Rogue waves of the Hirota and the Maxwell-Bloch equations

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In this paper, we derive a Darboux transformation of the Hirota and the Maxwell-Bloch (H-MB) system which is governed by femtosecond pulse propagation through an erbium doped fiber and further generalize it to the matrix form of the n-fold Darboux transformation of this system. This n-fold Darboux transformation implies the determinant representation of nth solutions of $(E^{[n]},p^{[n]},\eta^{[n]})$ generated from the known solution of (E,p,η). The determinant representation of $(E^{[n]},p^{[n]},\eta^{[n]})$ provides soliton solutions, positon solutions, and breather solutions (both bright and dark breathers) of the H-MB system. From the breather solutions, we also construct a bright and dark rogue wave solution for the H-MB system, which is currently one of the hottest topics in mathematics and physics. Surprisingly, the rogue wave solution for p and η has two peaks because of the order of the numerator and denominator of them. Meanwhile, after fixing the time and spatial parameters and changing two other unknown parameters α and β, we generate a rogue wave shape.

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I. INTRODUCTION

In the past four decades, nonlinear science has experienced an explosive growth with the invention of several exciting and fascinating new concepts such as solitons, dromions, positons, rogue waves, similaritons, supercontinuum generation, complete integrability, fractals, chaos, etc. Many of the completely integrable nonlinear partial differential systems (NPDEs) admit one of the most striking aspects of nonlinear phenomena, described as a soliton, which has a universal character and is also of great mathematical interest. The study of the solitons and other related solutions such as positons has become one of the most exciting and extremely active areas of research in the field of nonlinear sciences.

Among all concepts, in addition to solitons and positons [1–4], rogue waves have also been not only the subject of intensive research in oceanography [5–7], but also they have been studied extensively in several other areas, such as matter rogue waves [8,9] in Bose-Einstein condensates, rogue waves in surface and space plasmas [10], and financial rogue waves describing the possible physical mechanisms in financial markets and related fields [11]. In some of the above fields, a soliton system such as the nonlinear Schrödinger (NLS) equation [12], derivative NLS systems [13,14], and so on are considered and reported to admit rogue wave solutions under a certain specific choice of parameters. It has been proved that modulational instability is one of the main generating mechanisms for the rogue waves [12–17] and can be well described by the analytical expressions for the spectra of breather solutions at the point of extreme compression.

In 1967, McCall and Hahn [18] explored a special type of lossless pulse propagation in two-level resonant media. They have discovered the self-induced transparency (SIT) effect, which can be explained by using the Maxwell-Bloch (MB) system. If we consider these effects in an erbium doped nonlinear fiber, the system will be governed by a coupled system of the NLS and the MB equation (NLS-MB system) [19–23].

Rogue waves have been reported in different branches of physics, where the system dynamics is governed mostly by a single nonlinear partial differential equation [13,14]. But our main interest is to analyze the possibility of rogue waves in coupled nonlinear systems. The higher-order NLS and Maxwell-Bloch (HNLS-MB) system as a higher-order correction of the NLS-MB system was shown to admit a Lax pair and soliton-type pulse propagation [25–27]. Kodama [24] has shown that with a suitable transformation, the higher-order NLS equation can be reduced to the Hirota equation [28], whose rogue wave solution has already been reported in Refs. [29,30]. In a similar way, after a suitable choice of self-steepening and self-frequency effects, we obtain the H-MB system in the following form [31]:

$$E_z = i\alpha\left(\tfrac{1}{2}E_{tt} + |E|^2E\right) + \beta(E_{ttt} + 6|E|^2E_t) + 2p, \tag{1.1}$$

$$p_t = 2i\omega p + 2E\eta, \tag{1.2}$$

$$\eta_t = -(Ep^* + E^*p), \tag{1.3}$$

where E is the normalized slowly varying amplitude of the complex field envelope, p is the polarization, η means the population inversion, (ω,α,β) are three real constants, and $*$ represents a complex conjugate. β represents the strength of the higher-order linear and nonlinear effects.

The H-MB system has been shown to be integrable and also admits the Lax pair and other required properties of complete integrability [31]. Among many analytical methods, it is well known that the Darboux transformation is an efficient method to generate the soliton solutions for integrable systems [32]. The determinant representation of the n-fold Darboux transformation of the Ablowitz-Kaup-Newell-Segur (AKNS) system was given in Refs. [33,34]. The main task of this paper will be to construct an n-fold Darboux transformation of the H-MB system and find different kinds of solutions of the H-MB system by using the Darboux transformation.

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The paper is organized as follows. In Sec. II, the Lax representation of the H-MB system is introduced. In Sec. III, we derived the onefold Darboux transformation of the H-MB system. In Sec. IV, the generalization of the onefold Darboux transformation to n-fold Darboux transformation of the H-MB system will be given. Using these Darboux transformations, one-soliton, two-soliton, and positon solutions are derived in Secs. V and VI by assuming trivial seed solutions. In Sec. VII, starting from a periodic seed solution, the breather solution of the H-MB system is provided. A Taylor expansion from the breather solution will help us to construct the rogue wave solution in Sec. VIII. Section IX is devoted to conclusion and discussions.

II. LAX REPRESENTATION OF THE H-MB SYSTEM

In this section, we will concentrate on the linear eigenvalue problem of the Hirota and the Maxwell-Bloch (H-MB) system. The linear eigenvalue problem is expressed in the form of the Lax pair U and V as

$$\Phi_t = U\Phi, \quad \Phi_z = V\Phi, \tag{2.1}$$

where

$$U = \lambda\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix} = -i\lambda\sigma_3 + U_0, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2}$$

$$\begin{aligned} V &= \lambda^3\begin{pmatrix} 4i\beta & 0 \\ 0 & -4i\beta \end{pmatrix} + \lambda^2\begin{pmatrix} -\alpha i & -4\beta E \\ 4\beta E^* & \alpha i \end{pmatrix} + \lambda\begin{pmatrix} -2\beta i|E|^2 & \alpha E - 2\beta i E_t \\ -\alpha E^* - 2\beta i E_t^* & 2\beta i|E|^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{\alpha}{2}i|E|^2 - \beta(EE_t^* - E_tE^*) & 2\beta|E|^2E + \frac{\alpha}{2}iE_t + \beta E_{tt} \\ -2\beta|E|^2E^* + \frac{\alpha}{2}iE_t^* - \beta E_{tt}^* & -\frac{\alpha}{2}i|E|^2 + \beta(EE_t^* - E_tE^*) \end{pmatrix} + i\frac{1}{\lambda+\omega}\begin{pmatrix} \eta & -p \\ -p^* & -\eta \end{pmatrix} \end{aligned} \tag{2.3}$$

$$:= \lambda^3V_3 + \lambda^2V_2 + \lambda V_1 + V_0 + i\frac{1}{\lambda+\omega}V_{-1}, \tag{2.4}$$

$$\Phi = \Phi(\lambda) = \begin{pmatrix} \Phi_1(\lambda,t,z) \\ \Phi_2(\lambda,t,z) \end{pmatrix}, \tag{2.5}$$

is an eigenfunction associated with eigenvalue parameter λ of the linear Eq. (2.1), and V_i denotes the coefficient matrix of term λ^i. We obtain the classical Hirota and the Maxwell-Bloch system when $\alpha = 2, \beta = -1$. Being different from the AKNS system, only a part of the V matrix is polynomials in terms of E and its t derivatives in this system. Using the above linear system of the H-MB system, a onefold Darboux transformation will be introduced in the next section.

III. ONEFOLD DARBOUX TRANSFORMATION FOR THE H-MB SYSTEM

In this section, we construct and prove the onefold Darboux transformation for the H-MB system. First, we consider the transformation about the linear function Φ in the form

$$\Phi' = T\Phi = (\lambda A - S)\Phi, \tag{3.1}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}. \tag{3.2}$$

The new function Φ' satisfies

$$\Phi'_t = U'\Phi', \tag{3.3}$$

$$\Phi'_z = V'\Phi'. \tag{3.4}$$

Then the matrix T should satisfy the following identities:

$$T_t + TU = U'T, \tag{3.5}$$

$$T_z + TV = V'T. \tag{3.6}$$

Substituting the matrices A and S into Eq. (3.5) and comparing the coefficients of both sides will lead to the following conditions:

$$a_{12} = a_{21} = 0, \quad (a_{11})_t = (a_{22})_t = 0. \tag{3.7}$$

For our further discussions, we choose $A = I$ and $T = (\lambda I - S)$. The relation between old solutions (E,p,η) and new solutions (E',p',η'), which is called the Darboux transformation, can be obtained by using Eqs. (3.5) and (3.6).

From Eq. (3.5), we have

$$E' = E - 2is_{12}, \quad s_{21} = -s_{12}^* \tag{3.8}$$

$$S_t = \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}S - S\begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix} + i[S,\sigma_3]S. \tag{3.9}$$

Similarly, using Eq. (3.6), we obtain the following set of relations:

$$\begin{aligned} -S_z + (\lambda I - S)&\left(\lambda^3V_3 + \lambda^2V_2 + \lambda V_1 + V_0 + \frac{iV_{-1}}{\lambda+\omega}\right) \\ &= \left(\lambda^3V_3 + \lambda^2V'_2 + \lambda V'_1 + V'_0 + \frac{iV'_{-1}}{\lambda+\omega}\right)(\lambda I - S). \end{aligned} \tag{3.10}$$

Multiplying both sides of Eq. (3.10) by $\lambda I - S$ will lead to

$$\begin{aligned} &-S_z(\lambda+\omega) + (\lambda I - S)(-4i\lambda^3(\lambda+\omega)\sigma_3 \\ &\quad + \lambda^2(\lambda+\omega)V_2 + \lambda(\lambda+\omega)V_1 + V_0(\lambda+\omega) + iV_{-1}) \\ &= (-4i\lambda^3(\lambda+\omega)\sigma_3 + \lambda^2(\lambda+\omega)V'_2 \\ &\quad + \lambda(\lambda+\omega)V'_1 + V'_0(\lambda+\omega) + iV'_{-1})(\lambda I - S). \end{aligned}$$

Collecting the different powers of λ, we obtain the following set of identities:

$$\lambda^0: \quad S_z = (V_0' + i\omega^{-1}V_{-1}')S - S(i\omega^{-1}V_{-1} + V_0), \quad (3.11)$$

$$\lambda: \quad S_z = (\omega V_0 + iV_{-1}) - S(\omega V_1 + V_0) + (\omega V_1' + V_0')S + (-\omega V_0' - iV_{-1}'), \quad (3.12)$$

$$\lambda^2: \quad (\omega V_1 + V_0) - S(\omega V_2 + V_1) = (\omega V_1' + V_0') - (\omega V_2' + V_1')S. \quad (3.13)$$

$$\lambda^3: \quad (\omega V_2 + V_1) - S(\omega V_3 + V_2) = (\omega V_2' + V_1') - (\omega V_3 + V_2')S, \quad (3.14)$$

$$\lambda^4: \quad V_2' = V_2 - [S, V_3]. \quad (3.15)$$

From the above identities, after simplifications, we get

$$E' = E - 2is_{12}, \quad (3.16)$$

$$V_{-1}' = (S + \omega)V_{-1}(S + \omega)^{-1}, \quad (3.17)$$

which later gives a onefold Darboux transformation of the H-MB system.

We suppose

$$S = H\Lambda H^{-1}, \quad (3.18)$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

$$H = \begin{pmatrix} \Phi_1(\lambda_1,t,z) & \Phi_1(\lambda_2,t,z) \\ \Phi_2(\lambda_1,t,z) & \Phi_2(\lambda_2,t,z) \end{pmatrix} := \begin{pmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{pmatrix}.$$

In order to satisfy the constraints of S and V_{-1}', which is similar to V_{-1}, i.e., $s_{21} = -s_{12}^*$, the following constraints will be used:

$$\lambda_2 = \lambda_1^*, \quad s_{11} = s_{22}^*, \quad (3.19)$$

$$H = \begin{pmatrix} \Phi_1(\lambda_1,t,z) & -\Phi_2^*(\lambda_1,t,z) \\ \Phi_2(\lambda_1,t,z) & \Phi_1^*(\lambda_1,t,z) \end{pmatrix}. \quad (3.20)$$

After tedious calculations, Eqs. (3.16)–(3.20) and (2.1) will lead to Eqs. (3.5) and (3.6), i.e., the transformation equations (3.16) and (3.17) with the conditions (3.19), where Eq. (3.20) is the Darboux transformation of the H-MB system.

The detailed form of the onefold Darboux transformation of the H-MB system in terms of eigenfunctions will be given in the next section.

IV. DETERMINANT REPRESENTATION OF *n*-FOLD DARBOUX TRANSFORMATION

In this section, we will construct the determinant representation of the n-fold Darboux transformation of the H-MB system. For this purpose, we introduce n eigenfunctions,

$$\begin{pmatrix} \Phi_{1,i} \\ \Phi_{2,i} \end{pmatrix} = \Phi|_{\lambda=\lambda_i}, \quad i = 1,2,\ldots,2n, \quad (4.1)$$

with the following constraint on the eigenvalues, $\lambda_{2n-1} = \lambda_{2n}^*$, and the reduction conditions on eigenfunctions as $\Phi_{2,2n} = \Phi_{1,2n-1}^*$ and $\Phi_{2,2n-1} = -\Phi_{1,2n}^*$. For our further discussions, this reduction condition has been used.

For completeness, as the simplest Darboux transformation, the determinant representation of the onefold Darboux transformation of the H-MB system will be introduced in the following theorem using identities (3.16) and (3.17).

The onefold Darboux transformation of the H-MB system is expressed as

$$E^{[1]} = E - 2i\frac{(\lambda_1 - \lambda_1^*)\Phi_{1,1}\Phi_{2,1}^*}{\Delta_1}, \quad (4.2)$$

$$p^{[1]} = \frac{1}{T_{\Delta_1}}[2\eta[(-\omega - \lambda_1)|\Phi_{1,1}|^2 + (-\omega - \lambda_1^*)|\Phi_{2,1}|^2] \times (\lambda_1^* - \lambda_1)\Phi_{1,1}\Phi_{2,1}^* - p^*(\lambda_1^* - \lambda_1)^2\Phi_{1,1}^2\Phi_{2,1}^{*2} + p[(-\omega - \lambda_1)|\Phi_{1,1}|^2 + (-\omega - \lambda_1^*)|\Phi_{2,1}|^2]^2], \quad (4.3)$$

$$\eta^{[1]} = \frac{1}{T_{\Delta_1}}[\eta[(\omega + \lambda_1)|\Phi_{1,1}|^2 + (\omega + \lambda_1^*)|\Phi_{2,1}|^2] \times [(\omega + \lambda_1^*)|\Phi_{1,1}|^2 + (\omega + \lambda_1)|\Phi_{2,1}|^2] + (\lambda_1^* - \lambda_1)^2|\Phi_{1,1}|^2|\Phi_{2,1}|^2] - p^*(\lambda_1 - \lambda_1^*)\Phi_{1,1}\Phi_{2,1}^*[(\omega + \lambda_1^*)|\Phi_{1,1}|^2 + (\omega + \lambda_1)|\Phi_{2,1}|^2] + p[(\omega + \lambda_1)|\Phi_{1,1}|^2 + (\omega + \lambda_1^*)|\Phi_{2,1}|^2](\lambda_1 - \lambda_1^*)\Phi_{1,1}^*\Phi_{2,1}], \quad (4.4)$$

where

$$T_{\Delta_1} = [(\omega + \lambda_1)|\Phi_{1,1}|^2 + (\omega + \lambda_1^*)|\Phi_{2,1}|^2] \times [(\omega + \lambda_1^*)|\Phi_{1,1}|^2 + (\omega + \lambda_1)|\Phi_{2,1}|^2] - (\lambda_1^* - \lambda_1)^2|\Phi_{1,1}|^2|\Phi_{2,1}|^2. \quad (4.5)$$

It can be easily proved that the new solution $\eta^{[1]}$ is always real. This onefold transformation will be used to generate the one-soliton solution from trivial seed solutions of the H-MB system. Also this onefold Darboux transformation can be further generalized to construct the n-fold Darboux transformation of the H-MB system which is proposed in the following theorem.

Theorem 1. The n-fold Darboux transformation of the H-MB system can be represented as

$$T_n(\lambda;\lambda_1,\lambda_2,\lambda_3,\lambda_4,\ldots,\lambda_{2n}) \quad (4.6)$$

$$= \lambda^n I + t_{n-1}^{[n]}\lambda^{n-1} + \cdots + t_1^{[n]}\lambda + t_0^{[n]} \quad (4.7)$$

$$= \frac{1}{\Delta_n}\begin{pmatrix} (\mathbb{T}_n)_{11} & (\mathbb{T}_n)_{12} \\ (\mathbb{T}_n)_{21} & (\mathbb{T}_n)_{22} \end{pmatrix}, \quad (4.8)$$

where

$$\Delta_n = \begin{vmatrix} \Phi_{1,1} & \Phi_{2,1} & \lambda_1\Phi_{1,1} & \lambda_1\Phi_{2,1} & \cdots & \lambda_1^{n-1}\Phi_{1,1} & \lambda_1^{n-1}\Phi_{2,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2\Phi_{1,2} & \lambda_2\Phi_{2,2} & \cdots & \lambda_2^{n-1}\Phi_{1,2} & \lambda_2^{n-1}\Phi_{2,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3\Phi_{1,3} & \lambda_3\Phi_{2,3} & \cdots & \lambda_3^{n-1}\Phi_{1,3} & \lambda_3^{n-1}\Phi_{2,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4\Phi_{1,4} & \lambda_4\Phi_{2,4} & \cdots & \lambda_4^{n-1}\Phi_{1,4} & \lambda_4^{n-1}\Phi_{2,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1}\Phi_{1,2n-1} & \lambda_{2n-1}\Phi_{2,2n-1} & \cdots & \lambda_{2n-1}^{n-1}\Phi_{1,2n-1} & \lambda_{2n-1}^{n-1}\Phi_{2,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n}\Phi_{1,2n} & \lambda_{2n}\Phi_{2,2n} & \cdots & \lambda_{2n}^{n-1}\Phi_{1,2n} & \lambda_{2n}^{n-1}\Phi_{2,2n} \end{vmatrix},$$

$$(\mathbb{T}_n)_{11} = \begin{vmatrix} 1 & 0 & \lambda & 0 & \cdots & \lambda^{n-1} & 0 & \lambda^n \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1\Phi_{1,1} & \lambda_1\Phi_{2,1} & \cdots & \lambda_1^{n-1}\Phi_{1,1} & \lambda_1^{n-1}\Phi_{2,1} & \lambda_1^n\Phi_{1,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2\Phi_{1,2} & \lambda_2\Phi_{2,2} & \cdots & \lambda_2^{n-1}\Phi_{1,2} & \lambda_2^{n-1}\Phi_{2,2} & \lambda_2^n\Phi_{1,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3\Phi_{1,3} & \lambda_3\Phi_{2,3} & \cdots & \lambda_3^{n-1}\Phi_{1,3} & \lambda_3^{n-1}\Phi_{2,3} & \lambda_3^n\Phi_{1,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4\Phi_{1,4} & \lambda_4\Phi_{2,4} & \cdots & \lambda_4^{n-1}\Phi_{1,4} & \lambda_4^{n-1}\Phi_{2,4} & \lambda_4^n\Phi_{1,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1}\Phi_{1,2n-1} & \lambda_{2n-1}\Phi_{2,2n-1} & \cdots & \lambda_{2n-1}^{n-1}\Phi_{1,2n-1} & \lambda_{2n-1}^{n-1}\Phi_{2,2n-1} & \lambda_{2n-1}^n\Phi_{1,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n}\Phi_{1,2n} & \lambda_{2n}\Phi_{2,2n} & \cdots & \lambda_{2n}^{n-1}\Phi_{1,2n} & \lambda_{2n}^{n-1}\Phi_{2,2n} & \lambda_{2n}^n\Phi_{1,2n} \end{vmatrix}$$

$$(\mathbb{T}_n)_{12} = \begin{vmatrix} 0 & 1 & 0 & \lambda & \cdots & 0 & \lambda^{n-1} & 0 \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1\Phi_{1,1} & \lambda_1\Phi_{2,1} & \cdots & \lambda_1^{n-1}\Phi_{1,1} & \lambda_1^{n-1}\Phi_{2,1} & \lambda_1^n\Phi_{1,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2\Phi_{1,2} & \lambda_2\Phi_{2,2} & \cdots & \lambda_2^{n-1}\Phi_{1,2} & \lambda_2^{n-1}\Phi_{2,2} & \lambda_2^n\Phi_{1,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3\Phi_{1,3} & \lambda_3\Phi_{2,3} & \cdots & \lambda_3^{n-1}\Phi_{1,3} & \lambda_3^{n-1}\Phi_{2,3} & \lambda_3^n\Phi_{1,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4\Phi_{1,4} & \lambda_4\Phi_{2,4} & \cdots & \lambda_4^{n-1}\Phi_{1,4} & \lambda_4^{n-1}\Phi_{2,4} & \lambda_4^n\Phi_{1,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1}\Phi_{1,2n-1} & \lambda_{2n-1}\Phi_{2,2n-1} & \cdots & \lambda_{2n-1}^{n-1}\Phi_{1,2n-1} & \lambda_{2n-1}^{n-1}\Phi_{2,2n-1} & \lambda_{2n-1}^n\Phi_{1,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n}\Phi_{1,2n} & \lambda_{2n}\Phi_{2,2n} & \cdots & \lambda_{2n}^{n-1}\Phi_{1,2n} & \lambda_{2n}^{n-1}\Phi_{2,2n} & \lambda_{2n}^n\Phi_{1,2n} \end{vmatrix},$$

$$(\mathbb{T}_n)_{21} = \begin{vmatrix} 1 & 0 & \lambda & 0 & \cdots & \lambda^{n-1} & 0 & 0 \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1\Phi_{1,1} & \lambda_1\Phi_{2,1} & \cdots & \lambda_1^{n-1}\Phi_{1,1} & \lambda_1^{n-1}\Phi_{2,1} & \lambda_1^n\Phi_{2,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2\Phi_{1,2} & \lambda_2\Phi_{2,2} & \cdots & \lambda_2^{n-1}\Phi_{1,2} & \lambda_2^{n-1}\Phi_{2,2} & \lambda_2^n\Phi_{2,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3\Phi_{1,3} & \lambda_3\Phi_{2,3} & \cdots & \lambda_3^{n-1}\Phi_{1,3} & \lambda_3^{n-1}\Phi_{2,3} & \lambda_3^n\Phi_{2,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4\Phi_{1,4} & \lambda_4\Phi_{2,4} & \cdots & \lambda_4^{n-1}\Phi_{1,4} & \lambda_4^{n-1}\Phi_{2,4} & \lambda_4^n\Phi_{2,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1}\Phi_{1,2n-1} & \lambda_{2n-1}\Phi_{2,2n-1} & \cdots & \lambda_{2n-1}^{n-1}\Phi_{1,2n-1} & \lambda_{2n-1}^{n-1}\Phi_{2,2n-1} & \lambda_{2n-1}^n\Phi_{2,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n}\Phi_{1,2n} & \lambda_{2n}\Phi_{2,2n} & \cdots & \lambda_{2n}^{n-1}\Phi_{1,2n} & \lambda_{2n}^{n-1}\Phi_{2,2n} & \lambda_{2n}^n\Phi_{2,2n} \end{vmatrix},$$

$$(\mathbb{T}_n)_{22} = \begin{vmatrix} 0 & 1 & 0 & \lambda & \cdots & 0 & \lambda^{n-1} & \lambda^n \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1\Phi_{1,1} & \lambda_1\Phi_{2,1} & \cdots & \lambda_1^{n-1}\Phi_{1,1} & \lambda_1^{n-1}\Phi_{2,1} & \lambda_1^n\Phi_{2,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2\Phi_{1,2} & \lambda_2\Phi_{2,2} & \cdots & \lambda_2^{n-1}\Phi_{1,2} & \lambda_2^{n-1}\Phi_{2,2} & \lambda_2^n\Phi_{2,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3\Phi_{1,3} & \lambda_3\Phi_{2,3} & \cdots & \lambda_3^{n-1}\Phi_{1,3} & \lambda_3^{n-1}\Phi_{2,3} & \lambda_3^n\Phi_{2,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4\Phi_{1,4} & \lambda_4\Phi_{2,4} & \cdots & \lambda_4^{n-1}\Phi_{1,4} & \lambda_4^{n-1}\Phi_{2,4} & \lambda_4^n\Phi_{2,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1}\Phi_{1,2n-1} & \lambda_{2n-1}\Phi_{2,2n-1} & \cdots & \lambda_{2n-1}^{n-1}\Phi_{1,2n-1} & \lambda_{2n-1}^{n-1}\Phi_{2,2n-1} & \lambda_{2n-1}^n\Phi_{2,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n}\Phi_{1,2n} & \lambda_{2n}\Phi_{2,2n} & \cdots & \lambda_{2n}^{n-1}\Phi_{1,2n} & \lambda_{2n}^{n-1}\Phi_{2,2n} & \lambda_{2n}^n\Phi_{2,2n} \end{vmatrix}.$$

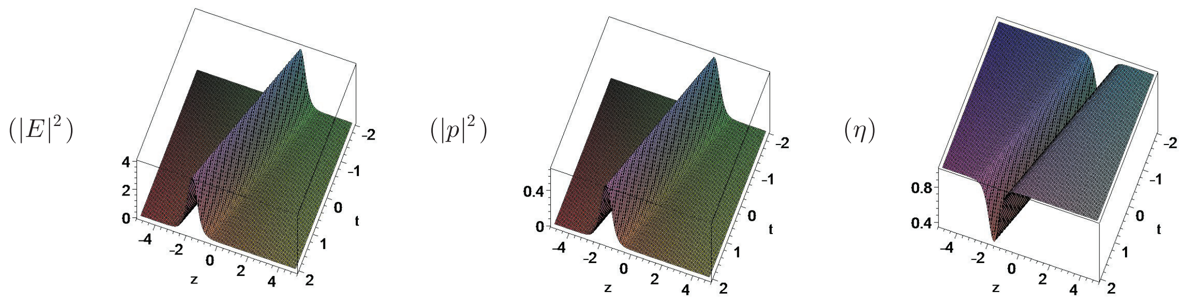

FIG. 1. (Color online) One-soliton solution (E,p,η) of the H-MB system with $\omega = 1.5$, $\alpha_1 = 0.5$, $\beta_1 = 1$, $\alpha = 2$, $\beta = -1$.

The following identity can be found:

$$T_n(\lambda;\lambda_1,\lambda_2,\lambda_3,\lambda_4,\ldots,\lambda_{2n})|_{\lambda=\lambda_i}\begin{pmatrix}\Phi_{1,i}\\ \Phi_{2,i}\end{pmatrix} = 0, \quad (4.9)$$

where $i = 1,2,3,\ldots,2n$. Similarly, the Darboux transformation for (E,p,η) can be constructed by using the following identities:

$$T_{nt} + T_n U = U^{[n]} T_n, \quad (4.10)$$

$$T_{nz} + T_n V = V^{[n]} T_n, \quad (4.11)$$

which can be further simplified as

$$U_0^{[n]} = U_0 + i\left[\sigma_3, t_{n-1}^{[n]}\right], \quad (4.12)$$

$$V_{-1}^{[n]} = T_n\big|_{\lambda=-\omega} V_{-1} T_n^{-1}\big|_{\lambda=-\omega}. \quad (4.13)$$

The proof for Eq. (4.13) is quite complicated in general when compared to the proof of one- and twofold Darboux transformations. However, this will become simple if we know the origin of T_n [33]. If we treat the n-fold Darboux transformation as a generalization of the $(n-1)$ Darboux transformation, the transformation will be multiplications of n onefold Darboux transformations at $\lambda = -\omega$. It is easy to prove that these multiplications can have a determinant representation as mentioned above.

The nth new solution after the n-fold Darboux transformation of the H-MB system will be

$$E^{[n]} = E + 2i\left(t_{n-1}^{[n]}\right)_{12}, \quad (4.14)$$

$$p^{[n]} = [2\eta(T_n)_{11}(T_n)_{12} - p^*(T_n)_{12}(T_n)_{12} + p(T_n)_{11}(T_n)_{11}]/ [(T_n)_{11}(T_n)_{22} - (T_n)_{12}(T_n)_{21}]|_{\lambda=-\omega}, \quad (4.15)$$

$$\eta^{[n]} = [\eta((T_n)_{11}(T_n)_{22} + (T_n)_{12}(T_n)_{21}) - p^*(T_n)_{12}(T_n)_{22} + p(T_n)_{11}(T_n)_{21}]/ [(T_n)_{11}(T_n)_{22} - (T_n)_{12}(T_n)_{21}]|_{\lambda=-\omega}, \quad (4.16)$$

where $(t_{n-1}^{[n]})_{12}$ is the element on the first row and second column in the matrix $t_{n-1}^{[n]}$. So far, we discussed the determinant construction of the nth Darboux transformation of the H-MB system. As an application of these transformations, the soliton and positon solutions of the H-MB system will be constructed in the next section.

V. SOLITON SOLUTIONS OF THE H-MB SYSTEM

In this section, having obtained the Daurboux transformation for our system, our next aim is to construct the one-soliton solution of the H-MB system by assuming suitable seed solutions. We assume trivial seed solutions as $E = 0$, $p = 0$, $\eta = 1$, and then the linear system becomes

$$\Phi_t = U\Phi, \quad (5.1)$$

$$\Phi_z = V\Phi, \quad (5.2)$$

where

$$\Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}, \quad (5.3)$$

$$U = \begin{pmatrix}-i\lambda & 0\\ 0 & i\lambda\end{pmatrix}, \quad (5.4)$$

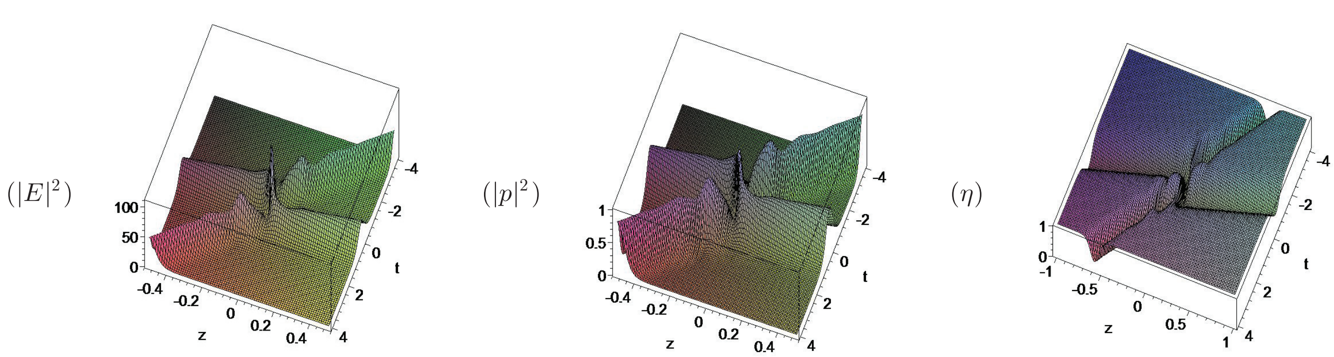

FIG. 2. (Color online) Two-soliton solution (E,p,η) of the H-MB system with $\omega = 1.5$, $\alpha_1 = 0.5$, $\beta_1 = 1$, $\alpha = 2$, $\beta = -1$.

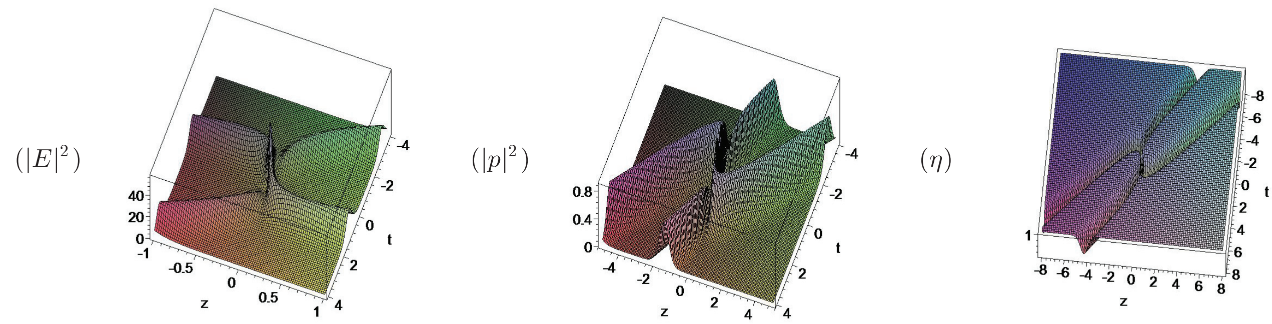

FIG. 3. (Color online) One-positon solution (E,p,η) of the H-MB system when $\omega = 1.5$, $\alpha_1 = 0.5$, $\beta_1 = 1$, $\alpha = 2$, $\beta = -1$.

$$V = \begin{pmatrix} 4i\beta\lambda^3 - \alpha i\lambda^2 & 0 \\ 0 & -4\beta i\lambda^3 + \alpha i\lambda^2 \end{pmatrix} + \frac{i}{\lambda+\omega}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.5}$$

From the above system, we construct the explicit eigenfunctions in the form

$$\Phi_1 = e^{-i\lambda t + (4\beta i\lambda^3 - \alpha i\lambda^2 + \frac{i}{\lambda+\omega})z + \frac{x_0+iy_0}{2}},$$
$$\Phi_2 = e^{i\lambda t + (-4\beta i\lambda^3 + \alpha i\lambda^2 - \frac{i}{\lambda+\omega})z - \frac{x_0+iy_0}{2} + i\theta},$$

where x_0, y_0, and θ are all arbitrarily fixed real constants. Substituting these two eigenfunctions into the onefold Darboux transformation equations (4.2)–(4.4) and choosing $\lambda = \alpha_1 + I\beta_1$, $x_0 = 0$, $y_0 = 0$, $\theta = 0$, then the following forms of soliton solutions are obtained:

$$E = 2\beta_1 e^{-2i\frac{C}{B}} \operatorname{sech}\left(2\beta_1 \frac{A}{B}\right), \tag{5.6}$$

$$p = \frac{i\beta_1((\alpha_1 - i\beta_1 + \omega)e^{-2\frac{D}{B}} + (\alpha_1 + \beta_1 i + \omega)e^{-2\frac{F}{B}})}{B} \times \operatorname{sech}^2\left(2\beta_1 \frac{A}{B}\right), \tag{5.7}$$

$$\eta = 1 - \frac{2\beta_1^2}{B}\operatorname{sech}^2\left(2\beta_1 \frac{A}{B}\right), \tag{5.8}$$

where A,B,C,D,F are explicitly given in Appendix A.

If we choose $\alpha = 1$, $\beta = 0$, the one-soliton solution is just the soliton solution of Eq. (15) of the NLS-MB system mentioned in Ref. [22] with $\alpha_1 = -\rho_1$, $\beta_1 = \nu_1$. Similarly, substituting these two eigenfunctions into the onefold Darboux transformation equations (4.2)–(4.4) and choosing $\alpha = 2$, $\beta = -1$, then the one-solition solutions of the classical H-MB system can be obtained, whose evolution is shown in Fig. 1, which clearly indicates that E and p are bright solitons because their waves are under the flat nonvanishing plane whereas η is a dark soliton.

Now let us discuss the construction of the two-soliton solutions of the H-MB system. For this purpose, we have to use two spectral parameters, $\lambda_1 = \alpha_1 + i\beta_1$ and $\lambda_2 = \alpha_2 + i\beta_2$. After the second Darboux transformation, we can construct the two-soliton solutions. As the general form of the two-soliton solution is quite tedious in nature, for simplicity, we are giving only the two-soliton solution of E with $\omega = 1.5$, $\alpha_1 = 0.5$, $\beta_1 = 1$, $\alpha_2 = 1$, $\beta_2 = 1.5$, $\alpha = 2$, and $\beta = -1$,

$$\begin{aligned} E_{2\text{-sol}} = & -i(17e^{-0.011\,764\,705\,88i(85t-1258z+1815iz+255it)} \\ & -(36+18i)e^{-0.011\,764\,705\,88i(170t-4385z+204iz+170it)} \\ & -7e^{0.011\,764\,705\,88i(-85t+1258z+1815iz+255it)} \\ & +(18i-12)e^{0.011\,764\,705\,88i(-170t+4385z+204iz+170it)} \\ & -16ie^{0.011\,764\,705\,88i(-85t+1258z+1815iz+255it)} \\ & +20ie^{-0.011\,764\,705\,88i(85t-1258z+1815iz+255it)})/ \\ & (-24\cos(0.999\,999\,999\,8t - 36.788\,235\,29z) \\ & +26\cosh(t + 18.952\,941\,18z) \\ & +2\cosh(-5t - 23.752\,941\,18z)). \end{aligned}$$

We also constructed the two-soliton solution for p and η in a similar manner. For completeness, instead of giving complicated forms of p and η, the graphical representation of them is shown in Fig. 2.

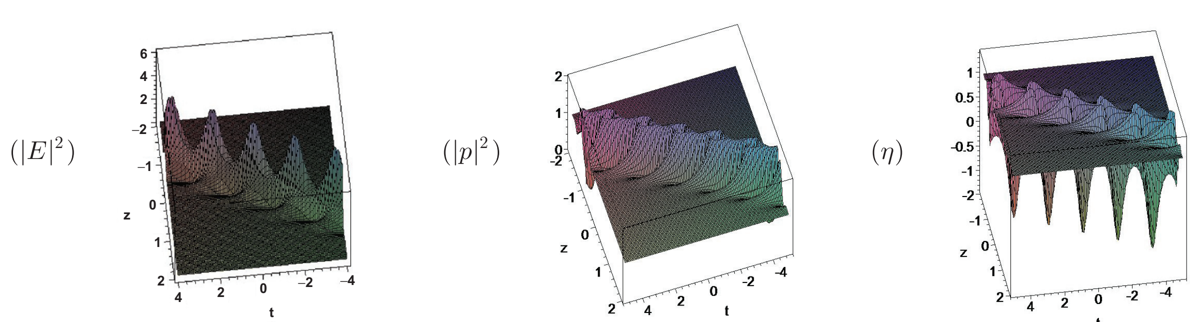

FIG. 4. (Color online) Breather solution (E,p,η) of the H-MB equation when $\omega = 0.5$, $d = 0.5$, $n_1 = 1$, $\alpha_1 = -1$, $\beta_1 = 1$, $\alpha = 2$, $\beta = -1$.

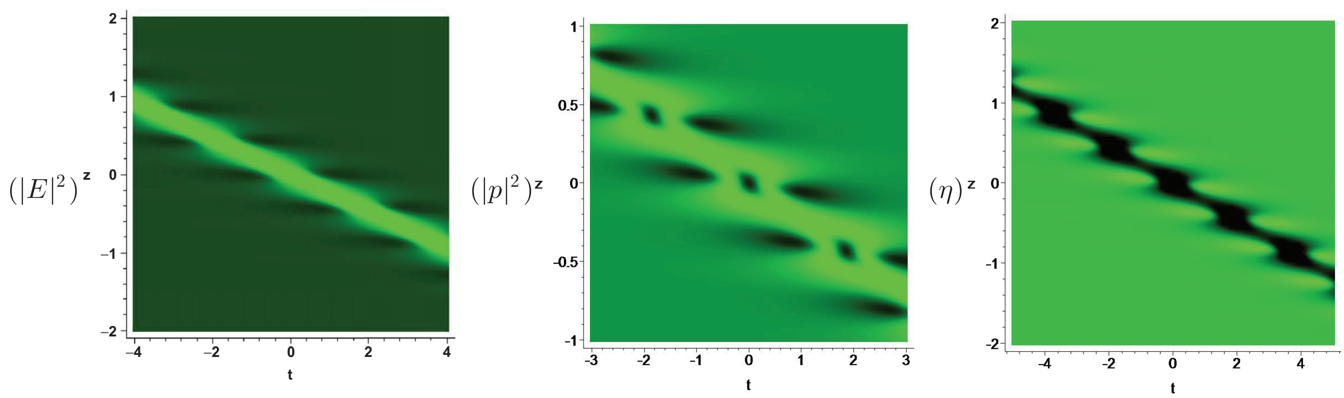

FIG. 5. (Color online) Breather solution (E,p,η) of the H-MB equation when $\omega = 0.5$, $d = 0.5$, $\alpha_1 = -1$, $\beta_1 = 1$, $\alpha = 2$, $\beta = -1$.

VI. BRIGHT AND DARK POSITON SOLUTIONS OF THE H-MB SYSTEM

In the case of the two-soliton solution constructed above, if the second spectral parameter λ_2 is assumed to be close to the first spectral parameter λ_1, and doing the Taylor expansion of the wave function to first order up to λ_1 will lead to a different kind of solution, which is called the degenerate soliton [34]–smooth positon solution. "Positon" was coined by Matvee [1,2,4] for the Korteweg–de Vries (KdV) equation by the same limiting approach. Note that the positon of the KdV is a singular solution. For the construction of positon solutions, the following four linear functions are needed to construct the second Darboux transformation and to generate the positon solutions:

$$\Phi_{1,1} = e^{-i\lambda_1 t+(4\beta i\lambda_1^3-\alpha i\lambda_1^2+\frac{i}{\lambda_1+\omega})z+\frac{x_0+iy_0}{2}},$$
$$\Phi_{2,1} = e^{i\lambda_1 t+(-4\beta i\lambda_1^3+\alpha i\lambda_1^2-\frac{i}{\lambda_1+\omega})z-\frac{x_0+iy_0}{2}+i\theta},$$
$$\Phi_{1,3} = e^{-i\lambda_3 t+(4\beta i\lambda_3^3-\alpha i\lambda_3^2+\frac{i}{\lambda_3+\omega})z+\frac{x_0+iy_0}{2}},$$
$$\Phi_{2,3} = e^{i\lambda_3 t+(-4\beta i\lambda_3^3+\alpha i\lambda_3^2-\frac{i}{\lambda_3+\omega})z-\frac{x_0+iy_0}{2}+i\theta}.$$

Now we take $\lambda_3 = \lambda_1 + \epsilon(1+i)$ and use the Taylor expansion of wave function ϕ_3 and ϕ_4 up to first order of ϵ in terms of λ_1. For example, choosing $\omega = 1.5, \alpha_1 = 0.5, \beta_1 = 1$, $\alpha = 2$, and $\beta = -1$, the positon solution E_p is constructed in the form

$$\begin{aligned} E_p = {} & 8ie^{14.8iz-it}[1582z\cosh(2t+2.40z) \\ & -688iz\sinh(2t+2.4z)-50i\cosh(2t+2.4z) \\ & +100it\sinh(2t+2.4z)]/[400t^2+119\,296z^2 \\ & -5504tz+50+50\cosh(4.8z+4t)]. \end{aligned}$$

In this case, the pictorial representation of the positon solutions (E_p,p_p,η_p) of the H-MB system is shown in Fig. 3.

From the above figures, we observe that two peaks of the positon solutions are at the same height, which is different from the two-soliton solution. Meanwhile the two waves depart at a relatively less speed after their collision. This is also different from the two solitons, which depart at a fixed speed. Here again, we find that E and p are bright positon solutions whereas η is a dark positon in all three cases discussed above.

VII. BRIGHT AND DARK BREATHER SOLUTIONS OF THE H-MB SYSTEM

In the last two sections, soliton solutions and positon solutions have been generated for the H-MB system. In this section, we will now focus on a different kind of solution, which is also derived from periodic solutions through the Darboux transformation. The resulting periodic solutions can be called breather solutions. Now, let us assume the seed

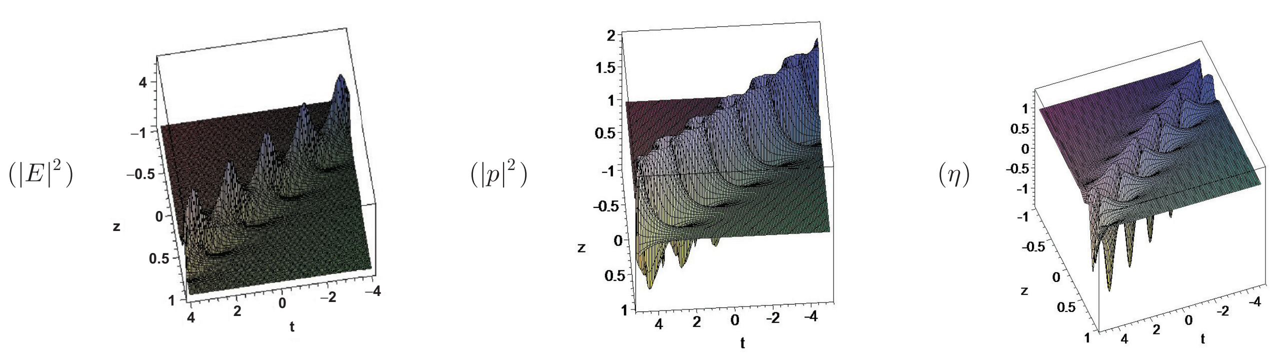

FIG. 6. (Color online) Breather solution (E,p,η) of the (0,1) CMKdV-MB equation when $\omega = 0.5$, $d = 0.5$, $\alpha_1 = -1$, $\beta_1 = 1$, $\alpha = 0$, $\beta = 1$.

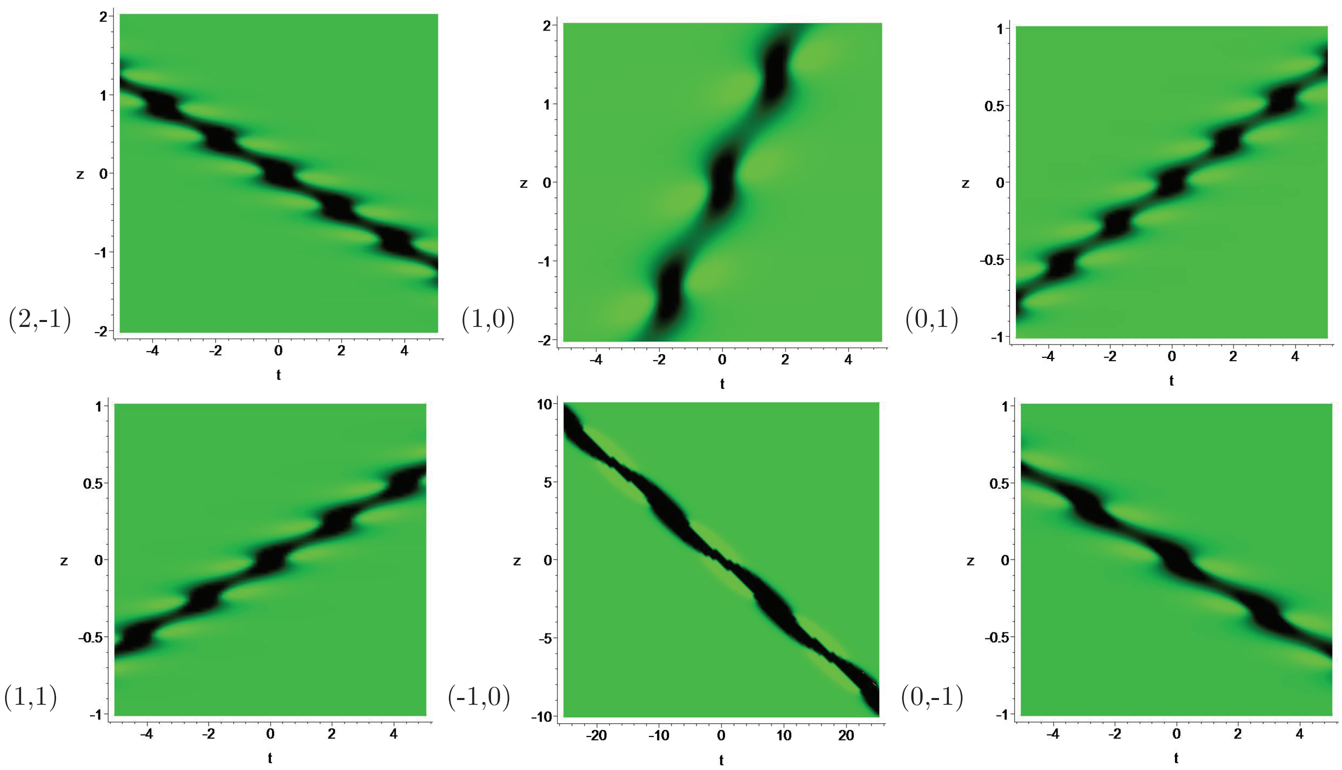

FIG. 7. (Color online) Breather solution η for different values of (α,β) of the H-MB system when $\omega = 0.5$, $d = 0.5$, $\alpha_1 = -1$, $\beta_1 = 1$.

solutions as $E = de^{i\rho}$, $p = ifE$, $\eta = 1$, $\rho = az + bt$, which admits the constraint in the form

$$2a + \alpha b^2 - 2\alpha d^2 + 2\beta b^3 - 12\beta d^2 b - 4f = 0, \tag{7.1}$$

$$fb - 2\omega f + 2 = 0. \tag{7.2}$$

Defining

$$R := \sqrt{-b^2 - 4b\lambda - 4\lambda^2 - 4d^2}, \tag{7.3}$$

the following wave function $\Phi = \binom{\phi_1}{\phi_2}$ is obtained in terms of R as

$$\phi_1(R) := e^{\frac{i(P_1 z + Q_1 t)}{4(b-2\omega)(\lambda+\omega)}}, \tag{7.4}$$

$$\phi_2(R) := \frac{1}{2d}(bi + 2i\lambda + R)e^{\frac{i(P_2 z + Q_2 t)}{4(b-2\omega)(\lambda+\omega)}}. \tag{7.5}$$

where P_1, Q_1, P_2, Q_2 are polynomials independent of t and z and their complete expressions are given in Appendix B.

If we use the above two wave functions to construct the two functions h_1 and h_2 as in Ref. [22], then the resulting solutions obtained through the Darboux transformation are found to have no meaning. Therefore, we would like to construct more complicated but physically meaningful solutions in the following part. By combining these two wave functions, we derive the functions h_1 and h_2 as follows:

$$h_1 := g_1(\lambda) - g_1(\lambda^*)^*, \quad h_2 := g_2(\lambda) + g_1(\lambda^*)^*, \tag{7.6}$$

where

$$g_1 := \phi_1(R) + \phi_1(-R), \quad g_2 := \phi_2(R) + \phi_2(-R). \tag{7.7}$$

It can be proved that h_1 and h_2 are also the solutions of the Lax equation with $\lambda := \alpha_1 + i\beta_1$. Using these two wave functions h_1 and h_2 in the onefold Darboux transformation will lead to the construction of breather solutions of the H-MB system. To simplify the calculations, we take $b = -2\alpha_1$ and use the second Darboux transformation discussed in the last section,

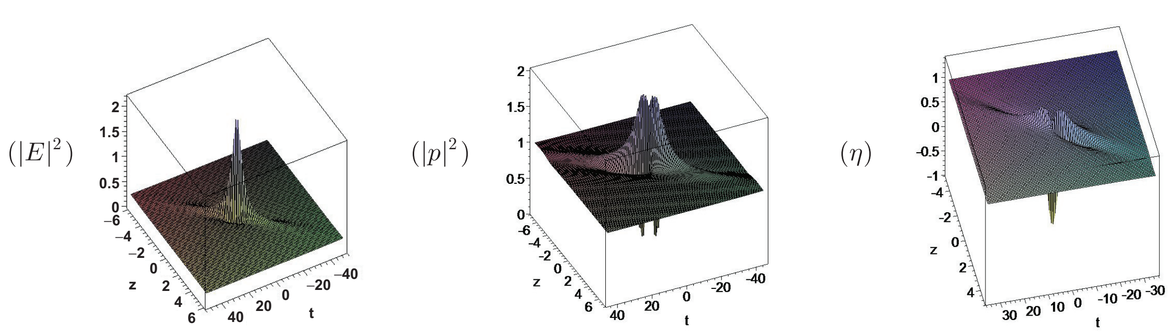

FIG. 8. (Color online) Rogue wave solution (E,p,η) of the H-MB system when $\omega = 0.5$, $d = 0.5$, $\alpha_1 = -1$, $\beta_1 = 1$, $\alpha = 2$, $\beta = -1$.

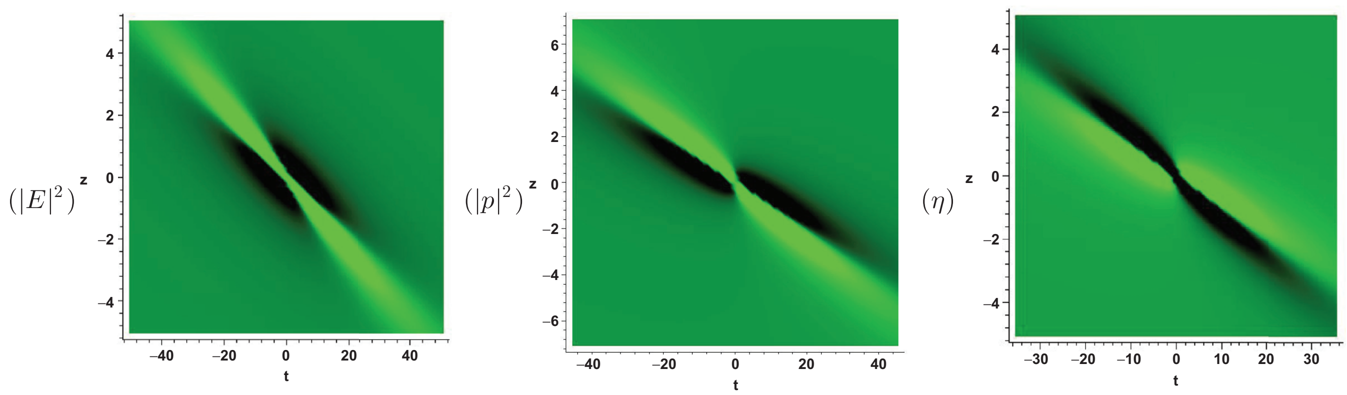

FIG. 9. (Color online) Rogue wave solution (E,p,η) of the H-MB equation when $\omega = 0.5$, $d = 0.5$, $\alpha_1 = -1$, $\beta_1 = 1$, $\alpha = 2$, $\beta = -1$.

then the final form of the breather solution E_b is obtained in the form

$$E_b = \Big[d^2\big(e^{i\frac{A_b(w)}{Y}} + e^{\frac{A_b(-w)}{Y}i}\big) + 2d\beta_1\big(e^{i\frac{C_b^*}{Y}} + e^{\frac{C_b}{Y}i}\big) - d\beta_1\big(e^{\frac{D_b}{X}i} + e^{\frac{D_b^*}{X}i}\big) + 2\beta_1(w-\beta_1)e^{\frac{F_b(w)}{Y}i} - 2\beta_1(w+\beta_1)e^{\frac{F_b(-w)}{Y}i}\Big]\Big/ \left[2d\cosh\left(2iw\beta_1 z\frac{G_b}{Y}\right) - 2\beta_1\cosh\left(2w\frac{H_b}{Y}\right)\right], \quad (7.8)$$

where

$$w = \sqrt{\beta_1^2 - d^2}, \quad (7.9)$$

$$X = (\alpha_1 + \beta_1 i + \omega)(-\alpha_1 + \beta_1 i - \omega), \quad (7.10)$$

$$Y = (\alpha_1 + \omega)(\alpha_1 + \beta_1 i + \omega)(-\alpha_1 + \beta_1 i - \omega), \quad (7.11)$$

and $A_b(w)$, C_b, D_b, $F_b(w)$, and G_b, H_b are polynomials of t, z, ω, w, a, b, d, α, β, α_1, and β_1, which are defined in Appendix B.

Similarly, the breather form of p and η can be constructed. For example, after taking the values

$$\omega = 0.5, \quad d = 0.5, \quad \alpha_1 = -1, \quad \beta_1 = 1, \quad \alpha = 2, \quad \beta = -1, \quad (7.12)$$

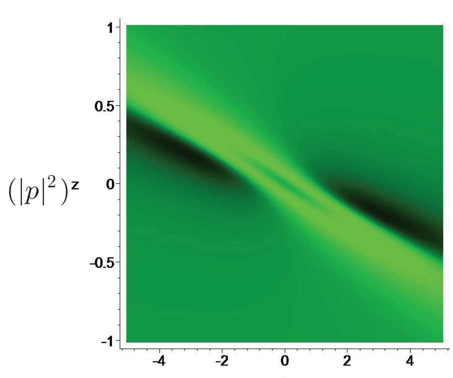

FIG. 10. (Color online) Enlarged graph of rogue wave solution p (in Fig. 9) of the H-MB system when $\omega = 0.5$, $d = 0.5$, $\alpha_1 = -1$, $\beta_1 = 1$, $\alpha = 2$, $\beta = -1$.

the breather solution of the H-MB system is plotted in Figs. 4 and 5.

Similarly, for the next choice of parameters $\omega = 1.5$, $\alpha_1 = 0.5$, $\beta_1 = 1$, $\alpha = 0$, $\beta = 1$, the breather solutions of the complex modified Korteweg–de Vries (CMKdV)-MB system are obtained. The picture of breather solutions is shown in Fig. 6. In addition to the above observation, we also find how the values of (α,β) change the direction of the breather solution η in the (t,z) plane (see Fig. 7).

Having constructed bright breathers for E and p and a dark breather for η, in the next section, our aim is to discuss the construction of rogue wave solutions of the H-MB system, which is in fact one single period of breather solutions.

VIII. BRIGHT AND DARK ROGUE WAVES IN THE H-MB SYSTEM

In this section, using the limit method of the NLS equation, we construct the rogue wave solutions of the H-MB system [34]. This kind of solution only appears in some special regions of time and distance and then will be drowned in one fixed nonvanishing plane. If we do the Taylor expansion to the breather solution (7.8) around $\beta_1 = d$, one rogue wave solution of E will be obtained and rogue waves for p and η can also be constructed in a similar way. In the following, for brevity, we only report the rogue wave for E_r in the form

$$E_r = \frac{\bar{A}(t,z)d}{\bar{B}(t,z)}\exp\bigg(\frac{i}{\alpha_1+\omega}\big(8z\beta\alpha_1^4 + 8z\beta\alpha_1^3\omega - 2z\alpha\alpha_1^2\omega + 2z - 2\alpha_1 t\omega + z\alpha d^2\omega - 12z\beta d^2\alpha_1\omega - 2z\alpha\alpha_1^3 - 2\alpha_1^2 t + z\alpha d^2\alpha_1 - 12z\beta d^2\alpha_1^2\big)\bigg), \quad (8.1)$$

where $\bar{A}(t,z), \bar{B}(t,z)$ are polynomials of t, z, ω, w, a, b, d, α, β, α_1, β_1, which are defined in Appendix C.

When we take $\omega = 0.5$, $d = 0.5$, $\alpha_1 = -1$, $\beta_1 = 1$, $\alpha = 2$, $\beta = -1$, the final forms of the rogue wave solutions will be

$$E_r = \left(\frac{8(1-3iz)}{4+(2t+17z)^2+36z^2} - \frac{1}{2}\right)e^{0.5i(-5z+4t)}, \quad (8.2)$$

$$p_r = e^{0.5i(-5z+4t)}(103\,149\,307\,3iz^4 + 103\,125\,00iz^2 - 531\,250\,0z - 625\,000t - 152\,343\,760z^3 - 187\,500\,0t^2z - 318\,750\,00tz^2$$

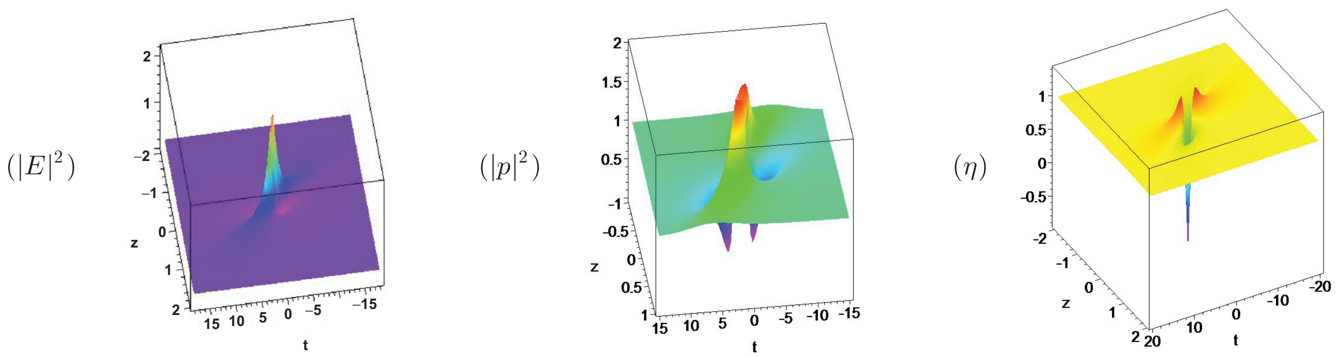

FIG. 11. (Color online) Rogue wave solution (E,p,η) of the (0,1) CMKdV-MB equation when $\omega=0.5$, $d=0.5$, $\alpha_1=-1$, $\beta_1=1$, $\alpha=0$, $\beta=1$.

$$+\,705\,468\,75it^2z^2+531\,250\,0it^3z+431\,640\,580itz^3+187\,500\,0itz+156\,250it^4-156\,250i)/(156\,250t^4+253\,906\,25z^2+312\,500t^2+431\,640\,580tz^3+705\,469\,00t^2z^2+531\,250\,0t^3z+156\,250+531\,250\,0tz+103\,149\,363\,6z^4), \quad (8.3)$$

$$\eta_r=(16t^4+64t^2+7224t^2z^2+105\,625z^4+2992z^2+442\,00tz^3-16+544t^3z+896zt)/(16t^4+32t^2+7224t^2z^2+105\,625z^4+2600z^2+442\,00tz^3+16+544t^3z+544zt). \quad (8.4)$$

From Eq. (8.2), it is clearly observed that the height of the background of $|E_r|^2$ is $\frac{1}{4}$ and the orders of the numerators and denominators of p_r and η_r are four. Because of these reasons, the graphs of the rogue waves for p_r and η_r have double peaks, which are shown in Fig. 8 and the corresponding density graph is plotted in Fig. 9.

For further understanding of our observations, we enlarge the above density in Fig. 9, and some zoomed portions of the above figures are clearly shown in Fig. 10. From the graph of $|p|^2$ shown in Fig. 10, we find one cave appears on top of the single peak with two caves on both sides of the peak.

To realize the significance of the different parameters α and β, we also consider the case when $\alpha=0$, $\beta=1$, i.e., the CMKdV-MB system (see Fig. 11). From the figures, we also find that the parameters α and β will change the shape, pulse width, etc., of the rogue wave. Therefore, in the following, we fix time t and distance z to see the role of parameters α and β and their impact on rogue wave dynamics.

We keep α and β as arbitrary parameters and choose $\omega=0.5$, $d=0.5$, $\alpha_1=-1$, $\beta_1=1$, $t=1$, $z=1$. This is also a rogue wave whose graph is portrayed in Fig. 12. Here we provide only the specific form of the rogue wave solution $E_{r\alpha\beta}$ as

$$E_{r\alpha\beta}=e^{-\frac{i}{4}(8+20\beta+7\alpha)}[40\alpha-17\alpha^2+156\beta-40-585\beta^2-192\alpha\beta+i(32+8\alpha+96\beta)]/(1170\beta^2-80\alpha+34\alpha^2-312\beta+112+384\alpha\beta). \quad (8.5)$$

This implies that after fixing the values of t and z, the solution depending on parameters α,β is also in the form of a rogue wave. This will give us some idea about how to modify the parameters α and β to visualize our theoretical results in terms of the experimental results in optics. From the above, one can easily conclude that E and p are bright rogue waves and η is a dark rogue wave.

All the solutions mentioned above including positons and the rogue wave solutions are indeed solutions of the H-MB system, which are verified by using the symbolic computation software MAPLE.

IX. CONCLUSION AND DISCUSSIONS

In this paper, after a suitable choice of self-steepening and self-frequency shift effects, we have derived the Darboux

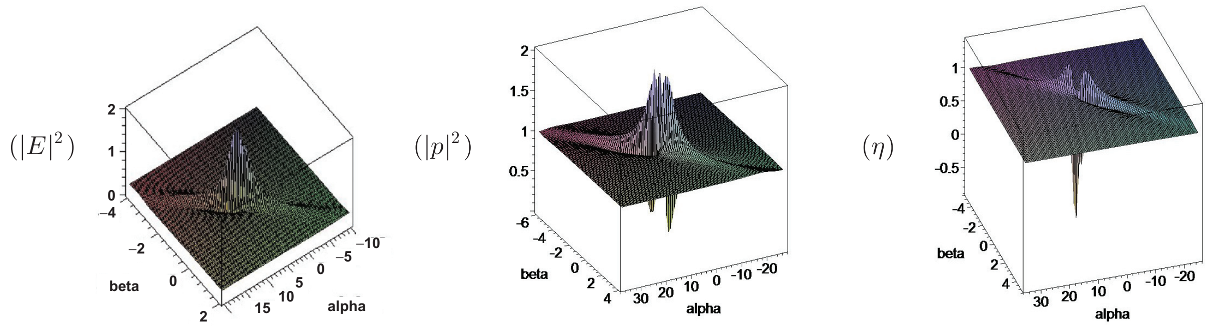

FIG. 12. (Color online) Rogue wave solution (E,p,η) of the H-MB equation when $\omega=0.5$, $d=0.5$, $\alpha_1=-1$, $\beta_1=1$, $t=1$, $z=1$.

transformation of the H-MB system which is governed by ultrashort pulse propagation through an erbium doped nonlinear optical waveguide and further generalized it to the matrix form of an n-fold Darboux transformation, which implies the determinant representation of $(E^{[n]},p^{[n]},\eta^{[n]})$ generated from a known solution (E,p,η). By choosing some special eigenvalues $\lambda_{2n-1}=\lambda_{2n}^*$ and eigenfunctions using the reduction conditions $\Phi_{1,2n-1}=\Phi_{2,2n}^*$, $\Phi_{2,2n-1}=-\Phi_{1,2n}^*$, the determinant representation of $(E^{[n]},p^{[n]},\eta^{[n]})$ provided some different solutions of the H-MB system. As examples, soliton solutions, breather solutions, and rogue wave solutions of the H-MB system have been constructed explicitly by using the Darboux transformation from trivial and periodic seed solutions. The rogue waves show interesting characteristics which might attract physicists to observe them in experiments with higher-order optical effects in the femtosecond regime. The interesting characteristics obtained contain the following two sides: (i) The rogue wave solution for p,η is surprisingly found by us to have two peaks because the order of the numerator and denominator of p,η in Eqs. (8.3) and (8.4) is four and (ii) after fixing the time and spatial parameter and by changing the other two unknown parameters α and β, we find a rogue wave shape also arises, as shown in Eq. (8.5). Still, there are a few interesting questions which are still unclear, such as, for example, the physical interpretations and observation of higher-order positon solutions, the role of higher-order rogue wave solutions and their applications in physics, in particular, the connection between rogue wave solutions and supercontinuum generation through modulation instability or soliton fission, etc., with higher-order optical effects.

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APPENDIX A: EXPLICIT EXPRESSIONS FOR *A*, *B*, *C*, *D*, AND *F*

$$
\begin{aligned}
A =& -24z\beta\alpha_1^3\omega+4z\alpha\alpha_1^2\omega-8z\beta\alpha_1^2\beta_1^2+2z\alpha\alpha_1\beta_1^2+4\beta_1^2z\beta\omega^2+2z\alpha\alpha_1\omega^2-12z\beta\alpha_1^2\omega^2+z\\
&+8z\beta\alpha_1\beta_1^2\omega+t\alpha_1^2+t\beta_1^2+t\omega^2+2t\alpha_1\omega-12z\beta\alpha_1^4+2z\alpha\alpha_1^3+4z\beta\beta_1^4,\\
B =& \alpha_1^2+2\alpha_1\omega+\omega^2+\beta_1^2,\\
C =& 2t\alpha_1^2\omega+t\alpha_1\omega^2+t\beta_1^2\alpha_1-4z\beta\alpha_1^5+z\alpha\alpha_1^4-z\alpha\beta_1^4-z\alpha_1+t\alpha_1^3+24z\beta\alpha_1^2\beta_1^2\omega+12z\beta\alpha_1\beta_1^2\omega^2-2z\alpha\alpha_1\beta_1^2\omega\\
&-8z\beta\alpha_1^4\omega-4z\beta\alpha_1^3\omega^2+8z\beta\alpha_1^3\beta_1^2+12z\beta\beta_1^4\alpha_1+2z\alpha\alpha_1^3\omega+z\alpha\alpha_1^2\omega^2-z\alpha\beta_1^2\omega^2-z\omega,\\
D :=& -24\beta_1z\beta\alpha_1^3\omega+4\beta_1z\alpha\alpha_1^2\omega+2\beta_1z\alpha\alpha_1\omega^2-12\beta_1z\beta\alpha_1^2\omega^2+8\beta_1^3z\beta\alpha_1\omega+\beta_1z+\beta_1^3t+4\beta_1^5z\beta+\beta_1t\omega^2+\beta_1t\alpha_1^2\\
&-8\beta_1^3z\beta\alpha_1^2+2\beta_1^3z\alpha\alpha_1+4\beta_1^3z\beta\omega^2+2\beta_1t\alpha_1\omega-12\beta_1z\beta\alpha_1^4+2\beta_1z\alpha\alpha_1^3+i\big(t\beta_1^2\alpha_1+z\alpha\alpha_1^4-z\alpha_1-z\omega+t\alpha_1^3+t\alpha_1\omega^2\\
&+2t\alpha_1^2\omega-z\alpha\beta_1^2\omega^2-4z\beta\alpha_1^3\omega^2-8z\beta\alpha_1^4\omega-4z\beta\alpha_1^5-z\alpha\beta_1^4-2z\alpha\alpha_1\beta_1^2\omega+8z\beta\alpha_1^3\beta_1^2+12z\beta\beta_1^4\alpha_1+2z\alpha\alpha_1^3\omega\\
&+24z\beta\alpha_1^2\beta_1^2\omega+12z\beta\alpha_1\beta_1^2\omega^2+z\alpha\alpha_1^2\omega^2\big),\\
F :=& 24\beta_1z\beta\alpha_1^3\omega-4\beta_1z\alpha\alpha_1^2\omega-2\beta_1z\alpha\alpha_1\omega^2+12\beta_1z\beta\alpha_1^2\omega^2-8\beta_1^3z\beta\alpha_1\omega-\beta_1z-\beta_1^3t-4\beta_1^5z\beta-\beta_1t\omega^2-\beta_1t\alpha_1^2+8\beta_1^3z\beta\alpha_1^2\\
&-2\beta_1^3z\alpha\alpha_1-4\beta_1^3z\beta\omega^2-2\beta_1t\alpha_1\omega+12\beta_1z\beta\alpha_1^4-2\beta_1z\alpha\alpha_1^3+i\big(t\beta_1^2\alpha_1+z\alpha\alpha_1^4-z\alpha_1-z\omega+t\alpha_1^3+t\alpha_1\omega^2+2t\alpha_1^2\omega\\
&-z\alpha\beta_1^2\omega^2-4z\beta\alpha_1^3\omega^2-8z\beta\alpha_1^4\omega-4z\beta\alpha_1^5-z\alpha\beta_1^4-2z\alpha\alpha_1\beta_1^2\omega+8z\beta\alpha_1^3\beta_1^2+12z\beta\beta_1^4\alpha_1+2z\alpha\alpha_1^3\omega\\
&+24z\beta\alpha_1^2\beta_1^2\omega+12z\beta\alpha_1\beta_1^2\omega^2+z\alpha\alpha_1^2\omega^2\big),
\end{aligned}
$$

APPENDIX B: EXPRESSIONS FOR P_1, Q_1, P_2, Q_2

$$
\begin{aligned}
P_1 =& 12\beta d^2b^2\lambda-4\alpha d^2\lambda\omega+4\beta b^3\lambda\omega+2\alpha d^2b\omega+2\alpha d^2b\lambda+2\alpha b^2\lambda\omega-24\beta d^2b\omega^2+12\beta d^2b^2\omega-8\lambda-8\omega-4\alpha d^2\omega^2\\
&-2\beta b^4\lambda-2\beta b^4\omega+2\alpha b^2\omega^2+4\beta b^3\omega^2-\alpha b^3\lambda-\alpha b^3\omega-24\beta d^2b\lambda\omega+2i\omega\beta Rb^3+2i\lambda\beta Rb^3+\lambda\alpha Rb^2i+\omega\alpha Rb^2i\\
&+16i\lambda^2\omega\beta Rb+4iR+8i\beta\lambda b\omega^2R-8i\beta\lambda b^2\omega R-4id^2\beta\omega Rb-4id^2\beta\lambda Rb-4i\alpha b\lambda R\omega+8id^2\beta\lambda R\omega-4i\beta b^2\omega^2R\\
&-16i\beta\lambda^3R\omega-2i\alpha\lambda^2Rb-2i\alpha b\omega^2R-16i\beta\lambda^2\omega^2R-4i\beta\lambda^2b^2R+4i\alpha\lambda^2R\omega+8i\beta\lambda^3Rb+8id^2\beta\omega^2R+4i\alpha\lambda\omega^2R,\\
Q_1 =& 2b^2\omega-4b\lambda\omega-4b\omega^2+2b^2\lambda-2iRb\omega+4iR\omega\lambda+4iR\omega^2-2iRb\lambda,\\
P_2 =& -12\beta d^2b^2\lambda+4\alpha d^2\lambda\omega-4\beta b^3\lambda\omega-2\alpha d^2b\omega-2\alpha d^2b\lambda-2\alpha b^2\lambda\omega+24\beta d^2b\omega^2-12\beta d^2b^2\omega+8\lambda+8\omega+4\alpha d^2\omega^2\\
&+2\beta b^4\lambda+2\beta b^4\omega-2\alpha b^2\omega^2-4\beta b^3\omega^2+\alpha b^3\lambda+\alpha b^3\omega+24\beta d^2b\lambda\omega+2i\omega\beta Rb^3+2i\lambda\beta Rb^3+\lambda\alpha Rb^2i+\omega\alpha Rb^2i\\
&+16i\lambda^2\omega\beta Rb+4iR+8i\beta\lambda b\omega^2R-8i\beta\lambda b^2\omega R-4id^2\beta\omega Rb-4id^2\beta\lambda Rb-4i\alpha b\lambda R\omega+8id^2\beta\lambda R\omega-4i\beta b^2\omega^2R\\
&-16i\beta\lambda^3R\omega-2i\alpha\lambda^2Rb-2i\alpha b\omega^2R-16i\beta\lambda^2\omega^2R-4i\beta\lambda^2b^2R+4i\alpha\lambda^2R\omega+8i\beta\lambda^3Rb+8id^2\beta\omega^2R+4i\alpha\lambda\omega^2R,\\
Q_2 =& 4b\lambda\omega+4b\omega^2-2b^2\omega-2b^2\lambda-2iRb\omega+4iR\omega\lambda+4iR\omega^2-2iRb\lambda.
\end{aligned}
$$

APPENDIX C: EXPRESSIONS FOR $A_b(w)$, C_b, D_b, $F_b(w)$, G_b, H_b, $\bar{A}(t,z)$, AND $\bar{B}(t,z)$

$$\begin{aligned}
A_b =& -2w\beta_1 z - 72z\beta\alpha_1^3\omega w\beta_1 - 72z\beta\alpha_1^2\omega^2 w\beta_1 + 6z\alpha\alpha_1^2\omega w\beta_1 + 6z\alpha\alpha_1\omega^2 w\beta_1 - 24z\beta\omega w\alpha_1\beta_1^3 - 24z\beta\alpha_1\omega^3 w\beta_1\\
&+ 2z\alpha\alpha_1^3 w\beta_1 - 24z\beta\alpha_1^4 w\beta_1 + 2z\alpha\omega^3 w\beta_1 - 24z\beta w\alpha_1^2\beta_1^3 + 2z\alpha w\alpha_1\beta_1^3 + 2z\alpha w\omega\beta_1^3 - \big(az\alpha_1^3 + az\omega^3 + bt\alpha_1^3 + bt\omega^3\\
&+ 3az\alpha_1^2\omega + 3az\alpha_1\omega^2 + 3bt\alpha_1^2\omega + 3bt\alpha_1\omega^2 + az\alpha_1\beta_1^2 + az\omega\beta_1^2 + bt\alpha_1\beta_1^2 + bt\omega\beta_1^2\big),\\
C_b =& -2z\omega^2 - 2z\beta_1^2 + 2\alpha_1^4 t - 2z\alpha_1^2 - 4z\omega\alpha_1 + 6\alpha_1^3 t\omega + 2z\alpha\alpha_1^5 - 8z\beta\alpha_1^6 + 6\alpha_1^2 t\omega^2 + 2\alpha_1 t\omega^3 - z\alpha d^2\omega^3 - z\alpha d^2\alpha_1^3\\
&+ 6z\alpha\alpha_1^4\omega + 6z\alpha\alpha_1^3\omega^2 + 2z\alpha\alpha_1^2\omega^3 - 24z\beta\alpha_1^5\omega - 24z\beta\alpha_1^4\omega^2 - 8z\beta\alpha_1^3\omega^3 + 12z\beta d^2\alpha_1^4 - 3z\alpha d^2\omega^2\alpha_1 - 3z\alpha d^2\alpha_1^2\omega\\
&+ 36z\beta d^2\alpha_1^3\omega + 36z\beta d^2\alpha_1^2\omega^2 + 12z\beta d^2\alpha_1\omega^3 - 8z\beta\alpha_1^4\beta_1^2 + 2z\alpha\alpha_1^3\beta_1^2 + 2\alpha_1 t\omega\beta_1^2 + 2\alpha_1^2 t\beta_1^2 + 4izd^2\beta w\omega\beta_1^2\\
&+ 4iz\alpha\alpha_1 w\omega\beta_1^2 + 4izd^2\beta w\alpha_1\beta_1^2 + 24iz\beta\alpha_1\omega^2 w\beta_1^2 + 12izd^2\beta\omega^2 w\alpha_1 + 12izd^2\beta w\alpha_1^2\omega + 2itw\omega\beta_1^2 + 2itw\alpha_1\beta_1^2\\
&+ 6itw\omega^2\alpha_1 + 6itw\alpha_1^2\omega + 4iz\alpha\alpha_1^4 w - 24iz\beta\alpha_1^5 w + 12z\beta d^2\alpha_1^2\beta_1^2 + 2z\alpha\alpha_1^2\omega\beta_1^2 - z\alpha d^2\alpha_1\beta_1^2 - z\alpha d^2\omega\beta_1^2\\
&- 8z\beta\alpha_1^3\omega\beta_1^2 + 12z\beta d^2\alpha_1\omega\beta_1^2 - 72iz\beta\alpha_1^4\omega w - 72iz\beta\alpha_1^3\omega^2 w - 24iz\beta\alpha_1^2\omega^3 w + 4iz\alpha w\alpha_1^2\beta_1^2 + 12iz\alpha\alpha_1^3\omega w\\
&+ 12iz\alpha\alpha_1^2\omega^2 w + 4iz\alpha\alpha_1\omega^3 w + 4izd^2\beta\omega^3 w + 4izd^2\beta w\alpha_1^3 - 16iz\beta\alpha_1^3 w\beta_1^2 + 8iz\beta\omega^3 w\beta_1^2 + 8iz\beta w\alpha_1\beta_1^4\\
&+ 8iz\beta w\omega\beta_1^4 + 2izw\alpha_1 + 2izw\omega + 2itw\omega^3 + 2itw\alpha_1^3,\\
D_b =&\ 8iz\alpha\alpha_1^2\omega w + 4iz\alpha\alpha_1\omega^2 w + 4izd^2\beta\omega^2 w + 4izd^2\beta w\alpha_1^2 + 8iz\beta\omega^2 w\beta_1^2 + 2izw + 4iz\alpha\alpha_1^3 w + 4itw\alpha_1\omega\\
&- 48iz\beta\alpha_1^3\omega w - 24iz\beta\alpha_1^2\omega^2 w + 2itw\omega^2 + 2itw\alpha_1^2 - 24iz\beta\alpha_1^4 w + 8izd^2\beta\omega w\alpha_1 - az\alpha_1^2 - az\omega^2 - bt\alpha_1^2 - bt\omega^2\\
&- 2az\alpha_1\omega - 2bt\alpha_1\omega - az\beta_1^2 - bt\beta_1^2 + 8iz\beta w\beta_1^4 + 16iz\beta\omega w\alpha_1\beta_1^2 + 4iz\alpha w\alpha_1\beta_1^2 - 16iz\beta\alpha_1^2 w\beta_1^2\\
&+ 4izd^2\beta w\beta_1^2 + 2itw\beta_1^2,\\
F_b =& -2z\omega^2 - 2z\beta_1^2 - 2w\beta_1 z + 2\alpha_1^4 t - 2z\alpha_1^2 - 4z\omega\alpha_1 + 6\alpha_1^3 t\omega + 2z\alpha\alpha_1^5 - 8z\beta\alpha_1^6 + 6\alpha_1^2 t\omega^2 + 2\alpha_1 t\omega^3 - z\alpha d^2\omega^3\\
&- z\alpha d^2\alpha_1^3 + 6z\alpha\alpha_1^4\omega + 6z\alpha\alpha_1^3\omega^2 + 2z\alpha\alpha_1^2\omega^3 - 24z\beta\alpha_1^5\omega - 24z\beta\alpha_1^4\omega^2 - 8z\beta\alpha_1^3\omega^3 + 12z\beta d^2\alpha_1^4 - 3z\alpha d^2\omega^2\alpha_1\\
&- 3z\alpha d^2\alpha_1^2\omega + 36z\beta d^2\alpha_1^3\omega + 36z\beta d^2\alpha_1^2\omega^2 + 12z\beta d^2\alpha_1\omega^3 - 72z\beta\alpha_1^3\omega w\beta_1 - 72z\beta\alpha_1^2\omega^2 w\beta_1 + 6z\alpha\alpha_1^2\omega w\beta_1\\
&+ 6z\alpha\alpha_1\omega^2 w\beta_1 - 24z\beta\omega w\alpha_1\beta_1^3 - 24z\beta\alpha_1\omega^3 w\beta_1 + 2z\alpha\alpha_1^3 w\beta_1 - 24z\beta\alpha_1^4 w\beta_1 + 2z\alpha\omega^3 w\beta_1 - 24z\beta w\alpha_1^2\beta_1^3\\
&+ 2z\alpha w\alpha_1\beta_1^3 + 2z\alpha w\omega\beta_1^3 - 8z\beta\alpha_1^4\beta_1^2 + 2z\alpha\alpha_1^3\beta_1^2 + 2\alpha_1 t\omega\beta_1^2 + 2\alpha_1^2 t\beta_1^2 + 12z\beta d^2\alpha_1^2\beta_1^2 + 2z\alpha\alpha_1^2\omega\beta_1^2\\
&- z\alpha d^2\alpha_1\beta_1^2 - z\alpha d^2\omega\beta_1^2 - 8z\beta\alpha_1^3\omega\beta_1^2 + 12z\beta d^2\alpha_1\omega\beta_1^2,\\
G_b =& -1 - 12\beta\alpha_1^4 + \alpha\alpha_1^3 + 3\alpha\alpha_1^2\omega - 36\beta\alpha_1^3\omega + \alpha\omega^3 + \alpha\omega\beta_1^2 - 36\alpha_1^2\beta\omega^2 - 12\alpha_1^2\beta\beta_1^2\\
&+ \alpha_1\alpha\beta_1^2 + 3\alpha_1\alpha\omega^2 - 12\alpha_1\omega\beta\beta_1^2 - 12\beta\alpha_1\omega^3,\\
H_b =& -12z\beta\alpha_1^4 - 24z\beta\alpha_1^3\omega + 2z\alpha\alpha_1^3 + 2z\beta d^2\alpha_1^2 - 12\alpha_1^2 z\beta\omega^2 + \alpha_1^2 t - 8\alpha_1^2 z\beta\beta_1^2 + 4z\alpha\alpha_1^2\omega + 2\alpha_1 t\omega + 4z\beta d^2\alpha_1\omega\\
&+ 2\alpha_1 z\alpha\beta_1^2 + 2\alpha_1 z\alpha\omega^2 + 8\alpha_1\omega z\beta\beta_1^2 + 2z\beta d^2\omega^2 + 4z\beta\omega^2\beta_1^2 + z + t\omega^2 + 4z\beta\beta_1^4 + t\beta_1^2 + 2zd^2\beta\beta_1^2,\\
\bar{A}(t,z) :=&\ 768d^2z^2\alpha\alpha_1^4\beta\omega^3 - 16d^2z^2\alpha^2\alpha_1^6 - 20d^4z^2\alpha^2\alpha_1^4 + 3\omega^4 + 96d^2z^2\alpha_1^4\beta + 192d^2z^2\alpha\alpha_1^3\beta\omega^4 + 3\alpha_1^4\\
&+ 384d^2tz\alpha_1^3\beta\omega^3 - 96d^6tz\beta\omega\alpha_1 - 64d^2tz\alpha\alpha_1^2\omega^3 - 16d^2tz\alpha\alpha_1\omega^4 + 1152d^2z^2\alpha\alpha_1^5\beta\omega^2 + 384d^2tz\alpha_1^5\beta\omega\\
&+ 576d^2tz\alpha_1^4\beta\omega^2 - 192d^4tz\beta\omega^2\alpha_1^2 - 192d^4tz\beta\omega^3\alpha_1 - 16d^2z^2\alpha\alpha_1^3 + 3\alpha_1^2 d^2 - 24d^2t^2\omega^2\alpha_1^2\\
&- 16d^2t^2\omega^3\alpha_1 - 16d^2t^2\alpha_1^3\omega - 144d^4z^2\beta\alpha_1^2 - 48d^4z^2\beta\omega^2 + 96d^2tz\beta\omega^4\alpha_1^2 - 144d^8z^2\beta^2\alpha_1^2 - 144d^8z^2\beta^2\omega^2\\
&- 8d^4t^2\omega\alpha_1 - 8d^2tz\alpha_1^2 - 8d^2tz\omega^2 - 64d^2tz\alpha\alpha_1^4\omega - 96d^2tz\alpha\alpha_1^3\omega^2 - 32d^4tz\alpha\alpha_1^2\omega - 16d^4tz\alpha\alpha_1\omega^2 + 8izd^2\alpha\omega^4\\
&- 96izd^2\beta\alpha_1^5 + 8izd^2\alpha\alpha_1^4 + 8id^4z\alpha\omega^2 + 8id^4z\alpha\alpha_1^2 - 96id^4z\alpha_1^3\beta - 4d^4t^2\omega^2 - 8izd^2\alpha_1 - 8izd^2\omega - 48d^4z^2\alpha^2\alpha_1^3\omega\\
&- 40d^4z^2\alpha^2\alpha_1^2\omega^2 - 2304d^2z^2\alpha_1^7\beta^2\omega - 48d^6tz\beta\alpha_1^2 - 48d^6tz\beta\omega^2 + 8d^4z^2\alpha\omega - 4d^6z^2\alpha^2\omega^2 - 4d^6z^2\alpha^2\alpha_1^2\\
&- 4d^4z^2\alpha^2\omega^4 + 8d^4z^2\alpha\alpha_1 - 864d^6z^2\alpha_1^2\beta^2\omega^2 - 576d^6z^2\alpha_1\beta^2\omega^3 - 32d^2z^2\alpha\alpha_1^2\omega - 16d^2z^2\alpha\alpha_1\omega^2 - 4d^2t^2\alpha_1^4\\
&- 4d^2t^2\omega^4 - 4d^4t^2\alpha_1^2 - 1152d^4z^2\alpha_1^5\beta^2\omega - 576d^6z^2\alpha_1^3\beta^2\omega - 576d^4z^2\alpha_1^4\beta^2\omega^2 + 6d^2\alpha_1\omega - 8d^6z^2\alpha^2\alpha_1\omega\\
&- 96id^4z\alpha_1\beta\omega^2 - 192id^4z\alpha_1^2\beta\omega + 16id^4z\alpha\alpha_1\omega - 384izd^2\beta\alpha_1^4\omega - 384izd^2\beta\alpha_1^2\omega^3 + 32izd^2\alpha\alpha_1^3\omega\\
&- 576izd^2\beta\alpha_1^3\omega^2 + 48izd^2\alpha\omega^2\alpha_1^2 + 32izd^2\alpha\omega^3\alpha_1 - 2304d^2z^2\alpha_1^5\beta^2\omega^3 - 3456d^2z^2\alpha_1^6\beta^2\omega^2 - 576d^2z^2\alpha_1^4\beta^2\omega^4\\
&+ 3d^2\omega^2 + 192d^2z^2\alpha\alpha_1^7\beta - 96izd^2\beta\alpha_1\omega^4 + 12\omega^3\alpha_1 + 18\alpha_1^2\omega^2 - 64d^2z^2\alpha^2\alpha_1^5\omega - 48d^4tz\beta\omega^4\\
&- 96d^2z^2\alpha^2\alpha_1^4\omega^2 - 64d^2z^2\alpha^2\alpha_1^3\omega^3 - 16d^2z^2\alpha^2\alpha_1^2\omega^4 + 48d^4tz\alpha_1^4\beta + 192d^4z^2\alpha\alpha_1^5\beta - 16d^4z^2\alpha^2\alpha_1\omega^3 + 96d^2tz\alpha_1^6\beta\\
&- 16d^2tz\alpha\alpha_1^5 - 16d^4tz\alpha\alpha_1^3 - 576d^2z^2\alpha_1^8\beta^2 - 576d^4z^2\alpha_1^6\beta^2 - 144d^6z^2\beta^2\omega^4 - 144d^6z^2\alpha_1^4\beta^2 - 288d^8z^2\beta^2\omega\alpha_1\\
&+ 192d^2z^2\alpha_1^3\beta\omega + 96d^2z^2\alpha_1^2\beta\omega^2 + 12\alpha_1^3\omega - 16d^2tz\omega\alpha_1 - 192d^4z^2\beta\omega\alpha_1 + 768d^2z^2\alpha\alpha_1^6\beta\omega + 192d^4z^2\alpha\alpha_1^3\beta\omega^2\\
&+ 384d^4z^2\alpha\alpha_1^4\beta\omega - 4z^2d^2,
\end{aligned}$$

$$
\begin{aligned}
\bar{B}(t,z) := {} & -768d^2z^2\alpha\alpha_1^4\beta\omega^3 + 16d^2z^2\alpha^2\alpha_1^6 + 20d^4z^2\alpha^2\alpha_1^4 + \omega^4 - 96d^2z^2\alpha_1^4\beta - 192d^2z^2\alpha\alpha_1^3\beta\omega^4 + \alpha_1^4 - 384d^2tz\alpha_1^3\beta\omega^3 \\
& + 96d^6tz\beta\omega\alpha_1 + 64d^2tz\alpha\alpha_1^2\omega^3 + 16d^2tz\alpha\alpha_1\omega^4 - 1152d^2z^2\alpha\alpha_1^5\beta\omega^2 - 384d^2tz\alpha_1^5\beta\omega - 576d^2tz\alpha_1^4\beta\omega^2 \\
& + 192d^4tz\beta\omega^2\alpha_1^2 + 192d^4tz\beta\omega^3\alpha_1 + 16d^2z^2\alpha\alpha_1^3 + \alpha_1^2d^2 + 24d^2t^2\omega^2\alpha_1^2 + 16d^2t^2\omega^3\alpha_1 + 16d^2t^2\alpha_1^3\omega + 144d^4z^2\beta\alpha_1^2 \\
& + 48d^4z^2\beta\omega^2 - 96d^2tz\beta\omega^4\alpha_1^2 + 144d^8z^2\beta^2\alpha_1^2 + 144d^8z^2\beta^2\omega^2 + 8d^4t^2\omega\alpha_1 + 8d^2tz\alpha_1^2 + 8d^2tz\omega^2 + 64d^2tz\alpha\alpha_1^4\omega \\
& + 96d^2tz\alpha\alpha_1^3\omega^2 + 32d^4tz\alpha\alpha_1^2\omega + 16d^4tz\alpha\alpha_1\omega^2 + 48d^4z^2\alpha^2\alpha_1^3\omega + 40d^4z^2\alpha^2\alpha_1^2\omega^2 + 2304d^2z^2\alpha_1^7\beta^2\omega \\
& + 48d^6tz\beta\alpha_1^2 + 48d^6tz\beta\omega^2 - 8d^4z^2\alpha\omega + 4d^6z^2\alpha^2\omega^2 + 4d^6z^2\alpha^2\alpha_1^2 + 4d^4z^2\alpha^2\omega^4 - 8d^4z^2\alpha\alpha_1 + 864d^6z^2\alpha_1^2\beta^2\omega^2 \\
& + 576d^6z^2\alpha_1\beta^2\omega^3 + 32d^2z^2\alpha\alpha_1^2\omega + 16d^2z^2\alpha\alpha_1\omega^2 + 4d^2t^2\alpha_1^4 + 4d^2t^2\omega^4 + 4d^4t^2\omega^2 + 4d^4t^2\alpha_1^2 + 1152d^4z^2\alpha_1^5\beta^2\omega \\
& + 576d^6z^2\alpha_1^3\beta^2\omega + 576d^4z^2\alpha_1^4\beta^2\omega^2 + 2d^2\alpha_1\omega + 8d^6z^2\alpha^2\alpha_1\omega + 2304d^2z^2\alpha_1^5\beta^2\omega^3 + 3456d^2z^2\alpha_1^6\beta^2\omega^2 \\
& + 576d^2z^2\alpha_1^4\beta^2\omega^4 + d^2\omega^2 - 192d^2z^2\alpha\alpha_1^7\beta + 4\omega^3\alpha_1 + 6\alpha_1^2\omega^2 + 64d^2z^2\alpha^2\alpha_1^5\omega + 48d^4tz\beta\omega^4 + 96d^2z^2\alpha^2\alpha_1^4\omega^2 \\
& + 64d^2z^2\alpha^2\alpha_1^3\omega^3 + 16d^2z^2\alpha^2\alpha_1^2\omega^4 - 48d^4tz\alpha_1^4\beta - 192d^4z^2\alpha\alpha_1^5\beta + 16d^4z^2\alpha^2\alpha_1\omega^3 - 96d^2tz\alpha_1^6\beta + 16d^2tz\alpha\alpha_1^5 \\
& + 16d^4tz\alpha\alpha_1^3 + 576d^2z^2\alpha_1^8\beta^2 + 576d^4z^2\alpha_1^6\beta^2 + 144d^6z^2\beta^2\omega^4 + 144d^6z^2\alpha_1^4\beta^2 + 288d^8z^2\beta^2\omega\alpha_1 - 192d^2z^2\alpha_1^3\beta\omega \\
& - 96d^2z^2\alpha_1^2\beta\omega^2 + 4\alpha_1^3\omega + 16d^2tz\omega\alpha_1 + 192d^4z^2\beta\omega\alpha_1 - 768d^2z^2\alpha\alpha_1^6\beta\omega - 192d^4z^2\alpha\alpha_1^3\beta\omega^2 - 384d^4z^2\alpha\alpha_1^4\beta\omega \\
& + 4z^2d^2.
\end{aligned}
$$

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