Diffusive process on a backbone structure with drift terms

E. K. Lenzi,[1,*] L. R. da Silva,[2] A. A. Tateishi,[1,3] M. K. Lenzi,[4] and H. V. Ribeiro[1]

[1]Departamento de Física, Universidade Estadual de Maringá, Avenida Colombo 5790, 87020-900 Maringá, PR, Brazil
[2]Departamento de Física, Universidade Federal do Rio Grande do Norte, 59072-970 Natal, RN, Brazil
[3]Section for Science of Complex Systems, Medical University of Vienna, Spitalgasse 23, 1090 Vienna, Austria
[4]Departamento de Engenharia Química, Universidade Federal do Paraná, Setor de Tecnologia, Jardim das Américas, Caixa Postal 19011, 81531-990 Curitiba, PR, Brazil

(Received 20 September 2012; revised manuscript received 9 December 2012; published 17 January 2013)

The effects of an external force on a diffusive process subjected to a backbone structure are investigated by considering the system governed by a Fokker-Planck equation with drift terms. Our results show an anomalous spreading which may present different diffusive regimes connected to anomalous diffusion and stationary states.

DOI: 10.1103/PhysRevE.87.012121 PACS number(s): 05.40.Jc, 05.10.Gg

I. INTRODUCTION

"The ant in the labyrinth" was the term used by P. G. de Gennes [1] to illustrate the random walk problem in an infinite percolation cluster. As is known, the random walk concept is intrinsically related to diffusion phenomena, which, depending on the conditions imposed by the system, can exhibit unusual characteristics. One of them is the presence of anomalous relaxation, e.g., a nonlinear time dependence on the mean square displacement such as $\langle (r-\langle r\rangle)^2\rangle \sim t^{\alpha}$ ($\alpha \neq 1$ is connected to anomalous diffusion). This power law has been widely found in several systems with disordered structures [2], fractals [3], and percolation clusters [4,5] with $\alpha = 2/d_W$, where d_W is the fractal dimension. In this scenario, a comb-like structure (see Fig. 1) was proposed as a model to investigate anomalous diffusion in percolation clusters with topological bias [6,7], wherein, according to Ref. [2], the branches of the comb play the same role as the dangling ends of the percolation cluster and the backbone of the comb is analogous to the quasilinear structure of the backbone of the cluster.

A diffusive process subject to a comb-like structure is described by the following Fokker-Planck equation, as reported in Ref. [8],

$$\frac{\partial}{\partial t}\rho(x,y;t) = \mathcal{D}_y\frac{\partial^2}{\partial y^2}\rho(x,y;t) + \delta(y)\mathcal{D}_x\frac{\partial^2}{\partial x^2}\rho(x,y;t), \tag{1}$$

where $\mathcal{D}_y$ and $\mathcal{D}_x$ are the diffusion coefficients in the x and y directions. The distributions obtained from this equation show that the diffusion in the x direction is not usual and can be connected to a fractional diffusion equation [9–11], as shown in Refs. [12–14]. This model, which is a simplified picture of highly disordered systems, can be connected to a rich class of anomalous diffusive processes [8,15–18] due to the geometric constraints [19] with the advantage of performing an exact analysis and providing analytical solutions [20,21], which can be used as a guide for more complex situations. In previous work, the subdiffusive ($\alpha < 1$) and superdiffusive ($\alpha > 1$) cases have been addressed in Refs. [20–26], a toy model to describe the dynamics of cancer proliferation is developed in Refs. [27,28], in Refs. [29] and [30] a fractional quantum dynamics is investigated, in Ref. [31] the usual form of the comb model is extended by incorporating fractal aspects, and in Ref. [32] ultracold atoms in a one-dimensional are investigated. Furthermore, different diffusive regimes have been reported in Ref. [33] as a consequence of the backbone structure incorporated in the diffusion equation. Here, we investigate the effects produced by an external force on the diffusive processes subjected to the structure represented in Fig. 1. In particular, we consider the Fokker-Planck equation

$$\begin{aligned}\frac{\partial}{\partial t}\rho(x,y;t) = {}& \mathcal{D}_y\frac{\partial^2}{\partial y^2}\rho(x,y;t) \\ & + \mathcal{D}_x\delta(y)\left(\frac{\partial^2}{\partial x^2} - \overline{v}_x\frac{\partial}{\partial x}\right)\rho(x,y;t) \\ & - \nabla\cdot(\vec{v}\rho(x,y;t)),\end{aligned} \tag{2}$$

with $\vec{v} = (v_x, v_y)$, where v_x, v_y, and $\overline{v}_x$ are constants. Equation (2) extends Eq. (1) by incorporating the drift term $\vec{F} = (v_x + \delta(y)\overline{v}_x, v_y)$, which represents an external force acting on the system. The boundary conditions used to investigate the solutions of Eq. (2) are $\rho(\pm\infty, y;t) = 0$ and $\rho(x,\pm\infty;t) = 0$; i.e., the backbone and the branches of the comb are not limited. We also consider an arbitrary initial condition given by $\rho(x,y;0) = \widehat{\rho}(x,y)$, where $\widehat{\rho}(x,y)$ is normalized. Our results show that the presence of the external force changes the diffusive process and may introduce different diffusive regimes depending on the choice of the parameters v_x, v_y, and $\overline{v}_x$.

This work is organized as follows. The next section, Sec. II, is devoted to investigating the effects of the drift forces outside and inside of the backbone. In Sec. III, we present a summary of the results and our conclusions.

II. DRIFT FORCES AND BACKBONE STRUCTURE

Let us start our analysis by considering the case where drift forces act outside of the backbone structure, i.e., $v_x \neq 0$, $v_y \neq 0$, with $\overline{v}_x = 0$. For this case, by using the Green function approach, the distribution obtained from Eq. (2) is given by

$$\rho(x,y;t) = -\int_{-\infty}^{\infty} d\overline{y}\,\widehat{\rho}(x,y)\mathcal{G}(x,y,\overline{y};t), \tag{3}$$

*eklenzi@dfi.uem.br

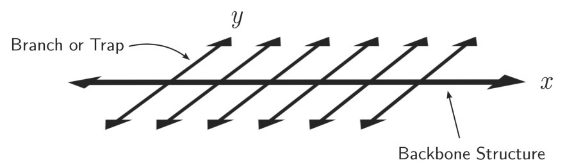

FIG. 1. Illustration of the backbone structure used in the comb model. Note that diffusion in the x direction only occurs when $y = 0$ and diffusion in the y direction is perpendicular to the x axis.

with the Green function,

$$\mathcal{G}(x,y,\overline{y};t) = -e^{\frac{v_y}{2\mathcal{D}_y}(y-\overline{y})}e^{-\frac{v_y^2}{4\mathcal{D}_y}t}\overline{\mathcal{G}}\,'(x,y,\overline{y};t),$$
$$\overline{\mathcal{G}}\,'(x,y,\overline{y};t) = \frac{1}{\sqrt{4\pi\mathcal{D}_y t}}\delta(x-v_x t)\big(e^{-\frac{(y-\overline{y})^2}{4\mathcal{D}_y t}} - e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y t}}\big)$$
$$+\frac{1}{\sqrt{8\mathcal{D}_x\sqrt{\mathcal{D}_y}}}\frac{|y|+|\overline{y}|}{\sqrt{4\pi\mathcal{D}_y}}\int_0^t d\overline{t}\,\frac{e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y(t-\overline{t})}}}{\big[(t-\overline{t})\overline{t}^{\frac{1}{2}}\big]^{\frac{3}{2}}}$$
$$\times \mathrm{H}_{1,1}^{1,0}\left[\sqrt{\frac{2}{\mathcal{D}_x}\sqrt{\frac{\mathcal{D}_y}{\overline{t}}}}|x-v_x\overline{t}|\,\middle|\begin{smallmatrix}(\frac{1}{4},\frac{1}{4})\\(0,\ 1)\end{smallmatrix}\right] \quad (4)$$

(details of calculations and useful properties of the H function can be found in Appendixes A and B). Now we address our attention to the relaxation of the system in order to characterize the effect produced by drift forces on system. In order to perform this analysis, we consider the mean square displacement for the x and y directions, for simplicity, by taking into account the initial condition $\hat{\rho}(x,y) = \delta(x)\delta(y-\tilde{y})$. After some calculation, it is possible to show that

$$\sigma_x^2(t) = \langle (x-\langle x\rangle)^2\rangle$$
$$= 2\mathcal{D}_x e^{-\frac{v_y}{2\mathcal{D}_y}\tilde{y}}\int_0^t \frac{d\tau}{\sqrt{4\pi\mathcal{D}_y\tau}}e^{-\frac{v_y^2}{4\mathcal{D}_y}\tau}e^{-\frac{\tilde{y}^2}{4\mathcal{D}_y\tau}} \quad (5)$$

and $\sigma_y^2(t) = 2\mathcal{D}_y t$. Equation (5) shows that the drift force v_x does not influence the backbone structure and that, for $v_y = 0$, we recover the mean square displacement presented in Ref. [20]. For $v_y \neq 0$, we observe a kind of confined diffusive regime for long times, also called a saturation regime, as shown in Fig. 2. This kind of behavior has been reported in Brownian dynamics simulations of a single polymer [34], in time series of continuous-time random walk and fractional Brownian motion [35], and also in living cells [36–39], where the crowded environment of the cytoplasm and the constrained diffusion are possible physical mechanisms of the anomalous diffusion. Moreover, once the comb model was proposed to mimic percolation-like structures, it is remarkable to note that the behavior of spreading obtained here is in good agreement with the diffusion reported in percolation clusters below the criticality, i.e., $p < p_c$ (where p_c is the critical probability threshold of percolation transition) and the clusters are considered finite (see Ref. [2]). Hence, due to the constant force acting on the y axis the system remains confined in the branches, i.e., the particles fall into a trap of the labyrinth and do not return to the backbone. In this manner, we verify that the drift term in the y direction changes the spreading of the system in the backbone and lead us to a stationary solution in the x direction.

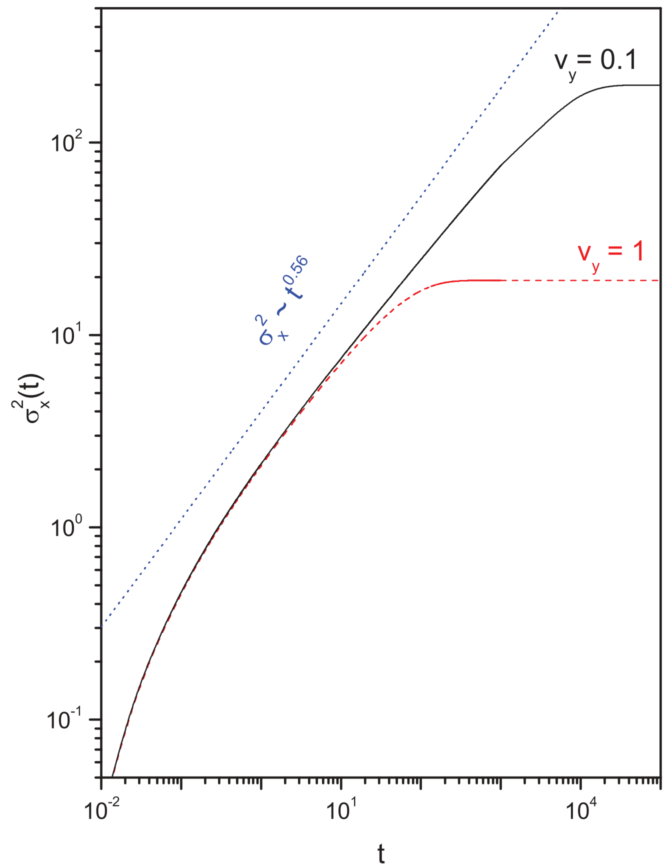

FIG. 2. (Color online) Time behavior of the mean square displacement with $v_x = 0$, $\mathcal{D}_y = 20$, $\mathcal{D}_x = 10$, and $\tilde{y} = 0.8$. The solid (black) line corresponds to the case $v_y = 0.1$ and the dashed (red) line is the case $v_y = 1$. Note that saturation in the x direction is reached more rapidly for large values of v_y and more slowly for small values of v_y. The dotted (blue) line was incorporated in order to illustrate the subdiffusive behavior presented by the system before reaching the stationary state.

Let us incorporate the drift force acting in the backbone structure, i.e., $\overline{v}_x \neq 0$, in the x direction. The presence of this term in the external force leads us to the following expression for the Green function:

$$\mathcal{G}(x,y,\overline{y};t) = -e^{-\frac{v_y^2}{4\mathcal{D}_y}t}e^{\frac{v_y}{2\mathcal{D}_y}(y-\overline{y})}\widetilde{\mathcal{G}}(x,y,\overline{y};t), \quad (6)$$

where

$$\widetilde{\mathcal{G}}(x,y,\overline{y};t) = \frac{1}{\sqrt{4\pi\mathcal{D}_y t}}\delta(x-v_x t)\big(e^{-\frac{(y-\overline{y})^2}{4\mathcal{D}_y t}} - e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y t}}\big)$$
$$+\frac{1}{t}\int_0^\infty du(|y|+|\overline{y}|+2\mathcal{D}_y u)$$
$$\times\mathcal{G}_y(|y|,|\overline{y}|,2\mathcal{D}_y u;u)\mathcal{G}_x(x,-\overline{v}_x u,-v_x t;t), \quad (7)$$

and $\mathcal{G}_\alpha(x,y,z;u) = e^{-\frac{1}{4\mathcal{D}_\alpha u}(x+y+z)^2}/\sqrt{4\pi\mathcal{D}_\alpha u}$ (details of calculations and useful properties of the H function are given in Appendixes A and B). Similarly to the previous case, we analyze the behavior of mean square displacement in the x and y directions to investigate the effect of the external force on

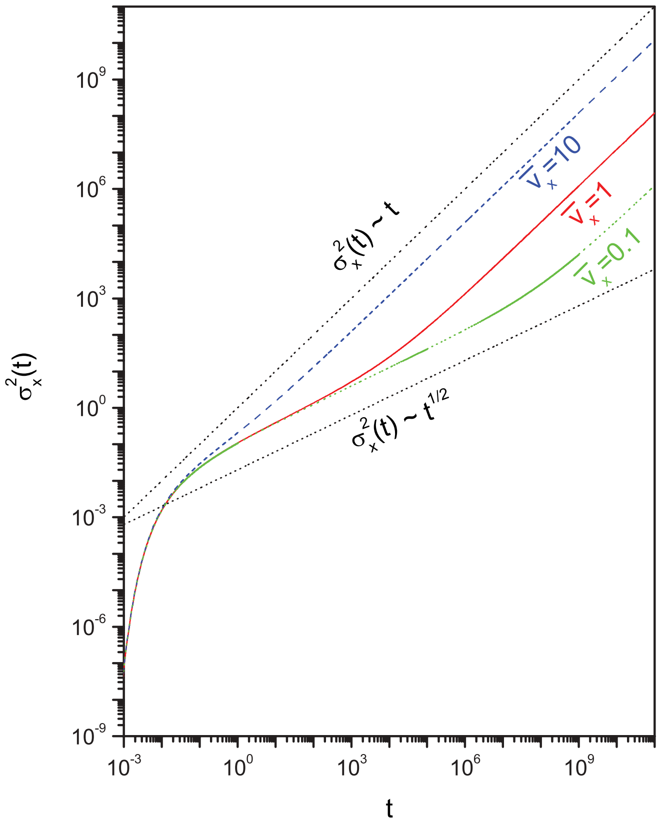

FIG. 3. (Color online) Time behavior of the mean square displacement for $v_y = 0$, $\mathcal{D}_y = 20$, $\mathcal{D}_x = 10$, and $\tilde{y} = 0.8$. The straight dotted lines were added in order to illustrate the usual and the subdiffusive behavior which can be manifested by the mean square displacement obtained from Eqs. (8) and (9). Dashed (blue), solid (red), and dotted (green) lines correspond to the cases $\overline{v}_x = 10$, $\overline{v}_x = 1$, and $\overline{v}_x = 0.1$, respectively, as indicated.

the relaxation of the system. For this case, we obtain

$$\langle x\rangle = v_x t + \frac{\overline{v}_x e^{-\frac{v_y}{2\mathcal{D}_y}\tilde{y}}}{\sqrt{4\pi\mathcal{D}_y}}\int_0^t \frac{d\overline{t}}{\sqrt{\overline{t}}} e^{-\frac{v_y^2}{4\mathcal{D}_y}\overline{t}} e^{-\frac{\tilde{y}^2}{4\mathcal{D}_y\overline{t}}} \tag{8}$$

and

$$\langle x^2\rangle = v_x^2 t^2 + \int_0^t \frac{d\overline{t}}{\sqrt{\pi\mathcal{D}_y\overline{t}}}(2v_x\overline{v}_x(t-\overline{t}) + \mathcal{D}_x) e^{-\frac{(\tilde{y}+v_y\overline{t})^2}{4\mathcal{D}_y\overline{t}}} + \frac{\overline{v}_x^2}{2\mathcal{D}_y}\int_0^t d\overline{t} e^{-\frac{v_y}{2\mathcal{D}_y}\tilde{y}} e^{-\frac{v_y^2}{4\mathcal{D}_y}\overline{t}} \operatorname{erfc}\left(-\frac{|\tilde{y}|}{2\sqrt{\mathcal{D}_y\overline{t}}}\right). \tag{9}$$

Figure 3 illustrates the behavior of the mean square displacement obtained for the previous equations by considering $v_y = 0$ and $v_x = 0$. An interesting feature is the presence of two different regimes after an initial transient. One of these regimes is subdiffusive and the other is usual. In this sense, note that the presence of the subdiffusive regime depends on the values of $\overline{v}_x$, e.g., $\overline{v}_x \gg 1$ yields an usual behavior, while $\overline{v}_x \ll 1$ yields a subdiffusive behavior. In addition, the same dynamical crossover between subdiffusion and normal diffusion is reported in Ref. [15], where the authors studied the random walk in a comb lattice by numerical simulations. This kind of crossover is also found in polymer physics, where, according to Ref. [40], the dynamics of a tagged monomer in a polymer must be anomalous until the terminal relaxation time, and this anomalous dynamics is connected to the mean relaxation response of the polymers to local strains [41]. In particular, for phantom Rouse polymers the mean square displacement of a tagged monomer behaves as $t^{1/2}$ until the terminal relaxation time τ, and only after that time does the dynamics of the polymer become diffusive [42]. Moreover, in the theory of percolation clusters diffusion is usual when percolation occurs, i.e., $p > p_c$, and the clusters are considered infinite [2]. In Fig. 4, we incorporate the drift term in the y direction, i.e., $v_y \neq 0$, and a stationary behavior is obtained for the distribution in the x direction, since $\sigma_x^2(t)$ is constant for long times as in the previous case worked out for the drift term outside of the the backbone structure.

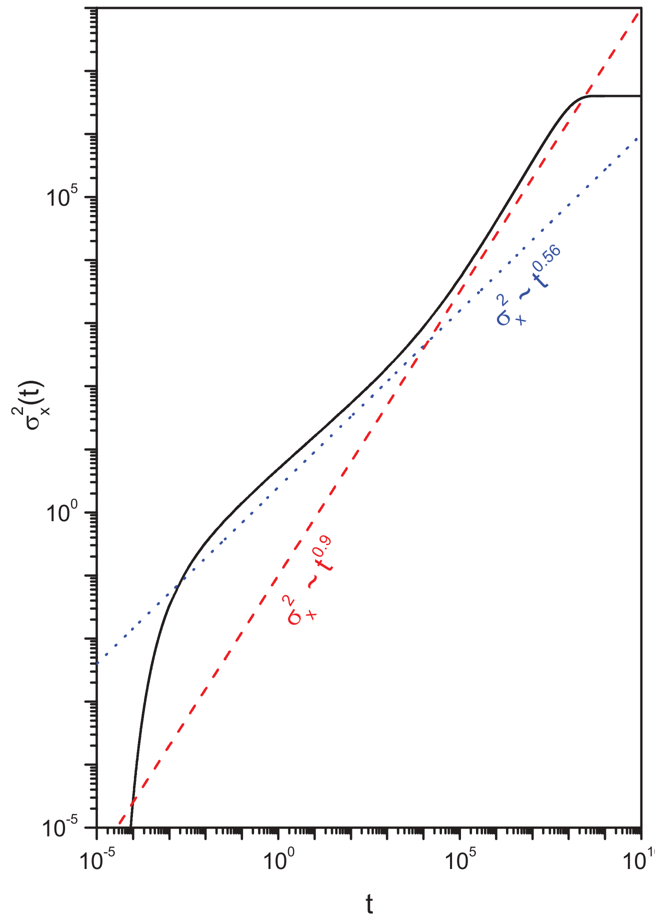

FIG. 4. (Color online) Time behavior of the mean square displacement for $v_y = 5\times 10^{-4}$, $v_x = 0$, $\overline{v}_x = 1$, $\mathcal{D}_y = 5$, $\mathcal{D}_x = 10$, and $\tilde{y} = 0.1$. The dotted (blue) and dashed (red) lines were incorporated in order to illustrate the subdiffusive behaviors presented by the system before reaching the stationary state.

III. DISCUSSION AND CONCLUSIONS

We have analyzed the Fokker-Planck equation in a backbone structure by considering the presence of drift terms acting in the y and x directions. The drift term has components inside and outside of this structure. For the case characterized by the drift terms with components outside of the backbone structure, the component dependent on v_x does not influence the spreading of the system. In this case, the presence of v_y leads us to a stationary solution for the x direction as shown in Fig. 2, and as expected, it does not have an influence on the y direction, which is characterized by the usual diffusion. For the case $v_x = 0$, $\overline{v}_x \neq 0$, and $v_y = 0$, we obtain an interesting behavior for the system when the x direction is analyzed. It presents different diffusive regimes, one of which is subdiffusive and the other the usual. The existence of the subdiffusive behavior depends on the values of $\overline{v}_x$, i.e., for $\overline{v}_x \ll 1$ the system remains in the subdiffusive regime for a long time, and for $\overline{v}_x \gg 1$ the subdiffusive regime may not exist. This fact suggests that, for large values of $\overline{v}_x$, the system remains in the backbone structure for a short time, which is governed by the subdiffusive case in connection to the fractional diffusion equation. In this scenario, the presence of v_y changes the last diffusive regime and leads us to a stationary solution as in the first case characterized by $\overline{v}_y = 0$ as illustrated in Fig. 2. Finally, we hope that the results presented here are useful to discuss diffusive processes connected to anomalous diffusion.

ACKNOWLEDGMENTS

We would like to thank CNPq and CAPES (Brazilian agencies) for partial financial support.

APPENDIX A: SOLUTIONS

Equation (2) subjected to the conditions $v_x \neq 0$, $v_y \neq 0$, with $\overline{v}_x = 0$, can be written as

$$\frac{\partial}{\partial t}\rho(x,y;t) = \mathcal{D}_y \frac{\partial^2}{\partial y^2}\rho(x,y;t) + \delta(y)\mathcal{D}_x \frac{\partial^2}{\partial x^2}\rho(x,y;t) - \nabla \cdot (\vec{v}\rho(x,y;t)). \tag{A1}$$

In order to obtain the solution for Eq. (A1) subjected to the conditions discussed in Sec. I in terms of the Green function approach, we use Laplace and Fourier transforms. Applying the Laplace transform ($\mathcal{L}\{\cdots\} = \int_0^\infty dt e^{-st}\cdots$ and $\mathcal{L}^{-1}\{\cdots\} = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} ds e^{st}\cdots$) in Eq. (A1), we obtain

$$\mathcal{D}_y \frac{\partial^2}{\partial y^2}\rho(x,y;s) + \delta(y)\mathcal{D}_x \frac{\partial^2}{\partial x^2}\rho(x,y;s) - \nabla\cdot(\vec{v}\rho(x,y;s)) = s\rho(x,y;s) - \widehat{\rho}(x,y). \tag{A2}$$

This equation can be simplified by employing the Fourier transform on the x variable ($\mathcal{F}_x\{\cdots\} = \int_{-\infty}^{\infty} dx e^{-ik_x x}\cdots$ and $\mathcal{F}_x^{-1}\{\cdots\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ik_x x}$), yielding the differential equation,

$$\mathcal{D}_y \frac{\partial^2}{\partial y^2}\rho(k_x,y;s) - \left(s + v_y\frac{\partial}{\partial y} + \delta(y)\mathcal{D}_x k_x^2 + ik_x v_x\right)\rho(k_x,y;s) = -\widehat{\rho}(k_x,y), \tag{A3}$$

which can be solved using the Green function approach. Using this approach the solution of Eq. (A3) is given by

$$\rho(k_x,y;s) = -\int_{-\infty}^{\infty} d\overline{y}\,\widehat{\rho}(k_x,y)\mathcal{G}(k_x,y,\overline{y};s), \tag{A4}$$

with the Green function obtained from the equation

$$\left(\mathcal{D}_y\frac{\partial^2}{\partial y^2} - v_y\frac{\partial}{\partial y} - \delta(y)\mathcal{D}_x k_x^2 - ik_x v_x - s\right)\mathcal{G}(k_x,y,\overline{y};s) = \delta(y-\overline{y}) \tag{A5}$$

subjected to the Dirichlet boundary condition, i.e., $\mathcal{G}(k_x,\pm\infty,\overline{y};s) = 0$. Equation (A5) may also be solved by using the Fourier transform with respect to variable y ($\mathcal{F}_y\{\cdots\} = \int_{-\infty}^{\infty} dy e^{-ik_y y}\cdots$ and $\mathcal{F}_y^{-1}\{\cdots\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik_y y}$). The solution in the Fourier space for Eq. (A5) is given by

$$\mathcal{G}(k_x,k_y,\overline{y};s) = -\frac{e^{-ik_y\overline{y}}}{\mathcal{D}_y k_y^2 + ik_y v_y + ik_x v_x + s} - \frac{\mathcal{D}_x k_x^2}{\mathcal{D}_y k_y^2 + ik_y v_y + ik_x v_x + s}\mathcal{G}(k_x,0,\overline{y};s). \tag{A6}$$

After some calculations, it is possible to show that

$$\mathcal{G}(k_x,0,\overline{y};s) = -\frac{e^{-\frac{v_y}{2\mathcal{D}_y}\overline{y}}e^{-\frac{\sqrt{\beta}}{2\mathcal{D}_y}|\overline{y}|}}{\sqrt{\beta} + \mathcal{D}_x k_x^2}, \tag{A7}$$

with $\beta = v_y^2 + 4\mathcal{D}_y s + 4i\mathcal{D}_y k_x v_x$. By substituting Eq. (A7) in Eq. (A6) and performing the inverse of Laplace transform, one obtains

$$\mathcal{G}(k_x,y,\overline{y};t) = -e^{\frac{v_y}{2\mathcal{D}_y}(y-\overline{y})}e^{-\frac{v_y^2}{4\mathcal{D}_y}t}e^{-ik_x v_x t}\left\{\frac{1}{\sqrt{4\pi\mathcal{D}_y t}}\left(e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y t}} - e^{-\frac{(y-\overline{y})^2}{4\mathcal{D}_y t}}\right)\right. \tag{A8}$$

$$\left. + \frac{1}{2\mathcal{D}_y}(|y|+|\overline{y}|)\int_0^t d\overline{t}\frac{e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y(t-\overline{t})}}}{\sqrt{4\pi\overline{t}(t-\overline{t})^3}} \times \mathrm{E}_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\mathcal{D}_x k_x^2}{2\sqrt{\mathcal{D}_y}}\sqrt{\overline{t}}\right)\right\}. \tag{A9}$$

Note the presence of the generalized Mittag-Leffler function in the last part of the previous equation [43]. Performing the inverse Fourier transform on the x variable and considering some identities of the Fox H function [44,45] (see, in particular, Appendix A), it is possible to show that the Green function is given by

$$\mathcal{G}(x,y,\overline{y};t) = -e^{\frac{v_y}{2\mathcal{D}_y}(y-\overline{y})}e^{-\frac{v_y^2}{4\mathcal{D}_y}t}\overline{\mathcal{G}}'(x,y,\overline{y};t),$$
$$\overline{\mathcal{G}}'(x,y,\overline{y};t) = \frac{1}{\sqrt{4\pi\mathcal{D}_y t}}\delta(x-v_x t)\left(e^{-\frac{(y-\overline{y})^2}{4\mathcal{D}_y t}} - e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y t}}\right) + \frac{1}{\sqrt{8\mathcal{D}_x\sqrt{\mathcal{D}_y}}}\frac{|y|+|\overline{y}|}{\sqrt{4\pi\mathcal{D}_y}}\int_0^t d\overline{t}\frac{e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y(t-\overline{t})}}}{[(t-\overline{t})\overline{t}^{\frac{1}{2}}]^{\frac{3}{2}}} \times \mathrm{H}_{1,1}^{1,0}\left[\sqrt{\frac{2}{\mathcal{D}_x}\sqrt{\frac{\mathcal{D}_y}{\overline{t}}}}|x-v_x\overline{t}|\,\middle|\begin{matrix}(\frac{1}{4},\frac{1}{4})\\(0,\,1)\end{matrix}\right]. \tag{A10}$$

Let us consider the solutions for the case $\overline{v}_x \neq 0$, $v_y \neq 0$, and $v_x \neq 0$. For this case, Eq. (2) is given by

$$\frac{\partial}{\partial t}\rho(x,y;t) = \mathcal{D}_y \frac{\partial^2}{\partial y^2}\rho(x,y;t) + \delta(y)\left(\mathcal{D}_x \frac{\partial^2}{\partial x^2} - \overline{v}_x \frac{\partial}{\partial x}\right)\rho(x,y;t) - v_y \frac{\partial}{\partial y}\rho(x,y;t) - v_x \frac{\partial}{\partial x}\rho(x,y;t). \tag{A11}$$

In order to solve this equation, it is possible to employ the same procedure used to obtain the solution of Eq. (A1). In this sense, we start by applying the Laplace transform and the Fourier transform on the x variable in Eq. (A11), which leads us to the equation

$$\mathcal{D}_y \frac{\partial^2}{\partial y^2}\rho(k_x,y;s) - \left(s + v_y \frac{\partial}{\partial y} + \delta(y)\left(\mathcal{D}_x k_x^2 + ik_x\overline{v}_x\right) + ik_x v_x\right)\rho(k_x,y;s) = -\widehat{\rho}(k_x,y), \tag{A12}$$

with the solution given by Eq. (A4) and the Green function governed by the equation

$$\mathcal{D}_y \frac{\partial^2}{\partial y^2}\mathcal{G}(k_x,y;s) - \left(s + v_y \frac{\partial}{\partial y} + \delta(y)\left(\mathcal{D}_x k_x^2 + ik_x\overline{v}_x\right) + ik_x v_x\right)\mathcal{G}(k_x,y;s) = \delta\left(y - \overline{y}\right), \tag{A13}$$

subjected to the boundary conditions $\mathcal{G}(k_x,\pm\infty,\overline{y};s) = 0$. By using the Fourier transform with respect to variable y and performing some calculations, it is possible to show that the solution in the Fourier space of Eq. (A13) is

$$\mathcal{G}(k_x,k_y,\overline{y};s) = -\frac{e^{-ik_y\overline{y}}}{\mathcal{D}_y k_y^2 + ik_y v_y + ik_x v_x + s} - \frac{\mathcal{D}_x k_x^2 + ik_x\overline{v}_x}{\mathcal{D}_y k_y^2 + ik_y v_y + ik_x v_x + s}\mathcal{G}(k_x,0,\overline{y};s) \tag{A14}$$

with

$$\mathcal{G}(k_x,0,\overline{y};s) = -\frac{e^{-\frac{v_y}{2\mathcal{D}_y}\overline{y}}e^{-\frac{\sqrt{\beta}}{2\mathcal{D}_y}|\overline{y}|}}{\sqrt{\beta} + \mathcal{D}_x k_x^2 + ik_x\overline{v}_x}. \tag{A15}$$

Applying, in previous equations, the inverse of Laplace and Fourier transforms, we obtain that

$$\mathcal{G}(x,y,\overline{y};t) = -e^{-\frac{v_y^2}{4\mathcal{D}_y}t}e^{\frac{v_y}{2\mathcal{D}_y}(y-\overline{y})}\widetilde{\mathcal{G}}(x,y,\overline{y};t), \tag{A16}$$

where

$$\widetilde{\mathcal{G}}(x,y,\overline{y};t) = \frac{1}{\sqrt{4\pi\mathcal{D}_y t}}\delta(x - v_x t)\left(e^{-\frac{(y-\overline{y})^2}{4\mathcal{D}_y t}} - e^{-\frac{(|y|+|\overline{y}|)^2}{4\mathcal{D}_y t}}\right) + \frac{1}{t}\int_0^\infty du\,(|y| + |\overline{y}| + 2\mathcal{D}_y u) \times \mathcal{G}_y(|y|,|\overline{y}|,2\mathcal{D}_y u;u)\,\mathcal{G}_x(x,-\overline{v}_x u,-v_x t;t), \tag{A17}$$

and $\mathcal{G}_\alpha(x,y,z;u) = e^{-\frac{1}{4\mathcal{D}_\alpha u}(x+y+z)^2}/\sqrt{4\pi\mathcal{D}_\alpha u}$.

APPENDIX B: H FUNCTION

The Fox H function (or H function) may be defined in terms of the Mellin-Branes-type integral [44,45]

$$\mathrm{H}^{m,n}_{p,q}\left[x\middle|^{(a_p,A_p)}_{(b_q,B_q)}\right] = \mathrm{H}^{m,n}_{p,q}\left[x\middle|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right] = \frac{1}{2\pi i}\int_L \chi(\xi)x^{-\xi}d\xi,$$
$$\chi(\xi) = \frac{\Pi_{j=1}^m\Gamma(b_j - B_j\xi)\Pi_{j=1}^n\Gamma(1 - a_j + A_j\xi)}{\Pi_{j=m+1}^q\Gamma(1 - b_j + B_j\xi)\Pi_{j=n+1}^p\Gamma(a_j - A_j\xi)}, \tag{B1}$$

where m, n, p, and q are integers satisfying $0 \leqslant n \leqslant p$ and $1 \leqslant m \leqslant q$. It may also be defined by its Mellin transform,

$$\int_0^\infty \mathrm{H}^{m,n}_{p,q}\left[ax\middle|^{(a_p,A_p)}_{(b_q,B_q)}\right]x^{\xi-1}dx = a^{-\xi}\chi(\xi). \tag{B2}$$

Here, the parameters have to be defined such that $A_j > 0$, $B_j > 0$, and $a_j(b_h + \nu) \neq B_h(a_j - \lambda - 1)$, where $\nu,\lambda = 0,1,2,\ldots$, $h = 1,2,\ldots,m$, and $j = 1,2,\ldots,m$. The contour L separates the poles of $\Gamma(b_j - B_j\xi)$ for $j = 1,2,\ldots,m$ from those of $\Gamma(1 - a_j + A_j\xi)$ for $j = 1,2,\ldots,n$ [44]. The H function is analytic in x if either (i) $x \neq 0$ and $M > 0$ or (ii) $0 < |x| < 1/B$ and $M = 0$, where $M = \sum_{j=1}^q B_j - \sum_{j=1}^p A_j$ and $B = \prod_{j=1}^p A_j^{A_j}\prod_{j=1}^q B_j^{-B_j}$.

Some useful properties of the Fox H function found in Ref. [44] are listed below.

(1) The H function is symmetric in the pairs $(a_1,A_1),\ldots,(a_p,A_p)$ and, likewise, $(a_{n+1},A_{n+1}),\ldots,(a_p,A_p)$ as well as in $(b_1,B_1),\ldots,(b_q,B_q)$ and in $(b_{n+1},B_{n+1}),\ldots,(b_q,B_q)$.

(2) For $k > 0$

$$\mathrm{H}^{m,n}_{p,q}\left[x\middle|^{(a_p,A_p)}_{(b_q,B_q)}\right] = k\mathrm{H}^{m\ n}_{p\ q}\left[x^k\middle|^{(a_p,kA_p)}_{(b_q,kB_q)}\right]. \tag{B3}$$

(3) The multiplication rule is

$$x^k\mathrm{H}^{m,n}_{p,;q}\left[x\middle|^{(a_p,A_p)}_{(b_q,B_q)}\right] = \mathrm{H}^{m,n}_{p,q}\left[x\middle|^{(a_p+kA_p,A_p)}_{(b_q+kB_q,B_q)}\right]. \tag{B4}$$

(4) For $n \geqslant 1$ and $q > m$,

$$\mathrm{H}^{m,n}_{p,q}\left[x\middle|^{(a_1,A_1)(a_2,A_2)\cdots(a_p,A_p)}_{(b_1,B_1)\cdots(b_{q-1},B_{q-1})(a_1,A_1)}\right] = \mathrm{H}^{m,n-1}_{p-1,q-1}\left[x\middle|^{(a_2,A_2)\cdots(a_p,A_p)}_{(b_1,B_1)\cdots(b_{q-1},B_{q-1})}\right]. \tag{B5}$$

(5) For $m \geqslant 2$ and $p > n$

$$\mathrm{H}^{m,n}_{p,q}\left[x\middle|^{(a_1,A_1)\cdots(a_{p-1},A_{p-1})(b_1,B_1)}_{(b_1,B_1)(b_2,B_2)\cdots(b_q,B_q)}\right] = \mathrm{H}^{m-1,n}_{p-1,q-1}\left[x\middle|^{(a_2,A_2)\cdots(a_{p-1},A_{p-1})}_{(b_2,B_2)\cdots(b_q,B_q)}\right]. \tag{B6}$$

(6) The relation between the generalized Mittag-Leffler function and the Fox H function is given by

$$\mathrm{E}_{\alpha,\beta}(x) = \mathrm{H}^{1,1}_{1,2}\left[-x\left|^{(0,1)}_{(0,1)(1-\beta,\alpha)}\right.\right]. \tag{B7}$$

(7) Under Fourier cosine transformation, the H function transforms as

$$\int_0^\infty \mathrm{H}^{m,n}_{p,q}\left[k\left|^{(a_p,A_p)}_{(b_q,B_q)}\right.\right]\cos(kx)dx = \frac{\pi}{x}\mathrm{H}^{n+1,m}_{q+1,p+2}\left[x\left|^{(1-b_q,B_q),(1,1/2)}_{(1,1),(1-a_p,A_p),(1,1/2)}\right.\right]. \tag{B8}$$

(8) If the poles of $\prod_{j=1}^{m}\Gamma(b_j - B_j\xi)$ are simple, the following series expansion is valid:

$$\mathrm{H}^{m,n}_{p,q}\left[x\left|^{(a_p,A_p)}_{(b_q,B_q)}\right.\right] = \sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu x^{(b_h+\nu)/B_h}}{\nu! B_h}\frac{\prod_{j=1,j\neq h}^{m}\Gamma\left(b_j - \frac{B_j}{B_h}(b_h+\nu)\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\frac{B_j}{B_h}(b_h+\nu)\right)} \times \frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\frac{A_j}{B_h}(b_h+\nu)\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\frac{A_j}{B_h}(b_h+\nu)\right)}. \tag{B9}$$

[1] P. G. de Gennes, J. Phys. (Paris) Lett. **37**, L-1 (1976).
[2] D. Ben-Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems* (Cambridge University Press, Cambridge, 2000).
[3] B. B. Mandelbrot, *The Fractal Geometry of Nature* (W. H. Freeman, San Francisco, 1982).
[4] Y. Gefen, A. Aharony, and S. Alexander, Phys. Rev. Lett. **50**, 77 (1983).
[5] H. E. Stanley, J. Phys. A **10**, L211 (1977).
[6] S. R. White and M. Barma, J. Phys. A **17**, 2995 (1984).
[7] A. Bunde, S. Havlin, H. E. Stanley, B. Trus, and G. H. Weiss, Phys. Rev. B **34**, 8129 (1986).
[8] V. E. Arkhincheev and E. M. Baskin, Sov. Phys. JETP **73**, 161 (1991).
[9] Y. Z. Povstenko, J. Thermal Stresses **28**, 83 (2005).
[10] Y. Z. Povstenko, Nonlin. Anal.: Real World Appl. **13**, 1229 (2012).
[11] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[12] M. A. Zahran, Appl. Math. Model. **33**, 3088 (2009).
[13] M. A. Zahran, E. M. Abulwafa, and S. A. El-Wakil, Physica A **323**, 237 (2003).
[14] S. A. El-Wakil, M. A. Zahran, and E. M. Abulwafa, Chaos Solitons Fractals **20**, 1113 (2004).
[15] D. Villamaina, A. Sarracino, G. Gradenigo, A. Puglisi, and A. Vulpiani, J. Stat. Mech. (2011) L01002.
[16] R. Burioni, D. Cassi, G. Giusiano, and S. Regina, Phys. Rev. E **67**, 016116 (2003).
[17] A. M. Reynolds, Physica A **334**, 39 (2004).
[18] O. A. Dvoretskaya and P. S. Kondratenko, Phys. Rev. E **79**, 041128 (2009).
[19] Y. Meroz, I. M. Sokolov, and J. Klafter, Phys. Rev. Lett. **107**, 260601 (2011).
[20] L. R. da Silva, A. A. Tateishi, M. K. Lenzi, E. K. Lenzi, and P. C. da Silva, Braz. J. Phys. **39**, 438 (2009).
[21] V. E. Arkhincheev, Physica A **280**, 304 (2000).
[22] V. E. Arkhincheev, Physica A **389**, 1 (2010).
[23] A. Iomin, Phys. Rev. E **83**, 052106 (2011).
[24] E. Baskin and A. Iomin, Phys. Rev. Lett. **93**, 120603 (2004).
[25] A. Iomin and E. Baskin, Phys. Rev. E **71**, 061101 (2005).
[26] K. Kruse and A. Iomin, New J. Phys. **10**, 023019 (2008).
[27] A. Iomin, Phys. Rev. E **73**, 061918 (2006).
[28] A. Iomin, J. Phys.: Conf. Ser. **7**, 57 (2005).
[29] A. Iomin, Phys. Rev. E **80**, 022103 (2009).
[30] A. Iomin, Chaos Solitons Fractals **44**, 348 (2011).
[31] A. Iomin, Phys. Rev. E **83**, 052106 (2011).
[32] A. Iomin, Phys. Rev. E **86**, 032101 (2012).
[33] A. A. Tateishi, E. K. Lenzi, H. V. Ribeiro, L. R. Evangelista, R. S. Mendes, and L. R. da Silva, J. Stat. Mech. (2011) P02022.
[34] S. C. Weber, J. A. Theriot, and A. J. Spakowitz, Phys. Rev. E **82**, 011913 (2010).
[35] J.-H. Jeon and R. Metzler, J. Phys. A **43**, 252001 (2010).
[36] W. F. Marshall, A. Straight, J. F. Marko, J. Swedlow, A. Dernburg, A. Belmont, A. W. Murray, D. A. Agard, and J. W. Sedat, Curr. Biol. **7**, 930 (1997).
[37] I. Golding and E. C. Cox, Phys. Rev. Lett. **96**, 098102 (2006).
[38] N. Leijnse, J.-H. Jeon, S. Loft, R. Metzler, and L. B. Oddershede, Eur. Phys. J.: Spec. Topics **204**, 75 (2012).
[39] J.-H. Jeon, V. Tejedor, S. Burov, E. Barkai, C. Selhuber-Unkel, K. Berg-Sorensen, L. Oddershede, and R. Metzler, Phys. Rev. Lett. **106**, 048103 (2011).
[40] D. Panja, J. Stat. Mech. (2010) P06011.
[41] D. Panja, J. Stat. Mech. (2010) L02001.
[42] D. Panja, J. Phys.: Condens. Matter **23**, 105103 (2011).
[43] I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, CA, 1999).
[44] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Function: Theory and Applications* (Springer, New York, 2009).
[45] R. Metzler and T. F. Nonnenmacher, Chem. Phys. **284**, 67 (2002).