Nonlinear Rayleigh-Taylor instability of rotating inviscid fluids

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(Received 23 October 2012; published 3 January 2013)

It is demonstrated theoretically that the nonlinear stage of the Rayleigh-Taylor instability can be retarded at arbitrary Atwood numbers in a rotating system with the axis of rotation normal to the acceleration of the interface between two uniform inviscid fluids. The Coriolis force provides an effective restoring force on the perturbed interface, and the uniform rotation will always decrease the nonlinear saturation amplitude of the interface at any disturbance wavelength.

DOI: 10.1103/PhysRevE.87.013001 PACS number(s): 47.20.Cq, 47.20.Ma, 47.20.Ky

I. INTRODUCTION

The instability of an interface, which separates two fluids and is accelerated into the heavier fluid, is known as the Rayleigh-Taylor instability (RTI) in honor of Rayleigh, who studied the unstable mode in a gravitational field [1], and Taylor, who considered the linear stability of unstable surface waves under acceleration [2]. RTI is of interest in a wide variety of fields, such as deep convection in oceans, inertial-confinement fusion (ICF), supernovae explosions, plasma physics, and vortex control. There are three stages in the RTI development. Interface disturbances in the linear stage of the classical RTI grow exponentially with a growth rate:

$$\gamma = \sqrt{gkA}, \quad A = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2},$$

where g is the interface acceleration, k is the wave number in the plane normal to g, and A is the Atwood number. ρ_1 and ρ_2 are the densities of the denser and lighter fluids, respectively. In the second stage, the interface is twisted into bubbles of lighter fluid and spikes of the heavier one penetrating into the lighter fluid along with the growth of perturbations. The bubble velocity saturates later on and the growth of the amplitude turns from exponential to linear in time. Such a transition is commonly referred to as a nonlinear saturation. At the last stage, the strong shear near the interface may cause vortices due to Kelvin-Helmholtz instability and then extensive interfacial mixing ensues. In order to study the nonlinear behavior of bubbles and spikes, weakly nonlinear theory has been developed [3–5] for irrotational flows.

Since RTI may dramatically reduce the performance of inertial confinement fusion (ICF) at both the initial implosion acceleration stage and the later deceleration stage by RT mixing [6], RTI should be suppressed as completely as possible [7–9]. For inviscid and irrotational uniform fluids, it has been shown that a uniform rotation with its axis parallel to the interface acceleration decreases the growth rate of normal mode disturbances [10]. Following this pioneering work a series of linear RTI analyses have been carried out to study the effect of viscosity [11], general rotation [12], and surface tension [13]. For exponentially stratified incompressible and viscous fluids, it is found that rotation tends to inhibit RTI through viscous effects [14]. Recent numerical simulations demonstrated that rotation with viscosity and diffusion, at least in the Boussinesq approximation or for low Atwood number flow, could retard RTI [15]. In the above study the rotation axis is parallel to the gravity, and the RTI suppression is explained by analyzing the vorticity field in the mixing zone. In other words, some studies of rotating RTI mainly focused on the case where the rotating vector is parallel to the interface acceleration, and hence the nonlinear influence of rotation about an axis normal to the direction of the acceleration on the bubble-spike formation was not studied theoretically. This is the motivation of the present paper.

II. PHYSICAL MODEL AND NONLINEAR THEORY

We consider an inner cylindrical fluid domain surrounded by an outer fluid. The inviscid fluids are immiscible and rotate around the z axis with the constant angular velocity Ω. y points in the radial direction and x is in the azimuthal direction. The acceleration g is in the same direction as y, and the densities of inner and outer fluids near the interface are assumed to be constant values ρ_1 and ρ_2, respectively. The interface at $r = R$ is accelerated to the inner fluid due to a disturbance as shown in Fig. 1(a). The disturbed flow near the interface can be described in the rectangular coordinate system fixed at the undisturbed interface as shown in Fig. 1(b) when the interface displacement η and the azimuthal wavelength are much smaller than R.

The two-dimensional potential flow of fluids is described by the hydrodynamical potentials ϕ_j and stream functions ψ_j; hence the disturbing velocities are

$$u_j = \frac{\partial \phi_j}{\partial x} = \frac{\partial \psi_j}{\partial y}, \quad v_j = \frac{\partial \phi_j}{\partial y} = -\frac{\partial \psi_j}{\partial x}, \tag{1}$$

where the subscript $j = 1, 2$ indicate inner and outer fluids, respectively. We assume that the instability is restricted to the neighborhood of the interface and hence apply the same boundary conditions as in the case of the traditional RTI. The

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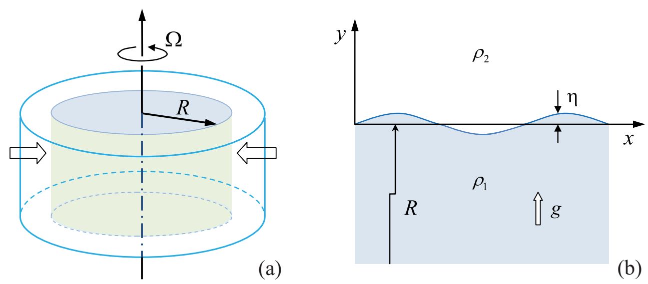

FIG. 1. (Color) (a) Schematic of the rotating system and (b) the coordinates used to study the interface instability. It is assumed that the interface displacement $\eta \ll R$, the radius of the interface.

governing equations and boundary conditions are

$$\nabla^2\psi_j = \nabla^2\phi_j = 0, \quad \begin{array}{ll} \nabla\psi_1 = \nabla\phi_1 = 0, & y \to -\infty, \\ \nabla\psi_2 = \nabla\phi_2 = 0, & y \to \infty, \end{array} \tag{2}$$

$$\partial_t\eta + \nabla\phi_j \cdot \nabla\eta = \phi_{j,y}, \quad y = \eta, \tag{3}$$

$$\rho_1\left\{\phi_{1,t} + \tfrac{1}{2}(\nabla\phi_1)^2 - gy + 2\Omega\psi_1\right\} = \rho_2\left\{\phi_{2,t} + \tfrac{1}{2}(\nabla\phi_2)^2 - gy + 2\Omega\psi_2\right\}, \quad y = \eta. \tag{4}$$

The above governing equations are the same as those of the classical RTI except for the Coriolis force terms. It is noted that the centrifugal effect brought by rotation is included in the acceleration g and will be discussed at the end of this paper. The interface displacement η and the disturbance variables ψ_j and ϕ_j can be expanded in powers of ϵ,

$$\begin{aligned}
\eta(x,t) &= \epsilon\eta^1(x,t) + \epsilon^2\eta^2(x,t) + \epsilon^3\eta^3(x,t) + O(\epsilon^4),\\
\phi_j(x,y,t) &= \epsilon\phi_j^1(x,y,t) + \epsilon^2\phi_j^2(x,y,t) + \epsilon^3\phi_j^3(x,y,t)\\
&\quad + O(\epsilon^4), \quad j = 1,2,\\
\psi_j(x,y,t) &= \epsilon\psi_j^1(x,y,t) + \epsilon^2\psi_j^2(x,y,t) + \epsilon^3\psi_j^3(x,y,t)\\
&\quad + O(\epsilon^4), \quad j = 1,2.
\end{aligned}$$

We insert this ansatz into Eqs. (2) to (4), which are expanded in a Taylor series about the undisturbed interfacial position ($\eta = 0$). By collecting the terms with the same powers in ϵ, we obtain the systems of order 1, 2, and 3, governing the linear instability and the second- and third-order corrections. In the first order of ϵ we obtain the equations

$$\nabla^2\psi_j^1 = \nabla^2\phi_j^1 = 0, \quad \begin{array}{ll} \nabla\psi_1^1 = \nabla\phi_1^1 = 0, & y \to -\infty, \\ \nabla\psi_2^1 = \nabla\phi_2^1 = 0, & y \to \infty, \end{array} \tag{5}$$

$$\partial_t\eta^1 = \phi_{j,y}^1, \quad y = 0, \tag{6}$$

$$\rho_1\left(\phi_{1,t}^1 - g\eta^1 + 2\Omega\psi_1^1\right) = \rho_2\left(\phi_{2,t}^1 - g\eta^1 + 2\Omega\psi_2^1\right), \quad y = 0. \tag{7}$$

Introducing normal mode of η^1, ψ_j^1, and ϕ_j^1 in the form of $\sim e^{i(kx+\omega t)}$, where k is the wave number in the x direction, we solve (5) to (7) and obtain the frequency for unstable mode $\omega = -i\gamma - \Omega A$, where $\gamma = \sqrt{gkA - A^2\Omega^2}$ is the linear growth rate and the Atwood number is $A = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$. For the case of a basic-mode initial disturbance of the form $\epsilon\cos(kx)$, the corresponding growing solution η is obtained up to the third order as follows:

$$\begin{aligned}
\eta = {} & \epsilon\cos(kx - A\Omega t)e^{\gamma t} + \epsilon^2 C_2\cos(2kx - 2A\Omega t)e^{2\gamma t}\\
& + \epsilon^3\big[C_{31}^c\cos(kx - A\Omega t) + C_{31}^s\sin(kx - A\Omega t)\\
& + C_3\cos(3kx - 3A\Omega t)\big]e^{3\gamma t},
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
C_2 &= \frac{kA}{2},\\
C_{31}^c &= -\frac{k^2}{16\gamma^2}(A^4\Omega^2 + A^2\Omega^2 + 3A^2\gamma^2 + \gamma^2),\\
C_{31}^s &= \frac{k^2A\Omega}{8\gamma}(A^2 + 1),\\
C_3 &= \frac{k^2}{8}(4A^2 - 1).
\end{aligned} \tag{9}$$

Different from the traditional RTI, the unstable modes are not stationary but traveling waves, though the phase velocity is much smaller than the rotation velocity ΩR. When the angular velocity Ω decreases to zero, the linear growth rate γ approaches the traditional value $\gamma_0 = \sqrt{gkA}$, and with $A = 1$ and $\Omega = 0$ the previous nonlinear solution of the interface position η is recovered [4].

The uniform rotation brings two additional forces, the Coriolis force and the centrifugal force. In order to analyze the Coriolis effect on the RTI, we first assume that the total acceleration of the interface g is a constant. According to the expression of the linear growth rate $\gamma = \sqrt{gkA - A^2\Omega^2}$, the Coriolis force always diminishes the growth of the RTI. It is shown in Fig. 2(a) that this retardation effect becomes stronger when the Atwood number A is increased. The reciprocal of the dimensionless angular velocity $\Omega^* = \Omega/\sqrt{gk}$ is the Rossby number. Secondly, since only the product gA appears in γ, the Coriolis suppression effect on the linear RTI works for both an acceleration stage ($g > 0$ and $A > 0$) and a deceleration stage ($g < 0$ and $A < 0$).

According to Eq. (8), the nonlinear evolution of the fundamental mode up to the third order can be expressed as $\eta_N\cos(kx - A\Omega t)$, where the nonlinear amplitude $\eta_N = (\eta_L + \eta_L^3 C_{31}^c)$ and the linear amplitude $\eta_L = \epsilon e^{\gamma t}$. Since $C_{31}^c < 0$ [see Eq. (9)], the third-order modulation always diminishes the growth of the fundamental mode. Such a nonlinear effect is described by the nonlinear suppression factor S:

$$S = \frac{\eta_N - \eta_L}{\eta_L^3 k^2} = -\frac{1}{16\gamma^2}(3A^2\gamma^2 + \gamma^2 + A^4\Omega^2 + A^2\Omega^2). \tag{10}$$

Apparently, S is a negative value; hence the larger $-S$, the stronger the nonlinear suppression exerted on the basic mode. It is shown in Fig. 2(b) that larger rotation velocity and Atwood number will cause stronger suppression of the nonlinear evolution of the RTI. The shape variation of spikes and bubbles including higher-order harmonics along with the uniform rotation are shown in Figs. 2(c) and 2(d).

The Coriolis suppression mechanism is straightforward and may be explained as follows. In first-order approximation, the disturbance velocity in the x (azimuthal) direction at

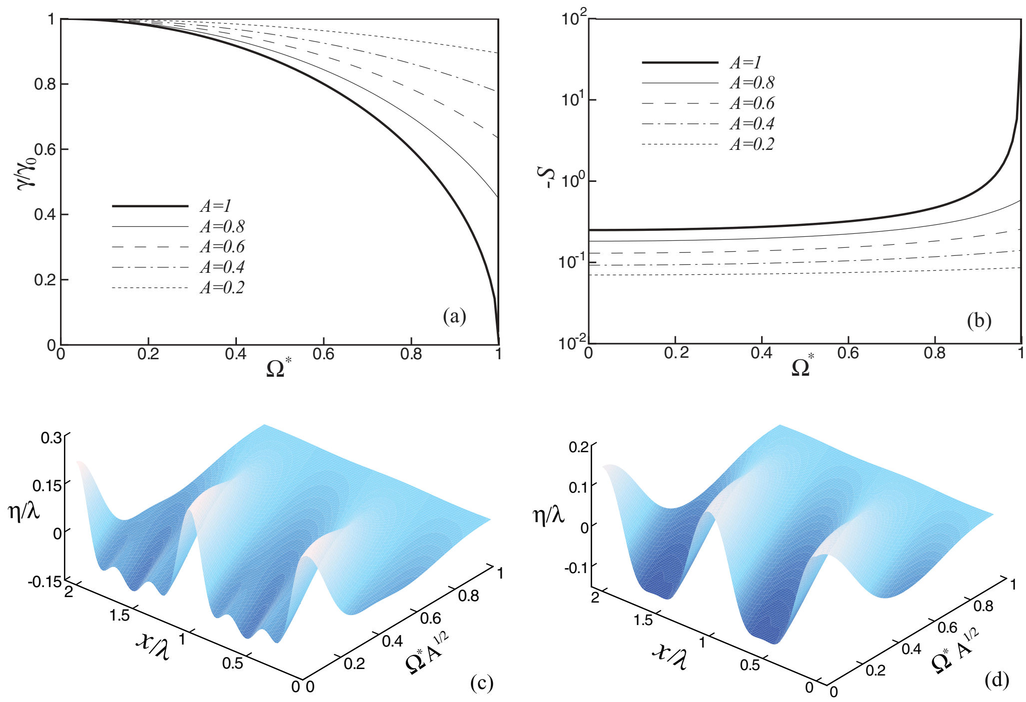

FIG. 2. (Color) (a) Linear growth rate γ and (b) the nonlinear suppression factor S of the fundamental mode as functions of the dimensionless rotation frequency. (c), (d) Interface displacements η at $\gamma_0 t = 5$ for $\epsilon k = 0.001$ in the cases $A = 1$ and $A = 0.4$, respectively. The definitions $\gamma_0 = \sqrt{gkA}$, $\Omega^* = \Omega/\sqrt{gk}$, and the wavelength $\lambda = 2\pi/k$ have been used.

the interface can be expressed as $u_1 \simeq -u_2 \simeq \epsilon e^{\gamma t}\cos(kx - A\Omega t + \theta)$, where θ is the rotation-induced phase difference between u and the interface perturbation η. $\cos\theta = A\Omega/\gamma_0 = \sqrt{A}\Omega^*$. It is shown in Fig. 3(a) that $\theta = \pi/2$ for the classical RTI case without rotation. However, a uniform rotation normal to the acceleration of the interface will change θ and cause a normal Coriolis force $-2\Omega u$. Since u_1 and u_2 share the same absolute value but with opposite directions at the interface, the direction of the effective normal Coriolis force F_n is the same as that of the heavier side. Consequently, the integrated F_n over every half wavelength behaves as a restoring force exerted on the distorted interface. When Ω^* increases to $1/\sqrt{A}$ or $\theta = 0$, u_1 is in the same phase as the interface displacement, and the effective normal Coriolis force, shown as arrows in Fig. 3(b), becomes a restoring force everywhere on the interface. It is easy to check that when $\Omega < 0$ the suppression mechanism of the Coriolis force works as well.

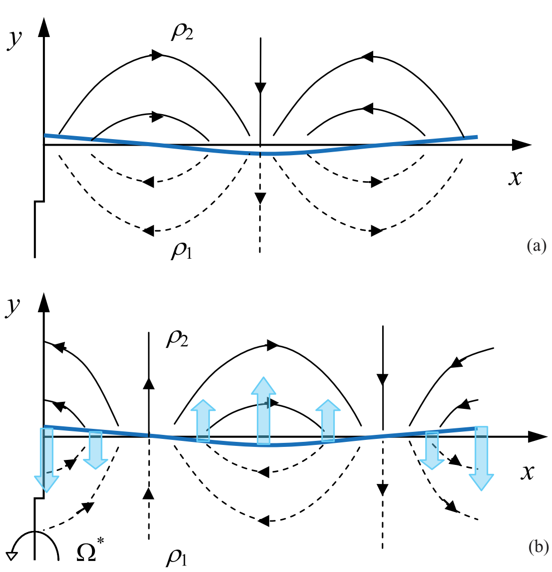

FIG. 3. (Color) Schematics illustrating the streamlines of inner (dashed lines) and outer (solid lines) fluids near the interface at (a) $\Omega^* = 0$ and (b) $\Omega^* A = 1$. The thick blue lines indicates the interface, and the arrows in (b) represent the effective Coriolis force.

In reality, the rotation induced centrifugal force changes the acceleration of the interface. In this paper we assume that the amplitude η and wavelength λ of the interface displacement are much smaller than R, and it is easy to find that Eq. (8) remains the same for cases including the centrifugal effect except that the interface acceleration $g = g_0 + R\Omega^2$ is used, where g_0 is the value without centrifugal effect. Consequently, the linear growth rate, including both the Coriolis effect and the centrifugal effect, is

$$\gamma = \sqrt{g_0 kA + R\Omega^2 kA - A^2\Omega^2}. \tag{11}$$

Since $Rk = 2\pi R/\lambda > 2\pi > A$, the effect of uniform rotation will enhance the linear RTI by increasing γ in the acceleration stage ($A > 0, g_0 > 0$) and retard the RTI by decreasing γ in the deceleration stage ($A < 0, g_0 < 0$). The perturbation enters the nonlinear regime when the amplitude is larger than a certain value. The shape of the interface changes from the sinusoidal form to broad thin bubbles and narrow thick spikes. It is convenient to define the transition into the nonlinear regime as occurring when the nonlinear saturation amplitude η_s is reached, where the amplitude of the fundamental mode is reduced by 10% due to nonlinear effects, i.e., $\eta_s = \eta_N = 0.9\eta_L$. From Eq. (8) we get

$$\frac{\eta_s}{\lambda} = \frac{1}{\pi}\sqrt{\frac{2}{5(3A^2+1)+5(A^4\Omega^2+A^2\Omega^2)/\gamma^2}}. \tag{12}$$

The appearance of an additional term $5(A^4\Omega^2 + A^2\Omega^2)/\gamma^2$ in the above equation is caused by the Coriolis force, and the centrifugal effect only changes the linear growth rate γ. Because this additional term is always positive, the surprising conclusion can be drawn that the uniform rotation will always decrease the nonlinear saturation amplitude at an arbitrary Atwood number and wavelength, during the acceleration stage as well as the deceleration stage. Bubbles and spikes will grow to a larger length scale before entering the nonlinear regime if the Atwood number and the rotation angular velocity are small. When $A = 1$ and $\Omega = 0$ the ratio $\eta_s/\lambda \simeq 0.1$, which is the widely used threshold for nonlinearity [16].

III. DISCUSSION

When the uniform rotation is significantly strong, i.e., $R\Omega^2 = |g_0|$, the RTI will be completely suppressed by rotation at the deceleration stage ($A < 0$ and $g_0 < 0$). For the acceleration stage ($A > 0$ and $g_0 > 0$), the additional term in Eq. (12) may be simplified as

$$\begin{aligned} 5(A^4\Omega^2 + A^2\Omega^2)/\gamma^2 &\simeq 5(A^4\Omega^2 + A^2\Omega^2)/(2g_0kA) \\ &= 5(A^3 + A)/(2Rk) \\ &\ll 5(A^3 + A)/4\pi < 1; \end{aligned}$$

hence the nonlinear saturation amplitude η_s is only slightly retarded by rotation. It is noted that the interface of the present model is a circular cylinder; hence the current stabilization mechanism is applicable to the equatorial region but not the polar regimes of a compressible rotating sphere. However, numerical simulations [15] have shown that when the gravity is parallel to the rotation, just as what happens at the polar regimes, the mixing zone generated by the RTI can be retarded by the effect of the Coriolis force on the vorticity field. Though these two mechanisms are different, the one proposed in this paper is applicable for potential flows and the other one is found in rotational flows; their cooperation makes it possible to suppress the Rayleigh-Taylor instability near a spherical interface by using rotation, and may have potential applications in the field of inertial confinement fusion, which includes both the acceleration and the deceleration stages.

ACKNOWLEDGMENTS

This work has been supported by the NSFC Grants (No. 10972007, No. 10921202, and No. 11225209).

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