Amplitude ratios for critical systems in the $c = -2$ universality class

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We study the finite-size corrections of the critical dense polymer (CDP) and the dimer models on $\infty \times N$ rectangular lattice. We find that the finite-size corrections in the CDP and dimer models depend in a crucial way on the parity of N, and a change of the parity of N is equivalent to the change of boundary conditions. We present a set of universal amplitude ratios for amplitudes in finite-size correction terms of critical systems in the universality class with central charge $c = -2$. The results are in perfect agreement with a perturbated conformal field theory under the assumption that all analytical corrections coming from the operators which belongs to the tower of the identity. Our results inspire many interesting problems for further studies.

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I. INTRODUCTION

Finite-size scaling and corrections for critical systems have attracted much attention in recent decades. The studied systems include the Ising model [1–5], the Potts model [6], the percolation model [7–9], the dimer model [10–15], the critical dense polymer (CDP) [16,17], etc. A central element of the modern theory of critical phenomena is the division critical systems into (bulk) universality classes so that critical systems in the same universality class have the same set of critical exponents [18], universal finite-size scaling functions [7], universal amplitude ratios [4], etc. For examples, critical liquid-gas systems [19,20], the three-dimensional Ising models [21], and the Lennard-Jones model system [22] have the same set of critical exponents.

Although many theoretical results are now known about the critical exponents and universal relations among the leading critical amplitudes, not much information is available on ratios among the amplitudes in finite-size correction terms [23,24]. In this paper, we present a set of universal ratios among amplitudes in finite-size correction terms for critical systems described by logarithmic conformal field theory (LCFT) with central charge $c = -2$, which can be realized in different critical model systems, including the CDP [16,17], the dimer model on a rectangular lattice [12], the Abelian sandpile model [25,26], the spanning tree (see Sec. III below), Hamiltonian walks on a Manhattan lattice [27], the rational triplet theory [28], symplectic fermions [29], the traveling salesman problem [30] as well as branching polymers [31].

Conformal invariance implies that for an infinitely long two-dimensional (2D) strip of finite width N at criticality, the eigenstates of the critical transfer matrix, associated with conformal states, have energies which scale with N like [32]

$$E_n = N f_b + f_s + \frac{\pi\zeta(\Delta_n - c/24)}{N} + O(N^{-2}), \tag{1}$$

where the bulk free energy density f_b, the surface free energy f_s, and the anisotropy factor ζ are nonuniversal constants; in contrast, the central charge c and the weight Δ_n of the conformal eigenstate, are universal (although Δ_n depends on boundary conditions). Higher order correction terms in (1) are nonuniversal; however it has been suggested recently [4], and partially checked for the Ising universality class ($c = 1/2$), that the asymptotic expansion of the eigenstates of the critical transfer matrix has the form ($n = 0$ will refer to the ground state),

$$E_n = N f_b + f_s + \sum_{p=1}^{\infty} \frac{a_p^{(n)}}{N^{2p-1}}, \tag{2}$$

and that the amplitude ratios $a_p^{(n)}/a_p^{(0)}$ are universal and depend only on the boundary conditions [4,33–36]. The case $p = 1$ readily follows from (1).

It has been shown [17] that for the CDP model on the strip with free boundary conditions, which is in the universality class with $c = -2$, the eigenstates of the critical transfer matrix has the same asymptotic expansion of Eq. (2). It has been found that the amplitude ratios for the coefficients of these series are universal and can be explained in the framework of the perturbative conformal field theory [17].

In some two-dimensional geometries, the values of $a_1^{(n)}$ are known [32,37], to be related to the central charge c and the conformal weights Δ_n of the scaling fields corresponding to the n-exited state,

$$a_1^{(n)} = 4\pi\zeta\left(\Delta_n - \frac{c}{24}\right), \quad \text{for cylinder geometry} \tag{3}$$

$$a_1^{(n)} = \pi\zeta\left(\Delta_n - \frac{c}{24}\right), \quad \text{for strip geometry.} \tag{4}$$

In this paper we will show that for the dimer model on rectangular lattice and for the CDP on a cylinder (all of these models belong to $c = -2$ universality class) the asymptotic expansion of the eigenstates of the critical transfer matrix can also be written in the form of Eq. (2) and find that the ratios among correction amplitudes are universal for $p = 1$ and 2, and depends only on the boundary conditions. In cylinder geometry we have Ramond and Neveu-Schwarz (R-NS) sectors and $\mathbb{Z}_4$ sector and amplitude ratios are given

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by

$$\frac{a_p^{(1)}}{a_p^{(0)}} = \frac{1}{2^{1-2p}-1}, \quad \text{R-NS sectors;} \tag{5}$$

$$\frac{a_p^{(1)}}{a_p^{(0)}} = 1 + \frac{2p}{4^{p-1}B_{2p}(1/2)}, \quad \mathbb{Z}_4 \text{ sector.} \tag{6}$$

In the strip geometry we have open-closed and closed-closed (open-open) boundary conditions and amplitude ratios are given by

$$\frac{a_p^{(1)}}{a_p^{(0)}} = 1 + \frac{p}{4^{p-1}B_{2p}(1/2)}, \quad \text{open-closed;} \tag{7}$$

$$\frac{a_p^{(1)}}{a_p^{(0)}} = 1 + \frac{2p}{B_{2p}}, \quad \text{closed-closed (open-open),} \tag{8}$$

where B_{2p} and $B_{2p}(1/2)$ are the Bernoulli polynomial $B_{2p}(\alpha)$ at $\alpha = 0$ and $\alpha = 1/2$, respectively. In the definition of the boundary conditions on the strip we have used the language of Abelian sandpile model [26,38].

For the critical dense polymer model on the strip the configuration space on which the transfer matrix acts can be divided into sectors $\mathcal{L}_{N,\ell}$, labeled by an integer $\ell \geqslant 0$, of the same parity as N. The overall ground state E_0, found in $\mathcal{L}_{N,0}$ or $\mathcal{L}_{N,1}$. The specific levels we will consider here are the two lowest-lying levels in each sector $\mathcal{L}_{N,\ell}$, denoted $E_0^{(\ell)}$ and $E_1^{(\ell)}$. It has been shown [12,13,39,40] that due to the certain nonlocal features present in these models, the finite-size corrections depend in a crucial way on the parity (even or odd) of N: A change of parity of N induces a change of boundary condition.

For even N (even $\ell = 2n+2$), which corresponds to the closed-closed (open-open) boundary conditions, the asymptotic expansions of $E_0^{(\ell)}$ can be written in the form of Eq. (2) and amplitude ratios are given by

$$\frac{a_p^{(n)}}{a_p^{(0)}} = 1 + 2p\frac{H_n^{(1-2p)}}{B_{2p}}, \quad \text{closed-closed (open-open).} \tag{9}$$

The asymptotic expansions of $E_1^{(\ell)}$ can be also written in the form of Eq. (2), where amplitude ratios are given now by

$$\frac{a_p^{(n)}}{a_p^{(0)}} = 1 + 2p\frac{H_{n+1}^{(1-2p)} - n^{2p-1}}{B_{2p}}, \quad \text{closed-closed (open-open),} \tag{10}$$

where $H_{n-1}^{(1-2p)}$ is harmonic number of order $(1-2p)$,

$$H_n^{(p)} = \sum_{k=1}^{n} k^{-p}, \tag{11}$$

and $I_n^{(p)}$ is given by

$$I_n^{(p)} = \sum_{k=1}^{n}\left(k - \frac{1}{2}\right)^{-p}. \tag{12}$$

For odd N (even $\ell = 2n+1$), which corresponds to the open-closed boundary condition, the asymptotic expansions of $E_0^{(\ell)}$ can be written in the form of Eq. (2) and amplitude ratios is given by

$$\frac{a_p^{(n)}}{a_p^{(0)}} = 1 + 2p\frac{I_n^{(1-2p)}}{B_{2p}(1/2)}, \quad \text{open-closed.} \tag{13}$$

The asymptotic expansions of $E_1^{(\ell)}$ can also be written in the form of Eq. (2), where amplitude ratios are given now by

$$\frac{a_p^{(n)}}{a_p^{(0)}} = 1 + 2p\frac{I_{n+1}^{(1-2p)} - \left(n - \frac{1}{2}\right)^{2p-1}}{B_{2p}(1/2)} \quad \text{open-closed.} \tag{14}$$

Note, that from Eqs. (13) and (9), in the case of $n = 1$, one can easily obtain Eqs. (7) and (8), respectively.

The conformal weights Δ_0 and Δ_1 depend on the boundary conditions and are given by

$$\Delta_0 = -\tfrac{1}{8}, \quad \Delta_1 = 0, \quad \text{R-NS sectors;} \tag{15}$$

$$\Delta_0 = -\tfrac{3}{32}, \quad \Delta_1 = \tfrac{5}{32}, \quad \mathbb{Z}_4 \text{ sector;} \tag{16}$$

$$\Delta_0 = -\tfrac{1}{8}, \quad \Delta_1 = \tfrac{3}{8}, \quad \text{open-closed;} \tag{17}$$

$$\Delta_0 = 0, \quad \Delta_1 = 1, \quad \text{closed-closed (open-open).} \tag{18}$$

In this paper we will consider two exactly solvable models in $c = -2$ LCFT, namely, the dimer model and the critical dense polymers (CDP) on a rectangular lattice.

II. CRITICAL DENSE POLYMERS (CDP) MODEL

We will consider an exactly solvable model of critical dense polymers on a square lattice [39]. The degrees of freedom are localized on elementary faces, which can be found in one of the following two configurations:

or , (19)

where the arcs represent segments of the polymer. Since the polymer segments pass uniformly through each face, this is a model of dense polymers. A typical configuration of dense polymers is shown in Fig. 1.

A lattice model of CDP has been solved exactly for finite strips [39] and for cylinders with finite circumference [40]. The result of [39] was proved in [41]. The CDP can be related to the spanning tree [42]. In this section, we study finite-size corrections for the CDP. In the next section, we study finite-size corrections for the dimer model [12,43].

Let us first consider the CDP model on a strip, with width N and height $2M$. The partition function of the CDP is defined in terms of a double-row transfer matrix $\mathbf{D}(u)$ and given by [39]

$$Z_{N,M} = \mathrm{Tr}\,\mathbf{D}(u)^M = \sum_n e^{-2ME_n(N;u)}, \tag{20}$$

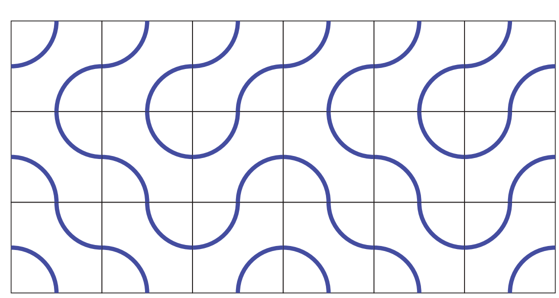

FIG. 1. (Color online) A typical configuration for dense polymer.

where the sum is over all eigenvalues of $\mathbf{D}(u)$, written as $e^{-2E_n(N;u)}$, $E_n(u)$ is the energy associated with the eigenvalue $D_n(u)$, and u is a spectral parameter related to the anisotropy factor by $\zeta = \sin 2u$.

The configuration space on which the transfer matrix acts can be divided into sectors $\mathcal{L}_{N,\ell}$, labeled by an integer $\ell \geqslant 0$, of the same parity as N. This number is related to a number $s = \ell + 1$ of defects. The range of ℓ is finite when N is finite, but will be considered as unbounded in the limit $N \to \infty$.

All eigenvalues have been determined in [39], for any finite value of N. After using the identities,

$$\prod_{n=1}^{\frac{N}{2}-1} \sin \frac{\pi(n+1/2)}{N} = \sqrt{N} 2^{\frac{1-N}{2}}, \quad \text{for even } N, \tag{21}$$

$$\prod_{n=0}^{\frac{N-3}{2}} \sin \frac{\pi(n+1/2)}{N} = 2^{\frac{1-N}{2}}, \quad \text{for odd } N, \tag{22}$$

the expressions for eigenvalues can be simplified and written as

$$E_n = -\frac{1}{2} \sum_{\substack{j=1 \\ j=N \bmod 2}}^{N-2} \log \left[\left(1 + \varepsilon_j \zeta \sin \frac{\pi j}{2N}\right) \left(1 + \mu_j \zeta \sin \frac{\pi j}{2N}\right) \right], \tag{23}$$

where the summation includes the integers j of the same parity as N. The eigenvalues also depend on anisotropy factor $\zeta = \sin 2u$ and on parameters ε_j, μ_j equal to ± 1, although not all choices of $\varepsilon_j, \mu_j = \pm 1$ are allowed. Exactly which sequences of $+1, -1$ correspond to actual eigenvalues, and for which sector, are given by the selection rules conjectured in [39], and proved very recently in [41].

A main result from [39] is that the set of eigenvalues of the transfer matrix in the sector $\mathcal{L}_{N,\ell}$ is such that it leads, through (1), to a set of conformal weights Δ_n whose values and degeneracies exactly match those of a quasirational representation $\mathcal{V}_{1,s}$ of highest weight $h_{1,s} = [(s-2)^2 - 1]/8$, with $s = \ell + 1$. This representation $\mathcal{V}_{1,s}$ is the quotient of the highest weight Verma module $V_{1,s}$ by the singular vector at level s.

The overall ground state E_0, found in $\mathcal{L}_{N,0}$ or $\mathcal{L}_{N,1}$, corresponds to all $\varepsilon_j = \mu_j = 1$ in (23):

$$E_0 = -\sum_{\substack{j=1 \\ j=N\,(2)}}^{N-2} \omega^{(\mathrm{CDP})}\left(\frac{\pi j}{2N}, u\right), \tag{24}$$

where

$$\omega^{(\mathrm{CDP})}(x,u) = \ln(1 + \sin 2u \, \sin x). \tag{25}$$

Using Taylor's theorem, the asymptotic expansions of the $\omega^{(\mathrm{CDP})}(x,u)$ can be written in the following form:

$$\omega^{(\mathrm{CDP})}(x,u) = \sum_{p=1}^{\infty} \frac{\lambda_p^{(\mathrm{CDP})}}{p!} x^p,$$

where $\lambda_1^{(\mathrm{CDP})} = \sin 2u$, $\lambda_2^{(\mathrm{CDP})} = -\sin^2 2u$, $\lambda_3^{(\mathrm{CDP})} = 2\sin^3 2u - \sin 2u, \ldots$.

From the Euler-MacLaurin summation formula, the asymptotic expansion of E_0 takes the form (2) with the coefficients,

$$a_p^{(0)} = \frac{\pi^{2p-1} B_{2p}(\alpha)}{(2p)!} \lambda_{2p-1}^{(\mathrm{CDP})}, \quad \alpha = \frac{N}{2} \bmod 1, \tag{26}$$

where $B_n(z)$ are the Bernoulli polynomials and $\alpha = 0, 1/2$. The bulk free energy is $f_b = f_b^{(\mathrm{CDP})}$ and surface free energy f_s are given by

$$f_b^{(\mathrm{CDP})} = \frac{1}{2}\ln 2 - \frac{1}{\pi}\int_0^{\pi/2} \ln\left(\frac{1}{\sin t} + \sin 2u\right) dt, \tag{27}$$

$$f_s = \frac{1}{2}\ln(1 + \sin 2u). \tag{28}$$

For $p = 1$, in particular, one finds $\Delta_0 - c/24 = 1/12$ for N even and $-1/24$ for N odd. Assuming $c = -2$, this gives $\Delta_0 = h_{1,1} = 0$ for N even, and $\Delta_0 = h_{1,2} = -1/8$ for N odd. Thus we can see that the CDP model on the strip with even N corresponds to the closed-closed boundary condition given by Eq. (18) and the CDP model on the strip with odd N corresponds to the open-closed boundary condition given by Eq. (17).

For $p = 2$ one finds from the previous formula,

$$a_2^{(0)} = \frac{\pi^3}{6}\left(\Delta_0^2 - \frac{1}{120}\right)\lambda_3^{(\mathrm{CDP})}, \tag{29}$$

with $\Delta_0 = 0$ for even N, and $\Delta_0 = -1/8$ for odd N.

The excited levels are obtained by switching the ε_j, μ_j from $+1$ to -1 in a way allowed by the selection rules [39,41]. The specific levels we will consider here are the two lowest-lying levels in each sector $\mathcal{L}_{N,\ell}$, denoted $E_0^{(\ell)}$ and $E_1^{(\ell)}$, for all $\ell \geqslant 1$ except $\ell = 2$ (which is somewhat special and requires a separate treatment; the following checks, however, work for them too). They are nondegenerate (within their sector) and are obtained by setting to -1 the parameters μ_j with the following indices (keeping those of the appropriate parity),

$$1 \leqslant j \leqslant \ell - 2 \quad \text{for} \quad E_0^{(\ell)}, \tag{30}$$

$$1 \leqslant j \neq \ell - 2 \leqslant \ell \quad \text{for} \quad E_1^{(\ell)}. \tag{31}$$

For even N (even $\ell = 2n + 2$), the asymptotic expansions of $E_0^{(\ell)}$ can be written in the form of Eq. (2), where coefficients $a_p^{(n)}$ are given by

$$a_p^{(n)} = \frac{\pi^{2p-1}\lambda_{2p-1}^{(\mathrm{CDP})}}{(2p-1)!}\left(\frac{B_{2p}}{2p} + H_n^{(1-2p)}\right). \tag{32}$$

That yields the first coefficients for $p = 1, 2$,

$$a_1^{(n)} = \pi\left(\frac{1}{12} + \frac{n(n+1)}{2}\right)\lambda_1^{(\mathrm{CDP})} = \pi\left(\frac{1}{12} + h_{1,2n+3}\right)\lambda_1^{(\mathrm{CDP})}, \tag{33}$$

$$a_2^{(n)} = \frac{\pi^3}{6}\left(\frac{n^2(n+1)^2}{4} - \frac{1}{120}\right)\lambda_3^{(\mathrm{CDP})} = \frac{\pi^3}{6}\left(h_{1,2n+3}^2 - \frac{1}{120}\right)\lambda_3^{(\mathrm{CDP})}. \tag{34}$$

From Eqs. (26) and (32) one can see that the ratio $a_p^{(n)}/a_p^{(0)}$ is independent from the function $\omega^{(\mathrm{CDP})}$, which means that this ratio can be universal and given by Eq. (9) for arbitrary n and by Eq. (8) for $n = 2$. By comparing the expressions given by Eqs. (26) and (33) for the case $n = 1$ with Eq. (4), one can find that the conformal weights Δ_0 and Δ_1 correspond to the closed-closed boundary condition given by Eq. (18).

The asymptotic expansions of $E_1^{(\ell)}$ can also be written in the form of Eq. (2), where coefficients $a_p^{(n)}$ are given now by

$$a_p^{(n)} = \frac{\pi^{2p-1}\lambda_{2p-1}^{(\text{CDP})}}{(2p-1)!}\left(\frac{B_{2p}}{2p} + H_{n+1}^{1-2p} - n^{2p-1}\right). \quad (35)$$

For $p = 1,2$

$$a_1^{(n)} = \pi\left(\frac{13}{12} + \frac{n(n+1)}{2}\right)\lambda_1^{(\text{CDP})} = \pi\left(\frac{1}{12} + h_{1,2n+3} + 1\right)\lambda_1^{(\text{CDP})}, \quad (36)$$

$$a_2^{(n)} = \frac{\pi^3}{6}\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{13n^2}{4} + 3n + \frac{119}{120}\right)\lambda_3^{(\text{CDP})} = \frac{\pi^3}{6}\left(h_{1,2n+3}^2 + 6h_{1,2n+3} + \frac{119}{120}\right)\lambda_3^{(\text{CDP})}. \quad (37)$$

For odd N (odd $\ell = 2n+1$), the asymptotic expansions of $E_0^{(\ell)}$ can be written in the form of Eq. (2), where coefficients $a_p^{(n)}$ are given by

$$a_p^{(n)} = \frac{\pi^{2p-1}\lambda_{2p-1}^{(\text{CDP})}}{(2p-1)!}\left(\frac{B_{2p}(1/2)}{2p} + I_n^{(1-2p)}\right). \quad (38)$$

For $p = 1,2$

$$a_1^{(n)} = \pi\left(\frac{11}{12} + \frac{n^2-1}{2}\right)\lambda_1^{(\text{CDP})} = \pi\left(\frac{1}{12} + h_{1,2n+2}\right)\lambda_1^{(\text{CDP})}, \quad (39)$$

$$a_2^{(n)} = \frac{\pi^3}{6}\left(\frac{n^2(2n^2-1)}{8} + \frac{7}{960}\right)\lambda_3^{(\text{CDP})} = \frac{\pi^3}{6}\left(h_{1,2n+2}^2 - \frac{1}{120}\right)\lambda_3^{(\text{CDP})}. \quad (40)$$

The asymptotic expansions of $E_1^{(\ell)}$ can also be written in the form of Eq. (2), where coefficients $a_p^{(n)}$ are given now by

$$a_p^{(n)} = \frac{\pi^{2p-1}\lambda_{2p-1}^{(\text{CDP})}}{(2p-1)!} \times \left(\frac{B_{2p}(1/2)}{2p} + I_{n+1}^{(1-2p)} - \left(n - \frac{1}{2}\right)^{2p-1}\right). \quad (41)$$

For $p = 1,2$

$$a_1^{(n)} = \pi\left(\frac{23}{24} + \frac{n^2}{2}\right)\lambda_1^{(\text{CDP})} = \pi\left(\frac{1}{12} + h_{1,2n+2} + 1\right)\lambda_1^{(\text{CDP})}, \quad (42)$$

$$a_2^{(n)} = \frac{\pi^3}{6}\left(\frac{n^4}{4} + \frac{23n^2}{8} + \frac{247}{960}\right)\lambda_3^{(\text{CDP})} = \frac{\pi^3}{6}\left(h_{1,2n+2}^2 + 6h_{1,2n+2} + \frac{119}{120}\right)\lambda_3^{(\text{CDP})}. \quad (43)$$

Thus working out the asymptotic expansion of $E_r^{(\ell)}$ for $r = 0,1$ yields the first coefficients,

$$a_1^{(n)} = \pi\left(h_{1,s} + r + \tfrac{1}{12}\right)\lambda_1^{(\text{CDP})}, \quad (44)$$

and

$$a_2^{(n)} = \frac{\pi^3}{6}\left(h_{1,s}^2 - \frac{1}{120}\right)\lambda_3^{(\text{CDP})}, \quad r = 0, \quad (45)$$

$$a_2^{(n)} = \frac{\pi^3}{6}\left(h_{1,s}^2 + 6\,h_{1,s} + \frac{119}{120}\right)\lambda_3, \quad r = 1, \quad (46)$$

with $s = 2n+3$ for even N and $s = 2n+2$ for odd N.

Let us now consider the CDP model on the cylinder. The partition function of CDP is defined in terms of a one-row transfer matrix $\mathbf{T}(u)$ and given by [40]

$$Z_{N,M} = \text{Tr}\,\mathbf{T}(u)^M = \sum_{n\geqslant 0} T_n(u)^M = \sum_{n\geqslant 0} e^{-ME_n(N;u)}, \quad (47)$$

where the sum is over all eigenvalues of $\mathbf{T}(u)$, written as $e^{-E_n(N;u)}$, $E_n(u)$ with $n = 0,1,2,\ldots$ is the energy associated with the eigenvalue $T_n(u)$, and u is a spectral parameter related to the anisotropy factor by $\zeta = \sin 2u$. The maximal eigenvalue $T_0(u)$ is labeled by $n = 0$.

Let us first consider the case with odd N. The largest eigenvalue Λ_0 and the second largest eigenvalue Λ_1 of the transfer matrix for the CDP model on the cylinder with odd N are the largest $[T_{0,1}(u)]$ and the second largest eigenvalues of the transfer matrix in the $\mathbb{Z}_4$ sector, respectively, and were found in [40]. After using the identity,

$$\prod_{n=0}^{N-1} \sin\frac{\pi(n+1/2)}{N} = 2^{1-N}, \quad (48)$$

the expression for the largest and second largest eigenvalues of the transfer matrix can be simplified and rewritten in the following form:

$$\ln\Lambda_0 = \ln T_{0,1}(u) = \frac{1}{2}\sum_{n=0}^{N-1}\omega^{(\text{CDP})}\left(\frac{\pi(n+1/2)}{N}, u\right), \quad (49)$$

$$\ln\Lambda_1 = \ln\Lambda_0 + \ln\left(\frac{1-\sin 2u\,\sin\frac{\pi}{2N}}{1+\sin 2u\,\sin\frac{\pi}{2N}}\right). \quad (50)$$

The sums in Eqs. (49) and (50) can be handled by using the Euler-Maclaurin summation formula [44]. After a straightforward calculation, we obtain that E_0 and E_1 can be written in the form of Eq. (2) where coefficients $a_p^{(0)}$ and $a_p^{(1)}$ are given by

$$a_p^{(0)} = \frac{\pi^{2p-1}B_{2p}(1/2)\lambda_{2p-1}}{(2p)!}, \quad a_1^{(0)} = -\frac{\pi\lambda_1}{24}, \quad a_2^{(0)} = \frac{7\pi^3\lambda_3}{5760}\cdots, \quad (51)$$

$$a_p^{(1)} = a_p^{(0)} + \frac{\pi^{2p-1}\lambda_{2p-1}}{2^{2p-2}(2p-1)!}, \quad a_1^{(1)} = \frac{23\,\pi\lambda_1}{24}, \quad a_2^{(1)} = \frac{247\pi^3\lambda_3}{5760}\cdots, \quad (52)$$

with $\lambda_{2p-1} = \lambda_{2p-1}^{(\text{CDP})}$. The bulk free energy is $f_b = f_b^{(\text{CDP})}$ and surface free energy f_s is zero. By comparing the expressions given by Eqs. (51) and (52) with the prediction of the conformal field theory for the cylinder geometry Eq. (3), we show that the central charge $c = -2$, while the conformal weights Δ_0 and Δ_1 correspond to the $\mathbb{Z}_4$ sector given by Eq. (16). The ratio $a_p^{(1)}/a_p^{(0)}$ is independent of the function ω and given by Eq. (6).

If Λ_0 and Λ_1 are the largest and the second-largest eigenvalues of the transfer matrix acting along the M directions, in the limit $M \to \infty$ the ground-state energy E_0 and the first exited state energy E_1, are given, respectively, by

$$E_0 = -\ln \Lambda_0 \quad \text{and} \quad E_1 = -\ln \Lambda_1.$$

Let us first consider the case of infinitely long cylinder with even N.

The largest (Λ_0) eigenvalue of the transfer matrix for the CDP model on infinitely long cylinder with even N is the largest $[T_{0,0}(u)]$ eigenvalue of the transfer matrix in the Ramond sector, and was found in [40]. After using the identity,

$$\prod_{n=0}^{\frac{N}{2}-1} \sin\frac{2\pi(n+1/2)}{N} = 2^{\frac{2-N}{2}}, \quad \text{for } N \text{ - even}, \tag{53}$$

the expression for the largest eigenvalue of the transfer matrix can be simplified and rewritten in the following form:

$$\ln \Lambda_0 = \ln T_{0,0}(u) = \sum_{n=0}^{\frac{N}{2}-1} \omega^{(\mathrm{CDP})}\left(\frac{2\pi(n+1/2)}{N}, u\right). \tag{54}$$

The second largest (Λ_1) eigenvalue of the transfer matrix for the CDP model is the largest $[T_{0,2}(u)]$ eigenvalue of the transfer matrix in the Neveu-Schwarz sector, and was found in [40]. After using the identity,

$$\prod_{n=1}^{\frac{N}{2}-1} \sin\frac{2\pi n}{N} = N 2^{-\frac{N}{2}}, \tag{55}$$

the expression for the largest eigenvalue of the transfer matrix can be simplified and rewritten in the following form:

$$\ln \Lambda_1 = \ln T_{0,2}(u) = \sum_{n=1}^{N/2-1} \omega^{(\mathrm{CDP})}\left(\frac{2\pi n}{N}, u\right). \tag{56}$$

The sums in Eqs. (54) and (56) can be handled by using the Euler-Maclaurin summation formula [44]. After a straightforward calculation, we obtain that E_0 and E_1 can be written in the form of Eq. (2) where coefficients $a_p^{(0)}$ and $a_p^{(1)}$ are given by

$$\begin{aligned} a_p^{(0)} &= \frac{2^{2p}\pi^{2p-1}B_{2p}(1/2)\lambda_{2p-1}}{(2p)!}, \\ a_1^{(0)} &= -\frac{\pi\lambda_1}{6}, \quad a_2^{(0)} = \frac{7\pi^3\lambda_3}{360}\cdots, \end{aligned} \tag{57}$$

$$\begin{aligned} a_p^{(1)} &= \frac{2^{2p}\pi^{2p-1}B_{2p}\lambda_{2p-1}}{(2p)!}, \\ a_1^{(1)} &= \frac{\pi\lambda_1}{3}, \quad a_2^{(1)} = -\frac{\pi^3\lambda_3}{45}\cdots, \end{aligned} \tag{58}$$

with $\lambda_{2p-1} = \lambda_{2p-1}^{(\mathrm{CDP})}$. The bulk free energy is $f_\infty = f_\infty^{(\mathrm{CDP})} = \ln\sqrt{2} - 1/\pi\int_0^{\pi/2}\omega^{(\mathrm{CDP})}(x,u)dx$ and surface free energy (f_s) is zero.

By comparing the expressions given by Eqs. (57) and (58) with Eq. (3), the central charge is readily seen to be $c = -2$, while the values of the conformal weights Δ_0 and Δ_1 correspond to the periodic boundary condition given by Eq. (15). Note that the anisotropy factor is equal to $\zeta = \sin 2u$ for the CDP model. Equations (57) and (58) imply that the ratios of the amplitudes of the N^{2p-1} corrections term in the first exited state energy E_1, and the ground-state energy E_0 expansion, that is, $a_p^{(1)}/a_p^{(0)}$ should not depend in detail on the function $\omega(x,z)$ as given by Eq. (5). In deriving Eq. (5) we have used the relation $B_{2p}(1/2) = (2^{1-2p}-1)B_{2p}$.

III. DIMER MODEL ON THE RECTANGULAR LATTICE

To check whether Eqs. (5)–(8) are still valid for other models in the $c = -2$ universality class, we proceed to study another exactly solvable model, namely dimer model on the rectangular lattice. The dimer problem originated from investigation of the thermodynamic properties of a system of diatomic molecules (called dimers) absorbed on the surface of a crystal. Dimer system is specified by a lattice G consisting of vertices (sites) connected by bonds.

Dimer can be placed on the bonds of G so that no vertex has more than one dimer (see Fig. 2). The "dimer problem" is to determine the number of ways of covering a given lattice with dimers, so that all sites are occupied and no two dimers overlap. The partition function $Z_{M,N}$ of the dimer model on a $M \times N$ lattice is given by

$$Z_{M,N}(z_v, z_h) = \sum z_v^{n_v} z_h^{n_h},$$

where summation is taken over all dimer covering configurations, z_v and z_h are, respectively, dimer weight in the horizontal and vertical directions, n_v and n_h are, respectively, the number of vertical and horizontal dimers. $Z_{M,N}(1,1)$ is the total number of different ways to cover the lattice by dimers. We consider a transfer matrix acting along the M direction.

The dimer model itself has no specific critical behavior, but serves as a way to construct different models with different universality classes (Ising model, Kasteleyn model, spanning trees, etc.). For example, the dimer model on square and

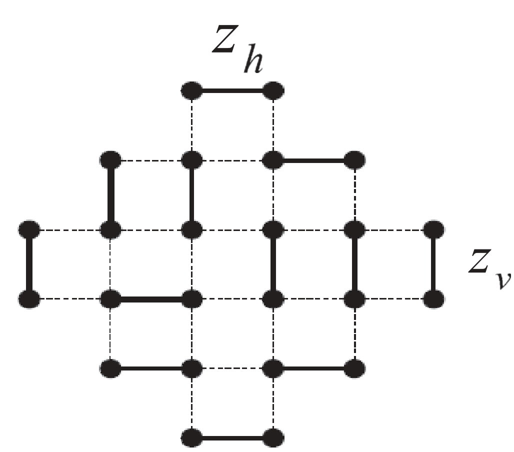

FIG. 2. A lattice G consisting of vertices connected by bonds. Dimers can be placed on the bonds of G so that no vertex has more than one dimer.

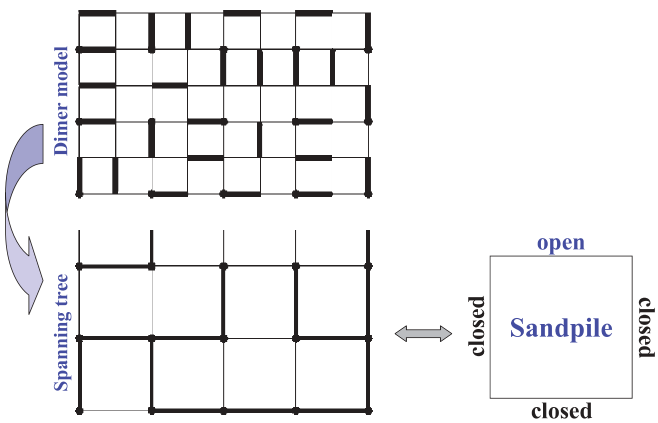

FIG. 3. (Color online) Mapping of dimer covering to a spanning tree on the odd sublattice and to the sandpile model with open top and closed bottom horizontal boundaries and closed vertical boundaries for an $M \times N = 6 \times 9$ lattice.

triangular lattices belongs to spanning tree universality class ($c = -2$) [12,15]; the dimer model on honeycomb lattice belongs to $q = 0$ Potts model universality class ($c = 1$) [45] and Fisher introduced a correspondence between the two-dimensional Ising model defined on a graph G, and the dimer model defined on a decorated version of this graph [46].

The finite-size corrections of the free energy of the dimer model show a strong dependence on the parity of the lattice. In our recent works [12,13], we have studied the finite-size corrections of the dimer model on an infinitely long strip of the square lattice with width N under two different boundary conditions: free and periodic. We have shown that changing the parity of N genuinely changes the boundary conditions, which is unusual in the world of conformal field theory. The correspondence between the change of boundary conditions and the change the parity of N is not apparent in the dimer model itself, but it becomes clear when one maps the dimer model onto the spanning tree model [42] or the sandpile model [25] (see Figs. 3 and 4). Thus we can see from Figs. 3 and 4 that in the limit $M \to \infty$ the dimer model on the infinitely long strip with odd N corresponds to the sandpile model on the infinitely long strip with closed-closed boundary conditions, while the dimer model on the infinitely long strip with even N corresponds to the sandpile model on the infinitely long strip with open-closed boundary conditions.

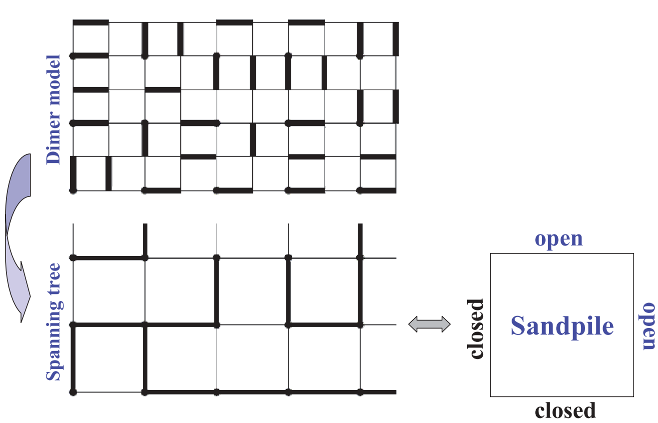

FIG. 4. (Color online) Mapping of dimer covering to a spanning tree on the odd sublattice and to the sandpile model with open top and closed bottom horizontal boundaries and with open right and closed left vertical boundaries for an $M \times N = 6 \times 10$ lattice.

Let us first consider the case of infinitely long cylinder with even N. For the dimer model the expressions for largest (Λ_0) and second largest (Λ_1) eigenvalues of the transfer matrix are given, respectively, by [43]

$$\ln \Lambda_0 = \sum_{n=0}^{N/2-1} \omega^{(\text{dimer})}\left(\frac{\pi(n+1/2)}{N/2}, z\right), \tag{59}$$

$$\ln \Lambda_1 = \sum_{n=1}^{N/2-1} \omega^{(\text{dimer})}\left(\frac{\pi n}{N/2}, z\right), \tag{60}$$

where

$$\omega^{(\text{dimer})}(x,z) = \text{arcsinh}(z \sin x), \tag{61}$$

and $z = z_v/z_h$.

Using Taylor's theorem, the asymptotic expansions of the $\omega^{(\text{dimer})}(x,z)$ can be written in the following form:

$$\omega^{(\text{dimer})}(x,z) = \sum_{p=1}^{\infty} \frac{\lambda_{2p-1}^{(\text{dimer})}}{(2p-1)!} x^{2p-1},$$

where $\lambda_1^{(\text{dimer})} = z$, $\lambda_3^{(\text{dimer})} = -z(1+z^2), \ldots$.

The sums in Eqs. (59) and (60) can be handled by using the Euler-Maclaurin summation formula [44]. After a straightforward calculation, we obtain that E_0 and E_1 can be written in the form of Eq. (2) where coefficients $a_p^{(0)}$ and $a_p^{(1)}$ are given by Eqs. (57) and (58) with $\lambda_{2p-1} = \lambda_{2p-1}^{(\text{dimer})}$. The bulk free energy is $f_\infty = f_\infty^{(\text{dimer})} = -1/2\pi \int_0^\pi \omega^{(\text{dimer})}(x,z)dx$ and surface free energy (f_s) is zero.

By comparing the expressions given by Eqs. (57) and (58) with Eq. (3), the central charge is readily seen to be $c = -2$, while the values of the conformal weights Δ_0 and Δ_1 correspond to periodic boundary condition given by Eq. (15). Note that the anisotropy factor is equal to $\zeta = z$ for dimer model. Equations (57) and (58) imply that the ratios of the amplitudes of the N^{2p-1} corrections term in the first excited state energy E_1, and the ground-state energy E_0 expansion, that is, $a_p^{(1)}/a_p^{(0)}$ should not depend in detail on the function $\omega(x,z)$ as given by Eq. (5). What is surprising is that the ratio $a_p^{(1)}/a_p^{(0)}$ for the dimer and CDP models on the cylinder with even N exactly coincides with the ratio $a_p^{(1)}/a_p^{(0)}$ for the Ising model in the case of periodic boundary conditions [4].

Following along the same line as in [43], we have calculated the largest and the second-largest eigenvalues of the transfer matrix for the dimer model on the infinitely long strip with even and odd N and on the infinitely long cylinder with odd N.

For the dimer model on the cylinder with odd N we find that

$$\ln \Lambda_0 = \sum_{n=1}^{(N-1)/2} \omega^{(\text{dimer})}\left(\frac{\pi n}{N/2}, z\right), \tag{62}$$

$$\ln \Lambda_1 = \sum_{n=1}^{(N-3)/2} \omega^{(\text{dimer})}\left(\frac{\pi n}{N/2}, z\right) = \ln \Lambda_0 - \omega^{(\text{dimer})}\left(\frac{\pi}{N}, z\right). \tag{63}$$

Now, using expressions for Λ_0 and Λ_1 for the dimer model on the cylinder with odd N given by Eqs. (62) and (63), with the help of the Euler-Maclaurin summation formula [44], we can write the asymptotic expansion of E_0 and E_1 in the form of Eq. (2), with coefficients $a_p^{(0)}$ and $a_p^{(1)}$ given by

$$a_p^{(0)} = \frac{\pi^{2p-1} B_{2p} \lambda_{2p-1}}{(2p)!}, \quad a_1^{(0)} = \frac{\lambda_1 \pi}{12}, \quad a_2^{(0)} = -\frac{\pi^3 \lambda_3}{720} \cdots, \tag{64}$$

$$a_p^{(1)} = \frac{\pi^{2p-1} \lambda_{2p-1}}{(2p-1)!}, \quad a_1^{(1)} = \lambda_1 \pi, \quad a_2^{(1)} = \frac{\pi^3 \lambda_3}{6} \cdots, \tag{65}$$

with $\lambda_{2p-1} = \lambda_{2p-1}^{(\text{dimer})}$. The bulk free energy is $f_\infty = f_\infty^{(\text{dimer})}$ and surface free energy f_s is zero. By comparing the expressions given by Eqs. (64) and (65) with Eq. (4), one can find that the central charge is $c = -2$ and the conformal weights Δ_0 and Δ_1 correspond to the open-open boundary condition given by Eq. (18). One can see that the ratio $a_p^{(1)}/a_p^{(0)}$ again is independent from the function ω, which means that this ratio can be universal and given by Eq. (8).

This is a very unusual situation. Although the dimer model is originally defined on a cylinder, it shows the finite-size corrections expected on a strip, and must really be viewed as a model on a strip [12,13].

For the dimer model on the strip with odd N we obtain

$$\ln \Lambda_0 = \sum_{n=1}^{(N-1)/2} \omega^{(\text{dimer})}\left(\frac{\pi n}{N+1}, z\right), \tag{66}$$

$$\ln \Lambda_1 = \sum_{n=2}^{(N-1)/2} \omega^{(\text{dimer})}\left(\frac{\pi n}{N+1}, z\right) = \ln \Lambda_0 - \omega^{(\text{dimer})}\left(\frac{\pi}{N+1}, z\right), \tag{67}$$

where $\omega^{(\text{dimer})}(x,z)$ is given by Eq. (61). Now, using expressions for Λ_0 and Λ_1 for the dimer model on the strip with odd N, with the help of the Euler-Maclaurin summation formula [44], the asymptotic expansion of the E_0 and E_1 can be written in the form of Eq. (2), where coefficients $a_p^{(0)}$ and $a_p^{(1)}$ are exactly the same as in the case of the dimer model on the cylinder with odd N and given by Eqs. (64) and (65), with $\lambda_{2p-1} = \lambda_{2p-1}^{(\text{dimer})}$. The only difference is that surface free energy for the dimer model on the strip with odd N is $f_s = \frac{1}{2}\ln(z + \sqrt{1+z^2})$, while for the case of the dimer model on the cylinder with odd N the surface free energy f_s is zero. The ratio $a_p^{(1)}/a_p^{(0)}$ is independent of the function ω and given by Eq. (7). Thus we can see that the dimer model on the strip with odd N corresponds to the closed-closed boundary condition given by Eq. (18).

Finally, let us consider the dimer model on the strip with even N. For the largest (Λ_0) and second largest (Λ_1) eigenvalues of the transfer matrix for the dimer model on the strip with even N we have obtained the following expressions:

$$\ln \Lambda_0 = \sum_{n=0}^{N/2-1} \omega^{(\text{dimer})}\left(\frac{\pi(n+1/2)}{N+1}, z\right), \tag{68}$$

$$\ln \Lambda_1 = \sum_{n=1}^{N/2-1} \omega^{(\text{dimer})}\left(\frac{\pi(n+1/2)}{N+1}, z\right) = \ln \Lambda_0 - \omega^{(\text{dimer})}\left(\frac{\pi}{2(N+1)}, z\right), \tag{69}$$

where $\omega^{(\text{dimer})}(x,z)$ is given by Eq. (61).

Now, using expressions for Λ_0 and Λ_1 for the dimer model on the strip with even N, the asymptotic expansion of E_0 and E_1 can be written in the form of Eq. (2), where coefficients $a_p^{(0)}$ and $a_p^{(1)}$ are given by Eqs. (51) and (52). The ratio $a_p^{(1)}/a_p^{(0)}$ is independent of the function ω and given by Eq. (7). Thus we can see that the dimer model on the strip with even N corresponds to the open-closed boundary condition given by Eq. (17).

IV. CONFORMAL FIELD THEORY

The finite-size corrections to Eq. (1) can be calculated by using a perturbated conformal field theory [47,48]. In general, any lattice Hamiltonian will contain correction terms to the critical Hamiltonian H_c,

$$H = H_c + \sum_k g_k \int_{-N/2}^{N/2} \phi_k(v) dv, \tag{70}$$

where g_k is a nonuniversal constant and $\phi_k(v)$ is a perturbative conformal field with scaling dimension x_k. The possible irrelevant operators for the Ising model was classified in [49]. Among these operators are those associated with the conformal block of the identity operator, the leading operator of which has the scaling dimension $x_l = 4$. To the first order in the perturbation, the energy gaps $(E_n - E_0)$ and the ground-state energy (E_0) can be written as

$$E_n - E_0 = \frac{2\pi}{N} x_n + 2\pi \sum_k g_k (C_{nkn} - C_{0k0}) \left(\frac{2\pi}{N}\right)^{x_k - 1},$$

$$E_0 = E_{0,c} + 2\pi \sum_k g_k C_{0k0} \left(\frac{2\pi}{N}\right)^{x_k - 1},$$

where C_{nkn} are universal structure constants. Note, that the ground-state energy E_0 and the first energy gap $(E_1 - E_0)$ are, respectively, the quantum analogies of the free energy f_N and inverse spin-spin correlation length ξ_N^{-1}; that is, $N f_N \Leftrightarrow E_0$, $\xi_N^{-1} \Leftrightarrow E_1 - E_0$.

Perturbation techniques have been studied for a long time, and successfully applied in concrete models [50]. Here it is sufficient that the state $|n\rangle$ be nondegenerate within its own representation since ϕ preserves each representation. Quite recently, Izmailian, Ruelle, and Hu [17] generalized the perturbation expansion to include Jordan cells, and examine whether the finite-size corrections are sensitive to the properties of indecomposable representations appearing in the conformal spectrum, in particular, their indecomposability parameters and find that the corrections do not depend on these parameters.

We will show below that for $c=-2$ LCFT the leading finite-size corrections $(1/N^3)$ can be described by the Hamiltonian given by Eq. (70) with a single perturbative conformal field $\phi_1(v)=L_{-2}^2(v)$ with scaling dimension $x_1=4$, which belongs to the tower of the identity. In the case of the cylinder geometry the spectra of the Hamiltonian (70) are built by the irreducible representation $\Delta,\bar{\Delta}$ of two commuting Virasoro algebras L_n and $\bar{L}_n$ [37]. Having in mind that we are interested in corrections coming from the conformal fields that belong to the conformal block of the identity operator, we are left with two possibilities [51]:

$$\phi_1(v)=L_{-2}^2(v)+\bar{L}_{-2}^2(v) \quad \text{and} \quad \phi_2(v)=L_{-2}(v)\bar{L}_{-2}(v).$$

The universal structure constants C_{nkn} (for $k=1,2$) can be obtained from the matrix elements $\langle n|\phi_k(0)|n\rangle=(2\pi/N)^{x_k}C_{nkn}$ [47], which have already been computed by Reinicke [51]:

$$C_{n1n}=(\Delta+r)\left(\Delta-\frac{2+c}{12}+\frac{r(2\Delta+r)(5\Delta+1)}{(\Delta+1)(2\Delta+1)}\right)+\frac{r}{30}[r^2(5c-8)-(5c+28)]\delta_{\Delta,0}+\left(\frac{c}{24}\right)^2+\frac{11c}{1440}+(\Delta\to\bar{\Delta},r\to\bar{r}), \tag{71}$$

$$C_{n2n}=(\Delta+r-c/24)(\bar{\Delta}+\bar{r}-c/24). \tag{72}$$

Let us consider the case $c=-2$. For the critical dense polymer and dimer models on the infinitely long cylinder with even N (periodic boundary condition) the ground state $|0\rangle$ and first excited state $|1\rangle$ are given by

$$|0\rangle=\left|\Delta_0=-\tfrac{1}{8},r=0;\bar{\Delta}_0=-\tfrac{1}{8},\bar{r}=0\right\rangle$$
$$|1\rangle=\left|\Delta_1=0,r=0;\bar{\Delta}_1=0,\bar{r}=0\right\rangle. \tag{73}$$

Thus for the universal structure constants C_{n1n} and C_{n2n} $(n=0,1)$ we can obtain from Eqs. (71) and (72) the following values:

$$C_{010}=7/480,\quad C_{111}=-1/60, \tag{74}$$

$$C_{020}=1/576,\quad C_{121}=1/144. \tag{75}$$

Then, the expansion of the ground-state energy (E_0) and the energy gaps (E_1-E_0) up to N^{-3} order can be written as

$$E_0-E_\infty=-\frac{\pi\zeta}{6N}+2\pi\left(\frac{7g_1}{480}+1/576g_2\right)\left(\frac{2\pi}{N}\right)^3, \tag{76}$$

$$E_1-E_0=\frac{\pi\zeta}{2N}-2\pi\left(\frac{g_1}{32}-1/192g_2\right)\left(\frac{2\pi}{N}\right)^3, \tag{77}$$

where E_∞ is the ground-state energy of the infinite lattice. A comparison of the expressions given by Eqs. (76) and (77) with the finite-size corrections for the critical dense polymer model and critical dimer model given by Eqs. (57) and (58), shows complete agreement for

$$g_2=0 \quad \text{and} \quad g_1=\lambda_3/(12\pi),$$

where $\lambda_3=-\sin 2u+2\sin^3 2u$ for critical dense polymer and $\lambda_3=-z(1+z^2)$ for dimer model. It is interesting to note that for the 2D Ising model, one finds [50] that the leading finite-size corrections $(1/N^3)$ can also be described by the Hamiltonian given by Eq. (70) with a single perturbative conformal field $\phi_1(v)=L_{-2}^2(v)+\bar{L}_{-2}^2(v)$.

For the critical dense polymer on infinitely long cylinder with odd N ($\mathbb{Z}_4$ sector) the ground state $|0\rangle$ and first excited state $|1\rangle$ are given by

$$|0\rangle=\left|\Delta_0=-\tfrac{3}{32},r=0;\bar{\Delta}_0=-\tfrac{3}{32},\bar{r}=0\right\rangle$$
$$|1\rangle=\left|\Delta_1=\tfrac{5}{32},r=0;\bar{\Delta}_1=\tfrac{5}{32},\bar{r}=0\right\rangle. \tag{78}$$

Thus for the universal structure constants C_{n1n} $(n=0,1)$ we can obtained from Eq. (71) the following values:

$$C_{010}=7/7680,\quad C_{111}=247/7680. \tag{79}$$

In the case of strip geometry the spectra can be understood in terms of irreducible representations Δ of a single Virasoro algebra. Having in mind that we are interested in corrections coming from the conformal fields that belong to the conformal block of the identity operator, we are left with one possibility [51]:

$$\phi_1(v)=L_{-2}^2(v),$$

and the universal structure constants C_{n1n} are now given by Eq. (71), where the dependence on $\bar{\Delta}$ and $\bar{r}$ are suppressed.

For the dimer model on the cylinder and strip with N-odd and the CDP on the strip with N-even [closed-closed (open-open) boundary condition] the ground state and the first excited state are given by

$$|0\rangle=|\Delta_0=0,r=0\rangle \quad \text{and} \quad |1\rangle=|\Delta_1=1,r=0\rangle. \tag{80}$$

Then, from Eq. (71) we can obtain the values of the universal structure constants:

$$C_{010}=-1/120 \quad \text{and} \quad C_{111}=119/120.$$

For the dimer model on the the strip with N-even and the CDP on the strip and cylinder with odd N (open-closed boundary condition) the ground state and first excited state are

$$|0\rangle=\left|\Delta_0=-\tfrac{1}{8},r=0\right\rangle \quad \text{and} \quad |1\rangle=\left|\Delta_1=\tfrac{3}{8},r=0\right\rangle \tag{81}$$

and from Eq. (71) we obtain

$$C_{010}=7/960 \quad \text{and} \quad C_{111}=127/960.$$

Thus the ratio of first-order corrections amplitudes for ξ_N and f_N is universal and equal to $(C_{111}-C_{010})/C_{010}$, which is consistent with Eqs. (5), (7), and (8) for the case $p=2$. Namely,

$$\frac{b_2}{a_2}=\frac{C_{111}-C_{010}}{C_{010}}=-15/7,\quad \text{R-NS sectors;}$$
$$\frac{b_2}{a_2}=\frac{C_{111}-C_{010}}{C_{010}}=240/7,\quad \mathbb{Z}_4\ \text{sector;}$$
$$\frac{b_2}{a_2}=\frac{C_{111}-C_{010}}{C_{010}}=-120,\quad \text{open-closed;}$$
$$\frac{b_2}{a_2}=\frac{C_{111}-C_{010}}{C_{010}}=120/7,\quad \text{closed-closed (open-open).}$$

V. CONCLUSION

We find that the finite-size corrections in the CDP and dimer models depend in a crucial way on the parity of N, and show that a change of the parity of N induces a change of boundary conditions: The dimer and the CDP model on the cylinder with even N belong to the periodic boundary condition (Ramond and Neveu-Schwarz sectors), the CDP model on the cylinder with odd N belongs to the periodic boundary condition ($\mathbb{Z}_4$ sector), the dimer model on the cylinder and strip with odd N and CDP model on the strip with even N belong to the closed-closed boundary condition, and the dimer model on the strip with even N and the CDP model on the strip with odd N belong to the open-closed boundary condition.

Taking a careful account of this, these unusual finite-size behaviors can be fully explained in the framework of the $c = -2$ logarithmic conformal field theory. We find that the ratios among correction amplitudes are universal for $p = 1$ and 2, and depend only on the boundary conditions [see Eqs. (5)–(8)].

The results of this paper inspire several problems for further studies: (i) Further work has to be done to possibly evaluate exactly all finite-size correction terms from perturbative conformal field theory. (ii) Can one obtain from the perturbated conformal field theory the value of the universal amplitude ratios b_p/a_p for $p > 2$? (iii) How do such amplitudes behave in other models, for example, in the three-state Potts model? (iv) Our results also present new challenges to scientists working on numerical studies of critical phenomena. For example, it is of interest to present accurate numerical evidences about whether the nonintegrable models in $c = -2$ universality classes has the same set of amplitude ratios.

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