Relation between optimal nonlinearity and non-Gaussian noise: Enhancing a weak signal in a nonlinear system

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In the study of stochastic resonance, it is often mentioned that nonlinearity can enhance a weak signal embedded in noise. In order to give a systematic proof of the signal enhancement in nonlinear systems, we derive an optimal nonlinearity that maximizes a signal-to-noise ratio (SNR). The obtained optimal nonlinearity yields the maximum unbiased signal estimation performance, which is known in the context of information theory. It is found that a linear system is optimal for a Gaussian noise, but for a non-Gaussian noise, there exist nonlinear systems that can achieve an SNR higher than that obtained from linear systems. This analysis refers to a system subjected to an additive non-Gaussian noise with a small signal input.

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I. INTRODUCTION

Since the discovery of *stochastic resonance* (SR) [1–4], positive applications of noise-mediated phenomena have been tentatively explored, especially in the field of signal processing [5–7]. Although a three-decade-long exploration of SR-based signal processing has found its potential applicability in a variety of systems, such as superconducting quantum interference devices (SQUIDs) [8,9] and sensory neurons [10–13], there still remains an open question on the performance of SR-based signal processing. In the occurrence of SR, a stable state of a nonlinear system is modulated by a small perturbation (weak input signal). The response of the system is naively obtained from linear response theory. In a more detailed analysis, the response of the system is affected by a nonperturbative effect, which is the cooperation of noise and nonlinearity. However, such a response in the presence of white Gaussian noise has been shown to be smaller than those of linear systems [14,15]. In contrast, a number of studies report that SR driven by non-Gaussian noise achieves a signal processing performance higher than those obtained by linear systems [16–27]. Then, the natural question is, *which type of nonlinearity is optimal for a given type of noise to maximize the signal processing performance?* As the consequence of the answer to this question, *we give a systematic proof of the existence of nonlinearities that exhibit signal processing performance higher than a linear system in the presence of non-Gaussian noise*.

In this paper, we derive the optimal nonlinearity to maximize the *signal-to-noise ratio* (SNR), which is often used as an indicator of signal processing performance. The improvement of SNR is one of the traditional subjects in filtering theory. We consider arbitrarily long time series of general types of additive noise, which may have temporal correlations. Hence, the obtained nonlinearity is regarded to be optimal over all filters. Conventionally well-known filters such as adaptive [28,29] and Kalman [30,31] filters satisfyingly work for a strong signal compared with noise intensity. However, the signal processing performances of these filters are deteriorated by strong noises. The improvement of the SNR for a weak signal is, thus, an important and challenging problem. In this paper, we consider such a weak signal input. Moreover, we assume that the probability density of the noise is known. All filters exploit the knowledge on noise statistics. For example, one of the main components of adaptive filters is the estimator for the noise characteristics. Kalman filters require the noise model. For this reason, we investigate the optimal nonlinearity for given noise statistics. Our approach establishes the basis of the exploitation of a broad class of noise types by using nonlinearities.

Surprisingly, the obtained nonlinearity yields the natural unbiased estimator for the input signal that achieves the highest estimation accuracy bounded by the Cramér-Rao inequality [32,33]. This estimation is a simple linear process, while the usual estimation as a postprocessing in the framework of information theory requires a highly nonlinear analysis [34–37]. Therefore, the optimization of nonlinearity of the system as preprocessing yields simple postprocessing calculations to obtain the estimated input signal with high accuracy.

In this paper, we consider a weak input signal. The conventional techniques such as linear filtering work well for a strong input signal. In contrast, the improvement of the SNR for a weak input signal is an important and challenging problem in various fields, not only in the field of traditional signal processing but also in fields such as particle physics [38], gravitational wave search [39], and medical science [40]. Moreover, we assume that the probability density of the noise is known. It is often mentioned that adaptive filters exhibit high signal processing performance when the noise statistics is unknown. However, adaptive filters exploit the temporal correlations of the received signals. This means that even adaptive filters exploit the portion of noise statistics. In general, an adaptive filter is regarded to estimate the probability density of the noise [28,29]. For this reason, we investigate the optimal nonlinearity for given noise statistics. Our approach establishes the basis of the exploitation of a broad class of noise types by using nonlinearities.

II. MODEL AND DEFINITION OF SNR

Generally, a signal processing device subjected to an additive noise is described by a system in a discrete time form

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as

$$x_i = F_i(\boldsymbol{s}+\boldsymbol{z}), \tag{1}$$

where x_i represents the output at time i, and the vectors $\boldsymbol{s}$ and $\boldsymbol{z}$ denote the time sequences of the input signal and noise of length N, respectively. The noise $\boldsymbol{z}$ is assumed to be additive, but not necessarily Gaussian or white. The output $\boldsymbol{x}$ is assumed to be given as a function of the input signal $\boldsymbol{s}$ added to the noise $\boldsymbol{z}$. The function F_i may depend on all components of the input sequence $\boldsymbol{s}+\boldsymbol{z}$ owing to the effect of memory and delay in the system. This is often the case in a filtering process. We are interested in the optimization of the function $\boldsymbol{F}$, which denotes the nonlinearity of our signal processing device, to maximize the SNR of the output $\boldsymbol{x}$ defined below.

In a weak-signal regime, the output x_i is described in the framework of linear response theory as

$$x_i = \langle F_i(\boldsymbol{z})\rangle + \langle \partial^j F_i\rangle s_j + \xi_i + O\left(s_i^2\right), \tag{2}$$

where $\langle \partial^j F_i\rangle \equiv \langle \partial F_i(\boldsymbol{z})/\partial z_j\rangle$ denotes the linear coefficient for the input, and $\xi_i \equiv F_i(\boldsymbol{z}) - \langle F_i(\boldsymbol{z})\rangle$ is the random part of the output. The Einstein summation convention $a^i b_i \equiv \sum_{i=1}^N a^i b_i$ is used in this paper. $\langle\cdot\rangle$ denotes the average over the noise, i.e., $\langle\cdot\rangle \equiv \int \cdot \rho(\boldsymbol{z}) d\boldsymbol{z}$, with the probability density of the input noise $\rho(\boldsymbol{z})$. Note that the response coefficient and the noise part are independent of the input signal in the linear response regime.

Considering the linear combination of $\{x_i\}$, $X = w^i x_i$, the SNR for X is given as

$$R = \frac{w^i w^j \langle \partial^k F_i\rangle\langle \partial^l F_j\rangle s_k s_l}{V_{ij} w^i w^j}, \tag{3}$$

where $V_{ij} \equiv \langle \xi_i \xi_j\rangle - \langle \xi_i\rangle\langle \xi_j\rangle$. The above SNR is maximized with respect to the weight w^k by solving $\partial R/\partial w^k = 0$. The result reads the SNR as

$$R = \langle \partial^k F_i\rangle V^{ij} \langle \partial^l F_j\rangle s_k s_l\,, \tag{4}$$

where we used the notation of the inverse matrix $C^{ij} C_{jk} = \delta^i{}_k$, with $\delta^i{}_j$ denoting Kronecker δ. This SNR is a natural extension of the conventional SNR defined by Fourier power spectrum in the presence of white noise. In this paper, we maximize this SNR with respect to the nonlinearity $\boldsymbol{F}$.

III. OPTIMAL NONLINEARITY

A. General case

Owing to the linear response theory, the SNR Eq. (4) yields the optimal nonlinearity regardless of the waveform of the input signal. The optimal nonlinearity is determined only by the probability density of the input noise $\rho(\boldsymbol{z})$ when the input signal is sufficiently weak.

The optimal nonlinearity $\boldsymbol{F}^*$ is determined by the functional derivative of the SNR $\delta R/\delta \boldsymbol{F}(\boldsymbol{z}) = 0$ as

$$F_i^*(\boldsymbol{z}) = A_i - B_{ik}\partial^k \ln \rho(\boldsymbol{z}), \tag{5}$$

where $\boldsymbol{A}$ is an arbitrary real constant vector and B is an arbitrary real regular matrix. This choice of the nonlinearity $\boldsymbol{F}^*$ yields the SNR as

$$R_* = \mathcal{I}^{ij}(0) s_i s_j,$$
$$\mathcal{I}^{ij}(\boldsymbol{s}) = \int \mu(\boldsymbol{y}|\boldsymbol{s}) \frac{\partial \ln \mu(\boldsymbol{y}|\boldsymbol{s})}{\partial s_i}\frac{\partial \ln \mu(\boldsymbol{y}|\boldsymbol{s})}{\partial s_j} d\boldsymbol{y}, \tag{6}$$

where $\mathcal{I}(\boldsymbol{s})$ is the Fisher information matrix, and $\mu(\boldsymbol{y}|\boldsymbol{s}) = \rho(\boldsymbol{y}-\boldsymbol{s}) = \rho(\boldsymbol{z})$ is the conditional probability density of the input $\boldsymbol{y} = \boldsymbol{s}+\boldsymbol{z}$ with the condition where the input signal is $\boldsymbol{s}$. Note that this SNR is independent of the arbitrary constants $\boldsymbol{A}$ and B. The constant $\boldsymbol{A}$ corresponds to the offset of the output $\boldsymbol{A} = \langle \boldsymbol{F}(\boldsymbol{z})\rangle$, and B determines the scale of the output. These constants play the role of gauge and are irrelevant to the SNR.

The optimal nonlinearity $\boldsymbol{F}^*$ given in Eq. (5) claims that *a linear system exhibits the highest SNR only when the input noise obeys a Gaussian distribution*. In contrast, *if the input noise is non-Gaussian, the optimal input-output characteristic $\boldsymbol{F}^*$ is nonlinear*. Thus, Eq. (5) confirms the existence of nonlinear systems with the SNR higher than that of linear systems in the presence of non-Gaussian noise that has been reported in the literature. Note that the term "Gaussian" here includes Ornstein-Uhlenbeck-like temporal correlations, since $\rho(\boldsymbol{z})$ represents the path probability of noise.

B. Homogeneous white noise

The joint probability of the homogeneous white noise $\boldsymbol{z}$ is given as $\rho(\boldsymbol{z}) = \prod_{i=1}^N \rho_0(z_i)$. Substituting this probability into Eq. (5) and choosing the gauge as $A_i = \alpha$ and $B_{ij} = \beta\delta_{ij}$, the optimal nonlinearity is expressed as

$$F_i^*(\boldsymbol{z}) = \alpha - \beta\partial^i \ln \rho_0(z_i) \tag{7}$$

for all $1 \leqslant i \leqslant N$. In this case, the Fisher information matrix is diagonalized. Correspondingly, the SNR in this case is given simply by the scalar Fisher information as

$$R_* = \mathcal{I}(0) P_\mathrm{s}, \quad \mathcal{I}(s) = \int \mu_0(y|s)\left[\frac{\partial \ln \mu_0(y|s)}{\partial s}\right]^2 dy, \tag{8}$$

where $P_\mathrm{s} = \sum_{i=1}^N s_i^2$ is the input signal power and $\mu_0(y|s) = \rho_0(y-s) = \rho_0(z)$ is the probability density of the input $y = s+z$ when the input signal is s. Note that R_*/P_s is independent of the waveform of the weak input signal $\boldsymbol{s}$, since all of the eigenvectors for the Fisher information matrix degenerate in this case.

In order to verify our results, we investigate the SNR for the optimal nonlinearity in the presence of homogeneous white non-Gaussian noise. As a simple example, we use a noise obeying a mixed Gaussian distribution $\rho_0(z) = p_+ \exp[-(z-z_+)^2/2\sigma_+^2]/\sqrt{2\pi\sigma_+^2} + p_- \exp[-(z-z_-^2)/2\sigma_-^2]/\sqrt{2\pi\sigma_-^2}$ with $p_+ + p_- = 1$. The SNR for the optimal nonlinearity is plotted as a function of σ_+ in Fig. 1. The parameters are set to be $p_+ = 0.7$, $z_+ = 0.1$, $z_- = -0.1$, $\sigma_- = 1.0$ fixed, and σ_+ varies. The gauge is fixed as $\alpha = 0$ and $\beta = 1$. For comparison, the SNRs for a linear system $F_i(\boldsymbol{z}) = z_i$ and a simple threshold system $F_i(\boldsymbol{z}) = \mathrm{sgn}(z_i - \theta)$, which is a typical system investigated in the study on SR phenomena, are displayed. The threshold is set to be $\theta = 0.5$.

In the simulation, the signal part of the output is evaluated by subtracting the offset $\langle F_i(\boldsymbol{z})\rangle = \alpha$ from the sample average of the output $\sum_{i=1}^N F_i(\boldsymbol{s}+\boldsymbol{z})/N$, according to the signal estimation algorithm proposed in Sec. IV. Similarly, the noise intensity of the output is evaluated by subtracting the estimated signal part of the output $S_\mathrm{est} = \sum_{i=1}^N F_i(\boldsymbol{s}+\boldsymbol{z})/N - \alpha$ from the obtained output data as $\sum_{i=1}^N [F_i(\boldsymbol{s}+\boldsymbol{z}) - S_\mathrm{est}]^2/N$. The simulation results displayed in Fig. 1 are obtained for the signal

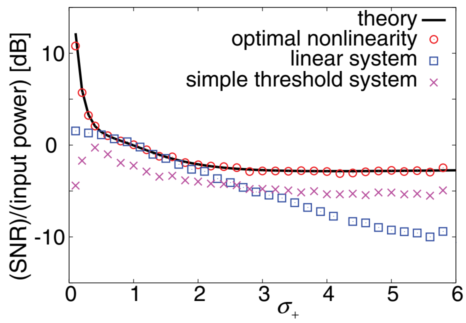

FIG. 1. (Color online) SNRs obtained from numerical simulations for the optimal nonlinearity (circles), a linear system (squares), and a simple threshold system (cross marks) in the presence of white mixed Gaussian noise. The solid line is the theoretically predicted SNR for the optimal nonlinearity.

$s_i = V$ for $i \leqslant N/2$, $s_i = -V$ for $i > N/2$, with $V = 0.1$ and $N = 1000$.

As seen in Fig. 1, the optimal nonlinearity exhibits a higher SNR than other systems. It is found that the SNR for the optimal nonlinearity decreases with the increase of σ_+ in the small σ_+ region, but slightly increases in the region of large σ_+. In contrast, the linear system shows a monotonically decreasing SNR with the increase of σ_+. In the above parameter settings, the optimal nonlinearity is approximated by a linear function as $F^*(z) \approx z/\sigma^2$ when $\sigma_+ \approx \sigma_- = \sigma$. The tangent appearance of the SNRs for the optimal nonlinearity and a linear system around $\sigma_+ = 1.0$ is due to this reason. Furthermore, it is found that a simple threshold system exhibits a higher SNR than that of a linear system in the large σ_+ region. Such an improvement of the SNR by using a nonlinear system cannot be expected in the presence of a Gaussian noise. This effect is due purely to the non-Gaussian property of the input noise.

C. Homogeneous Markovian noise

Consider the input noise generated from homogeneous Markovian process. If the initial condition is chosen as the stationary density for the Markovian process, the joint probability density of the noise is given as $\rho(\boldsymbol{z}) = [\prod_{i=1}^{N-1} T(z_{i+1}|z_i)]\rho_{\rm st}(z_1)$, where $T(z_{i+1}|z_i)$ denotes the transition probability and $\rho_{\rm st}$ is the stationary density of the noise. For simplicity, we choose the gauge as $A_i = \alpha$ and $B_{ij} = \beta\delta_{ij}$. In this case, the optimal nonlinearity $F_i^*(\boldsymbol{z})$ depends only on z_{i-1}, z_i, and z_{i+1}. Correspondingly, the Fisher information matrix becomes tridiagonal. Owing to the homogeneity and stationarity of the Markovian process, the Fisher information matrix is given as $\mathcal{I}^{ij} = \gamma_0\delta^{ij} + \gamma_1(\delta^{i,j-1} + \delta^{i,j+1})$, where

$$\gamma_0 = \int \rho(\boldsymbol{z})[\partial^1 \ln \rho(\boldsymbol{z})]^2 d\boldsymbol{z},$$
$$\gamma_1 = \int \rho(\boldsymbol{z})[\partial^1 \ln \rho(\boldsymbol{z})][\partial^2 \ln \rho(\boldsymbol{z})] d\boldsymbol{z}. \tag{9}$$

As seen in Eq. (6), the eigenvalue for the matrix $\mathcal{I}$ gives the SNR for the input signal whose waveform is the corresponding eigenvector. The eigenvalue is exactly obtained as $\gamma_0 + 2\gamma_1 \cos[k\pi/(N+1)]$ for the corresponding eigenvector $s_i^{(k)} = \sin[ik\pi/(N+1)]$, where $k = 1,\ldots,N$. Therefore, the upper bound of the SNR is given as $R \leqslant \gamma_0 + 2|\gamma_1|$ in the presence of homogeneous Markovian noise.

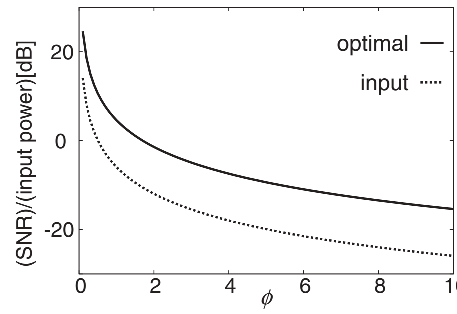

FIG. 2. SNRs obtained from numerical simulations for the optimal nonlinearity (solid line) and the input channel (dotted line) in the presence of noise generated from Markovian process. The parameter ϕ corresponds to the noise intensity.

In Fig. 2, the simulation results for Markovian noise are shown. The optimal SNR for a weak signal $s_i = V s_i^{(N)}$ per unit input signal power $R_*/P_{\rm s}$ is compared with that at the input channel, i.e., $\langle z_i^2\rangle^{-1}$. The amplitude of the signal is chosen as $V = 0.1$. In the simulation, the transition probability is given as $T(z_{i+1}|z_i) = \exp(-|z_{i+1} - g z_i|/\phi)/2\phi$ with $g = 0.7$. The result displayed in Fig. 2 is the average over 100 independent samples of the inputs and outputs of length $N = 1000$. It is numerically checked that the SNR for the optimal nonlinearity is approximated as $\gamma_0 + 2|\gamma_1|$.

IV. LINEAR ESTIMATION OF INPUT SIGNAL

As shown in the previous section, the SNR for the optimal nonlinearity Eq. (6) is expressed in terms of the Fisher information. The Fisher information connects the input signal $\boldsymbol{s}$, which is to be estimated, with the observed input with noise $\boldsymbol{y} = \boldsymbol{s} + \boldsymbol{z}$. In order to estimate the unknown input signal $\boldsymbol{s}$, the observed value of $\boldsymbol{y}$ and its conditional probability density $\mu(\boldsymbol{y}|\boldsymbol{s})$ are exploited. However, since $\boldsymbol{y}$ is a random variable, the estimated input signal $\hat{\boldsymbol{s}}$ is also a random variable, and the accuracy of the estimation is bounded by its covariance matrix $[{\rm Cov}(\hat{\boldsymbol{s}})]_{ij} \equiv \langle \hat{s}_i\hat{s}_j\rangle - \langle\hat{s}_i\rangle\langle\hat{s}_j\rangle$. It is well known that the Cramér-Rao inequality [32,33] gives the lower bound of the covariance ${\rm Cov}(\hat{\boldsymbol{s}})$ of the unbiased estimator, i.e., $\langle\hat{\boldsymbol{s}}\rangle = \boldsymbol{s}$, as

$${\rm Cov}(\hat{\boldsymbol{s}}) \geqslant \mathcal{I}^{-1}(\boldsymbol{s}), \tag{10}$$

which means that $\boldsymbol{v}^{\rm T}{\rm Cov}(\hat{\boldsymbol{s}})\boldsymbol{v} \geqslant \boldsymbol{v}^{\rm T}\mathcal{I}^{-1}(\boldsymbol{s})\boldsymbol{v}$ always holds for an arbitrary real vector $\boldsymbol{v}$. For a weak input signal $|\boldsymbol{s}| \to 0$, the accuracy of the estimation of the input signal is then bounded by the Fisher information matrix $\mathcal{I}(0)$.

Since the output $\boldsymbol{x}$ for the optimal nonlinearity is expressed in the linear response regime as $x_i = F_i^*(\boldsymbol{z}) + s_k\partial^k F_i^*(\boldsymbol{z}) + O(s^2) = A_i + s_k\mathcal{I}^{kl}(0)B_{li} + \xi_i + O(s^2)$, the natural unbiased estimator for the input signal $\hat{\boldsymbol{s}}$ is given by the linear transformation of the output $\boldsymbol{x}$ as $\hat{s}_i = (x_k - A_k)B^{kl}\mathcal{I}_{li}$. Accordingly, the covariance of the estimated signal is written as

$${\rm Cov}(\hat{\boldsymbol{s}}) = \mathcal{I}^{-1}(0). \tag{11}$$

Therefore, the optimal nonlinearity $\boldsymbol{F}^*$ achieves the lower bound of the unbiased estimator for the input signal. In this sense, *the optimal nonlinearity that maximizes the output SNR is the optimal unbiased estimator for the input signal*.

In the above framework, the probability density $\rho(z)$ is assumed to be known, and hence the optimal nonlinearity $\boldsymbol{F}^*$ is also given. $\boldsymbol{A}$, B, and $\mathcal{I}$ all are given. Then the optimal unbiased estimator, which is given by the linear transformation of the observed output $\boldsymbol{x}$, is fixed. This simple estimator has the advantage of easy calculations, as compared with other signal estimation algorithms that require complicated analysis [34–37].

V. CONCLUSIONS

We have derived the optimal nonlinearity to maximize the SNR of the output. The optimal nonlinearity is determined only by the probability density of the input noise. The concrete form of the optimal nonlinearity $\boldsymbol{F}^*$ given by Eq. (5) systematically claims that *a linear system is optimal only when the input noise is Gaussian, and for a non-Gaussian noise, there exist nonlinear systems that exhibit the SNR higher than that of a linear system*. Consequently, the existence of the nonlinearity exhibiting an SNR higher than that of a linear system allows the occurrence of stochastic resonance that yields the signal processing performance higher than a linear system in the presence of non-Gaussian noise, as reported in the literature [16–27].

Furthermore, the optimal nonlinearity obtained from the viewpoint of the maximization of the SNR has been found to yield the minimum error in the unbiased signal estimation process, and a simple estimation algorithm has been proposed. In general, the maximization of the SNR does not yield the optimal signal estimation nor the optimal information transmission performance. In the linear response regime, however, the signal is easily estimated using only the first moment of the output. Then the maximization of the SNR defined in terms of the first moment of the output coincides with the optimal unbiased signal estimation.

Our results are due to the knowledge on the noise statistics. In practical cases, the probability density of noise is often unknown. In such a case, the hybrid use of our method and the estimation for the noise statistics is required. This hybrid use is regarded as the extension of the conventional adaptive filters.

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