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Spatiotemporal order, disorder, and propagating defects in homogeneous system of relaxation oscillators

Rajeev Singh and Sitabhra Sinha
The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India

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A variety of complex spatial patterns relevant to chemical and biological systems can be generated through reaction-diffusion mechanisms. In this paper, we show that diffusive coupling through the inactivating component in a system of relaxation oscillators extends such complexity to the temporal domain, generating remarkable spatiotemporal phenomena. We provide analytic explanations of the antiphase synchronization and spatially patterned oscillatory death regimes. We report a chimera state where patches with distinct dynamics coexist and also observe propagating phase defects resembling persistent structures in cellular automata that may be used for computation.

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I. INTRODUCTION

The problem of understanding pattern formation across a variety of chemical and biological contexts [1,2] has stimulated much theoretical and experimental activity since the early work of Turing [3–5]. Studying the dynamics of coupled biochemical oscillators interacting through reaction-diffusion mechanisms constitutes a particularly promising approach to understanding the genesis of patterns in natural systems [6]. Generalizations of such processes involving differential excitatory and inhibitory interactions between elements as represented by the coupling terms have been used to represent a variety of complex systems [7–9]. They have also been proposed recently as a possible mechanism for computation in biological and chemical systems [10,11].

The stationary patterns exhibited by the models mentioned above represent only a fraction of the variety seen in nature, many of which exhibit periodic activity. Thus, extending ideas underlying reaction-diffusion mechanisms to systems of interacting relaxation oscillators should permit investigation of spatiotemporal patterns in biological systems, where oscillations are observed across many spatial and temporal scales, ranging from the periodic variations of intracellular molecular concentrations [12] to changes in the activity levels of different brain areas [13]. The coherent dynamics of these oscillators can produce functionally important collective behavior, such as synchronization [14], yielding different biological rhythms [15]. However, synchronized oscillations constitute only one of a number of possible collective phenomena that can emerge from such interactions. For example, a recent set of experiments on coupled chemical oscillators in a microfluidic device [16] have shown that antiphase synchronization as well as spatially heterogeneous oscillator death states [17] can occur in this system under different conditions. Extending the mechanism of coupling by lateral inhibition (e.g., via a rapidly diffusing inhibitory chemical species) to arrays of relaxation oscillators, used for modeling biological periodic activity, can be expected to reveal the underlying mechanism for a variety of spatiotemporal phenomena seen in natural systems.

In this paper, we study a generic model of relaxation oscillators, each comprising activator and inactivating components, coupled to nearest neighbors through lateral inhibition via diffusion of the inactivating component. Our model is capable of exhibiting a variety of spatiotemporal patterns that may be observed experimentally, while its simplicity allows an analytical understanding of their genesis. We provide a simple theoretical demonstration of the existence and stability of an antiphase synchronized state for coupled relaxation oscillators. In addition to reproducing some patterns reported earlier, we also observe novel states, such as attractors corresponding to spatially coexisting dynamically distinct configurations, which we term chimera states. Although homogeneous arrays of generic relaxation oscillators have been studied extensively, our observation of these spatially heterogeneous attractors for such systems is new to the best of our knowledge. We characterize basins of attraction for various patterns seen in the model, also demonstrating an unexpected robustness of the chimera states. This robustness suggests that the states we describe can be reproduced in suitably designed experiments. We report phase defect-like discontinuities moving ballistically through the system, producing complex patterns on collision with each other. We observe analogous structures in two-dimensional media that bear a striking resemblance to persistent configurations in cellular automata (CA) [18], which have been linked to the universal computation capabilities of such systems [19,20].

II. THE MODEL

Our model system comprises N relaxation oscillators interacting with each other in a specific topology. For the dynamics of individual relaxation oscillators, we use the phenomenological FitzHugh-Nagumo (FHN) equations, which are a generic model for such systems. Each oscillator is described by a fast activation variable u and a slow inactivation variable v:

$$\begin{aligned}\dot{u} &= f(u,v) = u\,(1-u)\,(u-\alpha) - v,\\ \dot{v} &= g(u,v) = \epsilon\,(k\,u - v - b),\end{aligned} \tag{1}$$

where $\alpha = 0.139$, $k = 0.6$ are parameters describing the kinetics, $\epsilon = 0.001$ characterizes the recovery rate of the medium, and b is a measure of the asymmetry of the oscillator (measured by the ratio of the time spent by the oscillator at high- and low-value branches of u). Parameter values are chosen such that the system is in the oscillatory regime. We have checked

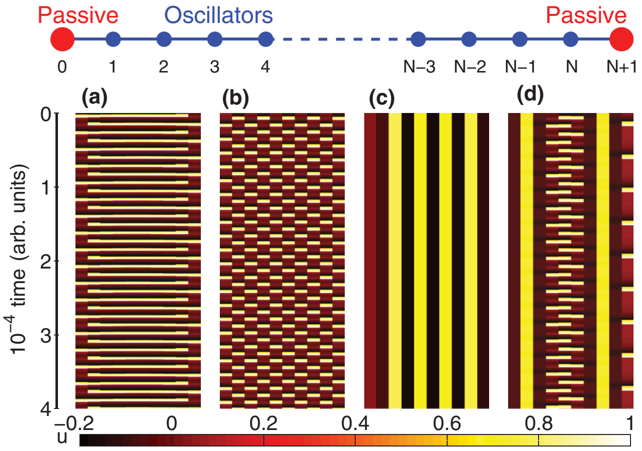

FIG. 1. (Color online) Spatiotemporal evolution of a one-dimensional array of coupled relaxation oscillators ($N = 10$) with passive elements at the boundaries (model system shown schematically on top). Pseudocolor plots of the activation variable u indicate different regimes characterized by (a) synchronized oscillations (SO), (b) antiphase synchronization (APS), (c) spatially patterned oscillation death (SPOD), and (d) chimera state (CS), i.e., co-occurrence of spatial patches with dynamically distinct behavior.

that small variations in the values do not affect our results qualitatively. To investigate spatial patterns generated by interaction between the oscillators, we arrange them in a one-dimensional chain [Fig. 1 (top)]. In the chemical experiments, the beads containing the reactive solution are suspended in a chemically inert medium that allows passage of only the inhibitory chemical species [16]. In our model, the oscillators are diffusively coupled via the inactivation variable v. The boundary conditions for the chain take into account the inert medium by including nonreactive passive elements at each end that are diffusively coupled to the neighboring oscillators. The inert medium between the oscillators is not considered explicitly, its volume being relatively small compared to the reservoirs at the boundary. We have verified that inclusion of intermediate nonreactive cells diffusively coupling each pair of oscillators does not affect the fixed-point equilibria of the system or their stability, once the diffusion constant is suitably scaled. The dynamics of the resulting system is described by

$$\begin{aligned} \dot{u}_i &= f(u_i,v_i), \\ \dot{v}_i &= g(u_i,v_i) + D_v\,(v_{i-1} + v_{i+1} - 2\,v_i), \\ \dot{v}_0 &= D_v\,(v_1 - v_0), \qquad \dot{v}_{N+1} = D_v\,(v_N - v_{N+1}), \end{aligned} \tag{2}$$

where $i = 1,2,\ldots,N$ and the diffusion constant D_v represents the strength of coupling between neighboring relaxation oscillators through their inactivation variables. For most results reported here, $N = 10$, although we have used larger values of N, up to 1000, to verify that our results are not sensitively dependent on system size. We have verified that the boundary conditions do not affect the results significantly by also considering periodic boundaries and observing patterns qualitatively identical to those reported here. The dynamical equations are solved using an adaptive Runge-Kutta scheme. The behavior of the system for each set of parameter values b and D_v is analyzed over many (10^4) initial conditions, with each oscillator having a random phase chosen from a uniform distribution.

III. RESULTS

Figures 1(a)–1(d) show a variety of asymptotic spatiotemporal patterns that we observe in the model system: (a) synchronized oscillations (SO), with all elements (except those at the boundary) having the same phase; (b) antiphase synchronization (APS), with neighboring elements in opposite phase; (c) spatially patterned oscillator death (SPOD) regime, where the oscillators are arrested in various stationary states; and (d) chimera states (CS), where oscillating regions coexist with patches showing negligible temporal variation. However, these do not exhaust the range of possible spatiotemporal phenomena that are observed, including propagating structures that are discussed later. Both APS and SPOD states have been observed experimentally in chemical systems [16]. Although the latter has been referred to as "Turing patterns" in the literature, we stress that SPOD is distinct as it is not obtained through destabilization of a homogeneous equilibrium (Turing instability) but occurs through a process of oscillator death [17]. There is a simple mathematical reason why the mechanism involved in generating SPOD *cannot* be Turing instability: the Jacobian matrix of the stable fixed point of the FHN model has the structure

$$\begin{pmatrix} - & - \\ + & - \end{pmatrix},$$

from which it immediately follows that the fixed point cannot be destabilized by the Turing mechanism [21].

To investigate the robustness of the observed patterns in detail, we numerically estimate the size of their basins of attraction in the (b,D_v) parameter space (Fig. 2). To identify distinct pattern regimes in (b,D_v) space [Fig. 2(a)] we introduce the following order parameters. The number of nonoscillating cells in the bulk of the system, $N_{\rm no}$, i.e., cells for which the variance with respect to time of the activation variable u, $\sigma_t^2(u_i)$, is zero, is used to characterize the SPOD ($N_{\rm no} = N$) and CS regimes ($0 < N_{\rm no} < N$). Both SO and APS states have all elements in the bulk oscillating. However, SO is distinguished by having all oscillators in the same phase as measured by the variance of the activation variables u, $\langle\sigma_i^2(u)\rangle_t = 0$, where $\langle\rangle_t$ represents time average. We can also define the synchronization among the oscillators in two distinct (even, odd) sublattices, as measured by the time-averaged variance of the activation variable, viz., $\langle\sigma^2_{\rm even}(u)\rangle_t$ and $\langle\sigma^2_{\rm odd}(u)\rangle_t$. This pair of order parameters is zero for both SO and APS states; however, if $\langle\sigma_i^2(u)\rangle_t > 0$, it signifies the APS regime. In practice, different regimes are characterized by thresholds whose specific values do not affect the qualitative nature of the results. Figure 2(a) indicates the parameter regions where SO, APS, SPOD, and CS states are observed for more than 50% of initial conditions (i.e., they have the largest basin). As mentioned earlier, the system also exhibits other regimes apart from the above ones, which occur in regions of (b,D_v) parameter space shown in white.

While diffusive coupling in a homogeneous system of oscillators is expected to promote the SO state [22], a striking observation from this phase diagram is that the APS state has

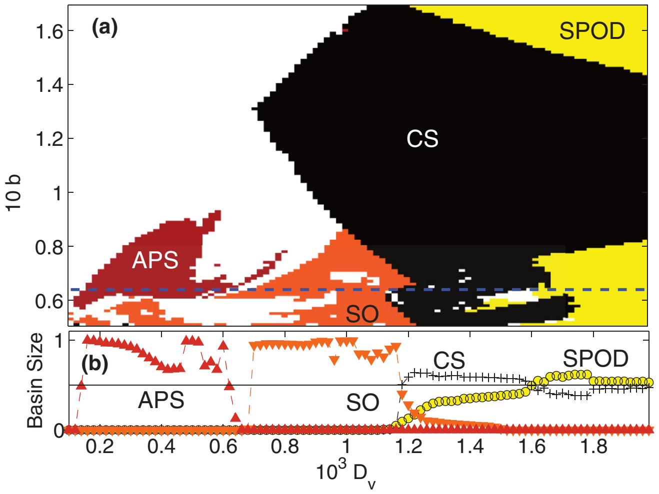

FIG. 2. (Color online) Different dynamical regimes of a one-dimensional array of coupled relaxation oscillators ($N = 10$) in the $D_v - b$ parameter plane, showing regions where the majority (>50%) of initial conditions result in synchronized oscillations (SO), antiphase synchronization (APS), spatially patterned oscillator death (SPOD), and chimera state (CS). (b) Variation of the attraction basin size for the different regimes mentioned above (measured as fraction of initial states reaching the attractor), with coupling strength D for $b = 0.064$ [i.e., along the broken line shown in panel (a)]. In practice, the regimes are distinguished by thresholds applied on the order parameters $\sigma_t^2(u_i)$, $\langle\sigma_i^2(u)\rangle_t$, $\langle\sigma_{\text{even}}^2(u)\rangle_t$ and $\langle\sigma_{\text{odd}}^2(u)\rangle_t$, which have been taken to be 0.05 for the present figure. Basin sizes have been estimated using 10^4 initial conditions.

a very large basin of attraction in certain regions [Fig. 2(b)]. The existence of APS is somewhat counterintuitive, as for diffusively coupled identical isochronous oscillators the only stable attractors are synchronized oscillations or oscillator death [22]. To understand the origin of such antiphase oscillations, we consider a simple model that captures the essence of relaxation oscillation phenomena and can be solved exactly. We consider the relaxation limit ($\epsilon \to 0$ in FHN system) and extreme asymmetry, where the limit cycle has a slow segment in which the system spends the entire duration of the oscillation period (the remaining segment of the cycle being traversed extremely fast). In this limit, we obtain the one-dimensional dynamical system: $\dot{x} = \omega(x)$, where x parameterizes the slow part of the limit cycle and can be redefined to belong to the interval (0,1). Figure 3(a) shows a schematic diagram of the trajectory of the limit cycle, where the system spends almost its entire oscillation period on the solid branch (the return from $x = 1$ to $x = 0$, shown by the broken line, is considered to be instantaneous). The model can be exactly solved if $\omega(x)$ is a constant ($= \omega$, say), although the geometrical argument is valid for any arbitrary positive definite function defined over the interval (0,1). By appropriate choice of time scale, we set the period $\omega^{-1} = 1$. A system of two such diffusively coupled oscillators can be described by

$$\dot{x}_1 = 1 + D\,(x_2 - x_1), \quad \dot{x}_2 = 1 + D\,(x_1 - x_2). \tag{3}$$

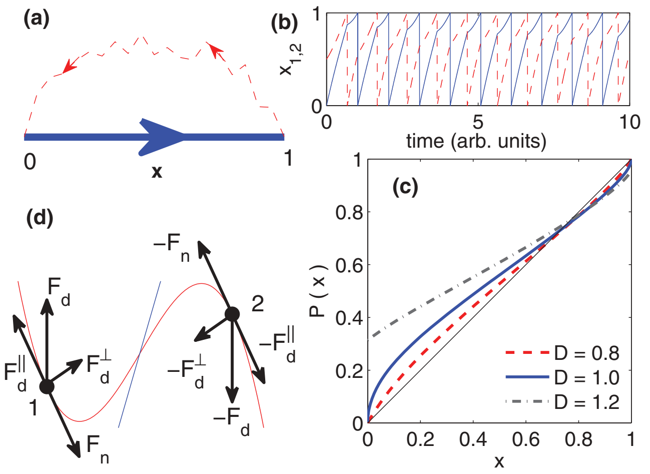

FIG. 3. (Color online) (a) Schematic diagram of a limit cycle trajectory for an oscillator in the relaxation limit ($\epsilon \to 0$) and extreme asymmetry (for details see text), such that the oscillator is on the solid line ($0 < x < 1$) for its entire period. (b) Time series of two such coupled oscillators [Eq. (3) with $D = 1$] and (c) the Poincare map for the system at different coupling strengths D showing stable antiphase synchronization. (d) Phase-plane diagram indicating the general mechanism (see text) for oscillator death in a system of two coupled oscillators (1 and 2).

Given the values of x_1,x_2 at some arbitrary initial time t', the solution of Eq. (3) at a later time t follows the relations

$$\begin{aligned} x_1(t) + x_2(t) &= x_1(t') + x_2(t') + 2(t - t'), \\ x_1(t) - x_2(t) &= [x_1(t') - x_2(t')]\,\exp[-2D(t - t')], \end{aligned} \tag{4}$$

till time t'' when max(x_1,x_2) reaches $x = 1$. After this, the larger of (x_1,x_2) is mapped back to $x = 0$ (because of the instantaneous nature of the remaining segment of the limit cycle) and t' is replaced by t''. Using the above exact solution of the coupled system Eq. (3), its Poincare map $P(x)$ is constructed in two steps. First, if x_1 starts at 0 and x_2 starts at some point $0 < x < 1$, we find the location of $x_1[= f(x)]$ at some time t when $x_2 = 1$ (which is then immediately mapped to $x_2 = 0$). Now, starting with $x_2 = 0$ and $x_1 = f(x)$, when $x_1 = 1$ we find the location of x_2: $x' = f[f(x)] = P(x)$. Using solution Eq. (4), with $x_1(t') = 0, x_2(t') = x, x_1(t) = f(x)$, and $x_2(t) = 1$, we get $f(x) = 1 + D^{-1}\,W\{-Dx\,\exp[D(x-2)]\}$, where W is the Lambert W function. Figure 3(c) shows the Poincare map $P(x) = f[f(x)]$ for different values of the coupling strength D. The map has one stable and one unstable fixed point, which correspond to the APS and SO states, respectively. Thus, for the model Eq. (3) we find that APS is the only stable state. Relaxing the extremal conditions under which this was derived may allow a stable SO state to coexist with the stable APS state [23]. This is a fundamental mechanism for generating APS states in any system of diffusively coupled oscillators exhibiting antiphase oscillations.

When the coupling D_v between oscillators in the array is increased to very high values, we observe that the oscillatory regimes (e.g., SO and APS) are replaced by stationary spatial patterns such as SPOD (Fig. 2). To understand the genesis

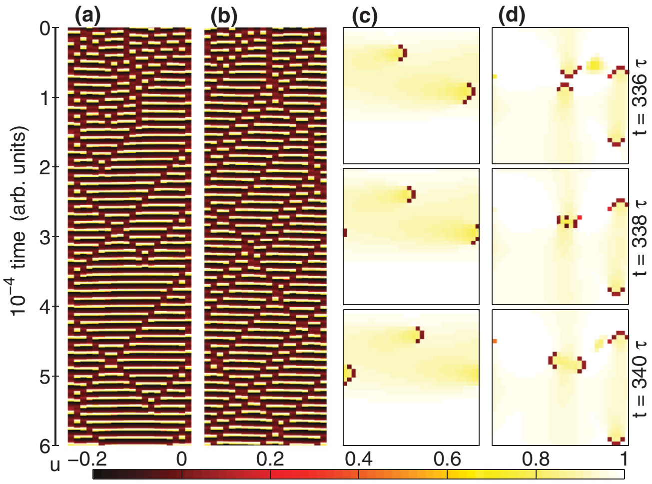

FIG. 4. (Color online) (a), (b) Spatio-temporal evolution of a system of coupled relaxation oscillators showing traveling waves of phase defects in (a) a linear array with passive elements at the boundaries and (b) with periodic boundary conditions. (c), (d) Propagating defects in two-dimensional media with periodic boundary condition showing (c) two horizontally moving "gliders" and (d) collision of two "gliders." For clear visualization of the motion of the spatially extended defects, snapshots of the two-dimensional medium are taken at intervals which are multiples of the oscillation period for the mean activity of the system τ.

of SPOD at strong coupling, we can again focus on a pair of coupled relaxation oscillators in the relaxation limit ($\epsilon \to 0$). The parameter b is chosen such that the v nullcline is placed symmetrically between the two branches of the u nullcline with the oscillator taking equal time to traverse each branch [Fig. 3(d)]. When the two oscillators (1 and 2) are in opposite branches (as shown in the schematic diagram), the two opposing forces acting on each oscillator, corresponding to the coupling [$F_d = D_v(v_2 - v_1)$] and the intrinsic kinetics (F_n), respectively, can exactly cancel when the coupling is strong resulting in oscillator death. Symmetry ensures that the force due to the intrinsic kinetics for the two oscillators is identical in magnitude but oppositely directed in the steady state. The occurrence and stabilization of this *heterogeneous* stationary state is the key to the occurrence of SPOD at strong coupling.

At intermediate values of coupling D_v in large arrays, the competition of the above mechanism with the intrinsic oscillatory dynamics dominant at low coupling, may give rise to chimera states. This CS regime is especially interesting, as the system exhibits a remarkable heterogeneous dynamical state in spite of the bulk being homogeneous. The occurrence of CS is not dependent on boundary conditions, as it is also reproduced with periodic boundaries. The observation of such states in a generic model of relaxation oscillators suggests that they should be present in a wide class of systems; similar phenomena have been recently reported in a specific chemical system model [24]. The chimera state described here comprises regions with dynamically distinct behavior, as opposed to its recent usage referring to the co-occurrence of coherent and noncoherent domains [25].

Aside from the spatiotemporal patterns in Figs. 1(a)–1(d), we also observe attractors having point-like "phase defects" (i.e., with a discontinuity of phase along the oscillator array at this point), moving in the background of system-wide oscillations. As seen from a typical example of such states [Fig. 4(a)], after initial transients these defects move in the medium with interactions between two such entities resulting in either the two being deflected in opposite directions, or either both or only one getting annihilated. This is unlike the situation of oppositely charged defects in nonoscillatory media, which typically annihilate on collision [26]. While the boundary for systems with passive elements at the ends is a source of new defects entering the medium, similar persistent structures are also seen in systems with periodic boundary conditions, where, in the steady state, a conserved number of defects can reflect off each other indefinitely [Fig. 4(b)].

To observe how these propagating defects manifest in higher dimensional systems, we consider a two-dimensional array of coupled oscillators defined on a torus. The system can have extremely complicated transient phenomena, and for simplicity we discuss only its asymptotic behavior. For a square lattice, we observe that there is a specific configuration of four sites that persists indefinitely (reminiscent of the *glider* configurations in the two-dimensional CA "Game of Life" [18]). These structures can move in horizontal or vertical directions [Fig. 4(c)]. The interaction of such "gliders" can produce complex spatiotemporal patterns, e.g., Fig. 4(d), which shows two "gliders" that on collision move off in directions perpendicular to their incident ones [27].

IV. CONCLUSIONS

To conclude, we have shown that a simple model of relaxation oscillators interacting via lateral inhibition-like coupling yields a variety of spatiotemporal patterns. Our model is simple and generic, suggesting that the patterns we predict may be observed in a range of experiments. These include coupled electronic circuits implementing relaxation oscillators [28] and Pt wire undergoing CO oxidation where the system is in an oscillatory regime [29] as well as the microfluidic chemical systems mentioned earlier. It will also be of great interest to see whether similar patterns occur in well-known generic models of chemical oscillators such as the Brusselator [30]. Recent theoretical work on trapped ions [31] suggests yet another system where such patterns can be found experimentally. Our initial exploration of propagating configurations in two-dimensional media suggests that systems of higher dimensions may exhibit yet more striking features. The possibility of using the propagating defects for computation is an intriguing one, especially as analogous entities have been used to construct logic gates in CA [18].

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