B19)}$$

With $\varepsilon = O(10^{-6})$ one sees for a mean temperature of, say, $\bar{T} = 300$ K, that any correction to the dominant term in the above expression is completely negligible for virtually all conceivable experiments.

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