Critical dynamics in glassy systems

Giorgio Parisi[1,2,3] and Tommaso Rizzo[1,2]
[1] *Dipartimento Fisica, Università "Sapienza," Piazzale A. Moro 2, I-00185 Rome, Italy*
[2] *IPCF-CNR, UOS Rome, Università "Sapienza," Piazzale A. Moro 2, I-00185 Rome, Italy*
[3] *INFN, Piazzale A. Moro 2, 00185 Rome, Italy*

(Received 28 May 2012; revised manuscript received 19 October 2012; published 3 January 2013)

Critical dynamics in various glass models, including those described by mode-coupling theory, is described by scale-invariant dynamical equations with a single nonuniversal quantity, i.e., the so-called parameter exponent that determines all the dynamical critical exponents. We show that these equations follow from the structure of the static replicated Gibbs free energy near the critical point. In particular, the exponent parameter is given by the ratio between two cubic proper vertexes that can be expressed as six-point cumulants measured in a purely static framework.

DOI: 10.1103/PhysRevE.87.012101 PACS number(s): 64.60.Ht, 64.70.Q−, 75.10.Nr

I. INTRODUCTION

In this paper, we will show that there is a deep connection between statics and dynamics in various glass models, including those described by mode-coupling theory. More precisely, we will derive under very general assumptions two equations [namely Eqs. (3) and (4) below] that connect the so-called dynamical parameter exponent λ with quantities that can be either computed or measured in a purely *static* framework. The first equation shows that the parameter λ is equal to the ratio w_2/w_1 of two cubic coefficients in the replicated Gibbs free energy and will be proven solving in parallel the statics (using replicas) and the dynamics (using the superfield formulation of Langevin dynamics). The ratio w_2/w_1 then will be further identified with the ratio ω_2/ω_1 of two cubic cumulants of the two-points order parameter that can be measured statically. The two equations can be used to determine the parameter λ without solving explicitly the dynamics of the system. In a set of recent publication this method has been applied (without justification) to a number of mean-field spin-glass models yielding new analytical predictions [1–3]. In some cases the dynamical exponent was already known from the explicit solution of the dynamics and the novel computations in Refs. [1–3] offered an *a posteriori* validation of Eq. (3), while in this paper we will present an *a priori* model-independent justification. In some other cases, the method instead yielded novel prediction for the parameter exponents that was compared with existing numerical work. Another recent application of Eq. (3) in the context of supercooled liquids has been presented in Ref. [4] while in Ref. [5] the same equation has been applied to Szamel's replicated Ornstein-Zernicke equations [6] showing that the replica method yields a characterization of critical dynamics in the β regime of supercooled liquids that is equivalent to the one of mode-coupling theory, both qualitatively and quantitatively.

Spin-glass (SG) dynamics in both the full replica-symmetry-breaking (f-RSB) and one-step RSB (1RSB) models exhibits critical slowing down [7–10]. In particular, it is well known that the dynamics of 1RSB models at the dynamical transition temperature T_d obeys the same dynamical equation of the schematic mode-coupling theory (MCT) developed in the context of supercooled liquids [11–15]. In the spin-glass context one considers the relaxation of $C(t)$, i.e., the overlap between a given initial equilibrium configuration of the dynamics and the configuration at time t. In the discontinuous 1RSB case at T_d the relaxation is a two-step process in which the system spends an increasing amount of time around a plateau value. Models with a discontinuous 1RSB transition include the p-SG either spherical or Ising, the Potts SG model, and the random orthogonal model (ROM). MCT predicts that there are two exponents controlling the dynamics around the plateau; in the early β regime, $C(t)$ approaches the plateau value with a power law $C(t) \approx q_{\rm EA} + c_a/t^a$, while in the late β (or early α) regime $C(t) = q_{\rm EA} - c_b t^b$ [14]. A well-known prediction of MCT is the following relationship between the decay exponents:

$$\frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \lambda, \tag{1}$$

where λ is the so-called exponent parameter which is the object of this work. In case of a continuous transition, there is no dynamic arrest and no b exponent is defined. Well-known instances are, e.g., the paramagnet to full-RSB SG transition along the de Almeida Thouless (dAT) line in mean-field (MF) SG models, either fully connected [7,8] or on random graphs [16], as well as the SG transition in Potts models with $p \leqslant 4$ (both fully connected [17] and on the Bethe lattice of any connectivity [18]) and in the p-spin spherical model with large external magnetic field [10].

We will show that the parameter exponent can be associated to physical observables computed in a static framework. We recall that, in general, the analytic treatment of the dynamics is more complicated than the statics and only a few models have been studied so far: the soft-spin Sherrington-Kirkpatrick (SK) model [7,8], schematic MCT's [12], soft-spin p spin, and Potts glass models, for which, notably, the connection with MCT was first identified [9]. This prompted to consider the spherical p-spin SG [10,19] in all details as a MF structural glass [20], even in the off-equilibrium regime below T_d [21]. In these cases, dynamics is explicitly solved and λ computed exactly. In particular, one finds that *it is not universal* and depends on the model and on the external parameters. On the other hand, its computation becomes difficult when we consider more complicated MF systems and notably finite-dimensional ones.

It is well known [22–24] that, similarly to the static transition, the dynamic one also can be located as the critical point of an appropriate potential that can be computed in a purely static framework. The properties of the free energy (for the continuous transition) of the potential (for the discontinuous transition) can be described in the replica framework by a replicated Gibbs free energy G,

$$G(\delta Q) = \frac{1}{2}\sum_{(ab),(cd)} \delta Q_{ab} M_{ab,cd} \delta Q_{cd} - \frac{w_1}{6}\mathrm{Tr}\,\delta Q^3 - \frac{w_2}{6}\sum_{ab}\delta Q_{ab}^3, \qquad (2)$$

with $a,b,c,d = 1,\ldots,n$. The order parameter δQ_{ab} is the deviation of the two-point function relevant for the given problem, e.g., the spin overlap in spin-glass models or the density-density fluctuations in supercooled liquids. The case $n = 0$ is relevant for the continuous transition [25–27], and the case $n = 1$ for the dynamic discontinuous transition [22–24,28]. The first result we will derive in this paper is that, both in the continuous and discontinuous case, the exponent parameter λ is given by the ratio between the coefficients w_1 and w_2 of the static replicated Gibbs free energy,

$$\lambda = \frac{w_2}{w_1}. \qquad (3)$$

We note that this ratio also yields the breaking point x in the case of continuous transition from RS to full-RSB [17].

As already stated, this result has been recently confirmed *a posteriori* considering fully connected models where the dynamics and the statics both can be obtained, in particular in the case of the de Almeida-Thouless line of the Sherrington-Kirkpatrick model and in the case of the spherical p-spin model [2,29]. Most importantly, it can be used to obtain the dynamical exponents without solving explicitly the dynamics. In particular, it has been applied to various models that were studied numerically in the past, notably the Potts SG [30] and the random-orthogonal-model SG [31]. Exploiting the fully connected nature of these models, the replicated Gibbs free energy was computed analytically by means of the saddle-point method and the previous numerical estimates were shown to be in fair agreement with the new analytical predictions [1,3].

In the following, we will argue that the above result holds provided the Gibbs free energy admits the expansion (2) near the critical point, including the case of finite-dimensional models above their upper critical dimension. In general, however, it is not possible to obtain an exact analytical expression of the Gibbs free energy. This problem is present not only in the case of finite-dimensional models but also for mean-field models defined on finite connectivity random lattices. Further progress can be made recalling that, in general, the Gibbs free energy is obtained as the Legendre transform of the free energy and, therefore, its (proper) vertexes can be associated to cumulants of the order parameter. This simple observation leads to the second and more fundamental result that we are going to derive in this paper: the parameter exponent is given by the ratio

$$\lambda = \frac{\omega_2}{\omega_1}, \qquad (4)$$

where ω_2 and ω_1 are six-point static functions of the microscopic variables s_i (that is, spins in spin-glass models or density fluctuations in structural glass models) defined by

$$\omega_1 \equiv \frac{1}{N}\sum_{ijk} \overline{\langle s_i s_j\rangle_c \langle s_j s_k\rangle_c \langle s_k s_i\rangle_c}, \qquad (5)$$

$$\omega_2 \equiv \frac{1}{2N}\sum_{ijk} \overline{\langle s_i s_j s_k\rangle_c^2}, \qquad (6)$$

where the overline denotes average over the disorder and the square bracket denotes thermal average. In the case of the discontinuous transition the above static averages are only defined in the glassy phase because of metastability. Most importantly, the glassy phase is characterized by the fact that there is an exponential number of metastable states and the above expression has to be interpreted in a slightly different way: Thermal averages have to be performed inside the same metastable state, while the overline denotes both the summation over all possible states and the standard disorder average. The above definitions therefore can be naturally extended to models with no quenched disorder (notably structural glasses); in this case, the overline denotes just a summation over the different metastable states. It is well known that the MCT singularity is characterized by the divergence of χ_4, a four-point susceptibility; our result shows that the parameter exponent is related to the ratio of two χ_6 susceptibilities.

We devised a method to compute analytically the above six-point functions for models defined on finite connectivity random lattices obtaining a prediction for the ratio w_1/w_2 that compares very well with the numerical simulations of the Bethe lattice SG [29,32]. In addition, the above cumulants can be computed numerically. As we will discuss in the paper, ω_1 and ω_2 both diverge at criticality and this may cause huge finite-size effect.

The paper is organized as follows. In Sec. II we discuss a simple fully connected spin-glass model where one can obtain closed equations for both the statics and the dynamics. In spite of its simplicity, the study of this model will not only confirm that (3) holds but will also help us understand why it is so. The key ingredient is that we have to study critical dynamics at large times in an expansion around the so-called fast-motion (FM) limit corresponding to the statics. In Sec. III we will argue that the result (3) holds also in more general cases; in particular, we will consider three different types of SG transitions (specified by the form of the Gibbs free energy) and the corresponding critical dynamical behavior. In Sec. IV we will discuss the second result (4), paying particular attention to the problems connected with the divergence of the physical observables ω_1 and ω_2. Technical details will be largely postponed to the appendices. In Sec. V we will give our conclusions. In particular, we will give discuss extensively the connection between our results and previous results obtained within MCT.

II. FULLY CONNECTED MODELS

A. The spherical model

In the case of the fully connected spherical p-spin models, one can obtain closed equations for the dynamics. The model is given by a set of N continuous spins with the spherical constraint $\sum_i s_i^2 = N$ and interacting through a Hamiltonian of the form

$$H = -\sum_{i_1<\cdots<i_p} J_{i_1\cdots i_p} s_{i_1}\cdots s_{i_p}, \tag{7}$$

where J are quenched random couplings with zero mean and variance $\overline{J^2} = p!/(2N^{p-1})$. The order parameter is $C(t)$, defined as the overlap between a given equilibrium configuration at time $t = 0$ and the configuration reached by the dynamics at time t averaged over the thermal noise and the disorder. In the high-temperature paramagnetic phase $C(t)$ decays to zero at large times and obeys the following equation [10,20]:

$$\dot{C}(t) = -TC(t) - \frac{p}{2T}\int_0^t C^{p-1}(t-u)\dot{C}(u)du, \tag{8}$$

with $C(0) = 1$. We will consider the case $p = 2$ corresponding to a continuous transition. In this case, it is convenient to rewrite the above equation in the form

$$T\dot{C}(t) = 2\tau C(t) - C^2(t) - \int_0^t (C(t-u) - C(t))\dot{C}(u)du;$$
$$\tau \equiv \frac{1-T^2}{2}. \tag{9}$$

Note that $\tau < 0$ in the high-temperature region where the above equation is valid. If the last term in the right-hand side were absent, we could easily solve the dynamical equation. We would have an exponential decay at any finite τ and a power-law decay $C(t) \propto 1/t$ at $\tau = 0$. The presence of the last term instead is a peculiar feature of SG dynamics and changes completely the dynamical behavior. One can check that it produces a term proportional to $\dot{C}(t)$ that dominates over the one on the left-hand side. Therefore, in order to study the large-time critical behavior at small τ, we can set it to zero. We can rescale the correlation as $C = -\tau f$ and obtain an equation that is a special case ($\lambda = 0$) of the general equation [10,12,33],

$$0 = 2f(x) + f^2(x)(1-\lambda) + \int_0^x [f(x-y) - f(x)]\dot{f}(y)dy. \tag{10}$$

The solution of the resulting equation is invariant under a rescaling of time $x \to bx$; it diverges as $1/x^a$ for small x and decays exponentially at large x. The exponent a can be determined by plugging the asynptotic form $1/x^a$ into the equation. This yields

$$\lambda = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)}. \tag{11}$$

Therefore, in the present case, $\lambda = 0$, we have $a = 1/2$.

In order to determine the actual time scale, we should match the large-time solution with the small-time solution. In general, we are only interested in the divergence of the time scale as τ goes to zero and this can be obtained, considering that the solution must not depend on τ at any finite time; therefore, we obtain

$$C(t) = \tau\, f(t/t^*) \quad t \ll 1, \quad t^* \propto \frac{1}{\tau^{\frac{1}{a}}}. \tag{12}$$

Note that in the present case $a = 1/2$ and $t^* \propto \tau^2$, which is a completely different result from the case in which the last term in the right-hand side is absent and where we would have $t^* \propto \tau$.

B. Soft-spin Sherrington-Kirkpatrick model

In this section, we will consider a more general fully connected spin-glass model. We will show that while it is not possible to obtain the dynamical equations in a closed simple form as in the spherical case, the dynamics *on large time scales* reduces to the above scale -invariant equation and shares the same critical properties. In spite of its simplicity, the derivation in this model contains all the ingredients that can be easily generalized to the more complex transitions considered in the following sections.

Our result, Eq. (3), connects dynamics on a large time scale with statics; therefore, in order to derive it, it is convenient to work in a framework where the similarity between statics and dynamics is apparent. The dynamical superfield approach has this property because it is based on a dynamical action similar to that of the static. A general presentation of the approach can be found in Ref. [34]. Its application to the SG problem was done by Kurchan in Ref. [35] and his results will be our starting point in the derivation.

The advantage of this formalism is that it makes apparent the equivalence between dynamics and the statics replica method. We consider a soft-spin version of the SK model with energy,

$$H = -\sum_{1\leqslant i\leqslant j\leqslant N} J_{ij} s_i s_j + \sum_{i=1}^{N} H_0(s_i), \tag{13}$$

where J_{ij} are quenched random couplings with zero mean and variance $1/N$ while $H_0(s_i)$ is a spin-length probability that dominates over the quadratic interaction for large values of s_i.

1. Statics by the replica method

The static of the problem can be solved by means of replicas and yields the equation,

$$Q_{ab} = \langle s_a s_b \rangle, \tag{14}$$

where the angled brackets denote that we sum over the spins with the following weight:

$$\exp\left[\frac{\beta^2}{2} Q_{ab} s_a s_b - \beta \sum_a H_0(s_a)\right]. \tag{15}$$

In the paramagnetic phase the solution is simply

$$Q_{ab} = q_d \delta_{ab}, \tag{16}$$

where q_d (d stands for diagonal) obeys the equation

$$q_d = \frac{\int s^2 e^{\frac{\beta^2}{2} q_d s^2 - \beta H_0(s)}\, ds}{\int e^{\frac{\beta^2}{2} q_d s^2 - \beta H_0(s)}\, ds}. \tag{17}$$

In order to study the continuous spin-glass transition Eq. (14) is expanded in powers of

$$\delta Q_{ab} \equiv Q_{ab} - q_d \delta_{ab}. \tag{18}$$

The coefficients of the expansion are written in terms of spin averages computed in the solution with $\delta Q_{ab} = 0$. At the first order, for instance, we need four spin averages that read

$$\langle s_a s_b s_c s_d \rangle = (q_4 - 3q_d^2)\delta_{abcd} + q_d^2(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{cb}), \tag{19}$$

where

$$q_4 = \frac{\int s^4 e^{\frac{\beta^2}{4} q_d s^2 - \beta H_0(s)}\, ds}{\int e^{\frac{\beta^2}{4} q_d s^2 - \beta H_0(s)}\, ds}. \tag{20}$$

To go to the second order in δQ_{ab} we need correlation functions among six replicas, and the final result is

$$0 = 2\tau \delta Q_{ab} + w_1 (\delta Q)^2_{ab}, \tag{21}$$

where

$$\tau \equiv \frac{\beta^2 q_d - 1}{2}, \quad w_1 \equiv q_d^3 \beta^4, \tag{22}$$

which shows that at $\tau = 0$ there is a continuous SG transition from a solution $\delta Q_{ab} = 0$ to a solution $\delta Q_{ab} = \tau / w_1$.

2. *Large-time dynamics as an expansion around the fast-motion solution*

Within the superfield formulation [34] the dynamical equations of the problem are very similar to those of the static replica treatment. They were derived by Kurchan in Ref. [35] and we will brefly quote his results, which are our starting point. In the superfield approach Langevin dynamics for the model is written as an action in terms of bosonic $(s_i(t),\hat{s}_i(t))$ and fermionic variables $(c_i(t),\overline{c}_i(t))$ at different times and sites. These variables are condensed in a single superfield $\phi(1)$ by means of the introduction of two auxiliary fermionic coordinates $(\theta,\overline{\theta})$,

$$\phi(1) \equiv s(t_1) + \overline{c}(t_1)\theta_1 + c(t_1)\overline{\theta}_1 + \theta_1\overline{\theta}_1 \hat{s}(t_1), \quad 1 \equiv (t_1,\theta_1,\overline{\theta}_1). \tag{23}$$

In terms of these coordinates, we obtain the equation [35]

$$Q(1,2) = \langle \phi(1)\phi(2) \rangle, \tag{24}$$

where the angled brackets denote the average computed with respect to the following action:

$$S = S_{\rm KIN} + S_{\rm POT}, \tag{25}$$

where $S_{\rm KIN}$ is a the dynamical part of the action

$$S_{\rm KIN} \equiv \frac{1}{\Gamma_0} \int d\theta\, d\overline{\theta}\, dt\, \partial_\theta \phi (\partial_{\overline{\theta}} \phi - \theta \partial_t \phi), \tag{26}$$

where Γ_0 is the inverse of the time scale of Langevin dynamics and changing Γ_0 amounts to a simple rescaling of time. The potential part of the action reads

$$S_{\rm POT} = \frac{\beta^2}{2} \int d1 d2 Q(1,2)\phi(1)\phi(2) + \beta \int d1 H_0(\phi(1)). \tag{27}$$

The similarity with the replica action (15) is evident. Transforming the term $\int d1 d2 Q(1,2)\phi(1)\phi(2)$ to the original variables one immediately recovers the well-known fact that the dynamic is equivalent to that of single spin with a correlated noise and a memory term [7,8,36].

First, we want to show that the dynamical treatment is completely equivalent to the static one. As noted already by Kurchan this can be seen by taking the so-called fast motion (FM) limit that amounts to consider a very fast microscopic dynamics ($\Gamma_0 \to \infty$) or, equivalently, large times. In this limit, the system evolves so fast that the distributions at different times are uncorrelated. Correspondingly, one can write

$$Q(1,2) = C(0)\delta(1,2), \tag{28}$$

where $\delta(1,2)$ is a δ function in superspace. Note the similarity with the replica ansatz (16). Plugging the above ansatz into the dynamical action (27), we obtain that the dynamics is equivalent to the FM dynamic in a static potential,

$$V(s) = C(0)s^2 + H_0(s), \tag{29}$$

and the dynamical equation reduces to an equation for $C(0)$ identical to that obeyed by q_d [Eq. (17)], consistently with the fact that $C(0)$ is equal to the thermal average of s^2 at equal times, i.e.,

$$C(0) = q_d. \tag{30}$$

The above simple result illustrates how, using the FM limit, one can recover in a dynamical formulation the results of the static obtained by means of replicas. It is easy to see that the equivalence must hold for any observable, in particular *any dynamical correlation function in the FM limit must have the same form of the corresponding correlation function in replica space*, provided one replaces δ functions in replica space with δ functions in superspace. We have already written the expression for the four-point function in the replica method, see Eq. (19). The corresponding dynamical expression in the FM limit can be obtained from it,

$$\begin{aligned}\langle \phi(1)\phi(2)\phi(3)\phi(4) \rangle_{\rm FM} = {}& (q_4 - 3q_d^2)\delta(1,2,3,4) \\ &+ q_d^2[\delta(1,2)\delta(3,4) + \delta(1,3)\delta(2,4) \\ &+ \delta(1,3)\delta(2,4)].\end{aligned} \tag{31}$$

The FM limit yields the static or infinite time limit; therefore, in order to study the dynamics on large but finite times scales we can perform an expansion around the FM limit. Technically, this can be viewed as an expansion in powers of $1/\Gamma_0$ around zero. The order parameter is expanded around the FM solution as

$$\delta Q(1,2) \equiv Q(1,2) - C(0)\delta(1,2). \tag{32}$$

Note that this is similar to the replica expansion (18) in terms of the off-diagonal order parameter. In the FM limit we have $\delta Q(1,2) = 0$, while we expect that, at finite times for a nonzero but small $1/\Gamma_0$, $\delta Q(1,2)$ will be also small. Note, however, that as soon as we consider a finite albeit small value of $1/\Gamma_0$, there is always a region of small times where $\delta Q(1,2)$ is not small and in this region it is not appropriate to employ a perturbative expansion. On the other hand, it can be checked self-consistently at the end that these large small-time-difference correlations do not contribute to the equation for the small large-times-difference correlations. The behavior of the correlations at large times will turn out to have a great deal of universality while the small-time behavior is strongly model dependent. However, because of this scale separation of the dynamics, the solution at large times will be obtained up to an irrelevant nonuniversal factor that should be determined through a matching with the short-time solution.

The expansion of the dynamical equation (24) around the FM solution is formally identical to the expansion of the static equation (14) around the paramagnetic solution. This is a consequence of the similarity between (27) and (15). The expansion is written in terms of averages with respect to a dynamics in the potential (29) with the given value of Γ_0. At leading order we have

$$\delta Q(1,2) = \int d3\, d4\, \langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle_{\Gamma_0}\delta Q(3,4) + O(\delta Q^2). \tag{33}$$

We have already seen that the in the FM limit, $\Gamma_0 \to \infty$, the dynamical correlations are formally the same of the replica treatment and the expansion of these correlations in terms of $1/\Gamma_0$ generates terms proportional to time derivatives of $\delta Q(1,2)$. On the other hand, we have seen in the previous section that the terms involving explicitly time derivatives play a subleading role in critical SG dynamics. Therefore, at leading order, we can replace the dynamical correlation at finite $1/\Gamma_0$ with their FM limit, e.g., Eqs. (31). Given that FM correlations are equivalent to correlations in replica space, it follows that the expansion of the dynamical equation is equivalent to the corresponding expansion (21) in replica space, i.e., it reads

$$0 = 2\tau\delta Q(1,2) + w_1 \int d3 \delta Q(1,3)\delta Q(3,2), \tag{34}$$

with the *same* values of τ and ω_1. The final step of our derivation is to show that the above dynamical equation is of the same form (10) obtained starting from the spherical model. In order to do that we recall that the order parameter $Q(1,2)$ is a two-point superfield correlation whose most general causal form is (see, e.g., Ref. [35])

$$Q(1,2) = \left\{1 + \frac{1}{2}(\overline{\theta}_1 - \overline{\theta}_2)\left[\theta_1 + \theta_2 - (\theta_1 - \theta_2)\epsilon(t_1 - t_2)\frac{\partial}{\partial t_1}\right]\right\} \times C(t_1 - t_2), \tag{35}$$

where $\epsilon(t)$ is the sign function. The above expression encodes causality, the fluctuation-dissipation-theorem (FDT), and time-translational invariance. From the above formula we see that time-traslational invariance and FDT imply that all the components of $Q(1,2)$ can be recovered from the boson-boson component $C(t)$; in other words, the whole function $Q(1,2)$ is actually described by a single real even function $C(t)$ that is the correlation $\langle s(t_1)s(t_2)\rangle$. Note also that from Eq. (35) we see that selecting the boson-boson component in $Q(1,2)$ is formally equivalent to evaluating it for $\theta_1 = 0, \overline{\theta}_1 = 0, \theta_2 = 0, \overline{\theta}_2 = 0$.

Note that the object $\delta Q(1,2) \equiv Q(1,2) - \delta(1,2)C(0)$ is also of the form (35) with a function $\delta C(t)$ such that $\delta C(0) = 0$ and $\delta C(t) = C(t)$ for $t \neq 0$. It can be also noted that the term $\int \delta Q(1,3)\delta Q(3,2)d3$ is also of the form (35) with an appropriate boson-boson component that we will express in terms of $C(t)$. These properties guarantee that if we satisfy the boson-boson component of Eq. (34), all other components will be automatically satisfied. As stated, this is equivalent to formally setting $\theta_1 = 0, \overline{\theta}_1 = 0, \theta_2 = 0, \overline{\theta}_2 = 0$ and the final result is

$$\left.\int \delta Q(1,3)\delta Q(3,2)d3\right|_{\theta_1=\theta_2=\overline{\theta}_1=\overline{\theta}_2=0} = -C^2(t) - \int_0^t [C(t-y) - C(t)]\dot{C}(y)dy, \tag{36}$$

where $t = t_1 - t_2$. The above simple result is the key ingredient of the statics-dynamics connection and it is derived in detail in Appendix A. Putting everything together, we see that the dynamical equation is the same as the spherical model obtained previously,

$$0 = 2\tau C(t) + w_1\left\{-C^2(t) - \int_0^t [C(t-u) - C(t)]\dot{C}(u)du\right\}. \tag{37}$$

To summarize, the results of this subsection are as follows:

(i) The dynamics on large time scales near the critical SG temperature can be obtained as an expansion around the FM solution.

(ii) The resulting dynamical equations do not depend on the terms that contain explicit time derivatives and, therefore, are invariant under a rescaling of times; independent of the details of the model, the equation reduces to Eq. (37).

(iii) The coefficients of the relevant terms of the equations can be computed in the FM limit and, thus, are the same of the corresponding replicated static equation. The peculiar nature of SG critical dynamics is determined by the presence of the term $w_1(\delta Q^2)_{ab}$ in the replica equations that comes from the term $w_1 \mathrm{Tr}\, Q^3$ in the static replicated action.

Since the dynamical equation is time-scale invariant, the large-time behavior is determined minus an overall time scale. This should be determined matching the large-time solution with the small-time solution, which depends on the details of the dynamics and of the model. On the other hand, the divergence of the time scale approaching the critical temperature can be obtained using matching arguments as in Eq. (12).

III. FROM REPLICATED GIBBS FREE ENERGY TO CRITICAL DYNAMICS

A. The general case

In the preceding section we have considered the case of fully connected models for which one can obtain both the static and the dynamics from the saddle-point method. We then took the FM limit in the dynamical equations and recovered the statics. The first correction to the FM limit yielded an equation invariant under a rescaling of time which determines critical behavior, notably the dynamical exponents controlling the time decay of the correlation at criticality and the divergence of the time scale as the control parameter (temperature, field) approaches its critical value. The explicit computation of the previous section shows that there is perfect equivalence between the dynamical equations and the statical equations in the sense that their coefficients are the same in the FM limit. This result could have been anticipated; after all, we knew from the beginning that the dynamics in the large time limit must reproduce the static. This observation allows us to generalize considerably the results of the previous section and prove, in full generality, Eqs. (3) and (4). In order to do so we will introduce the replicated Gibbs free energy and the dynamical Gibbs free energy and the corresponding equations of state in order to describe the statics and the dynamics of a general model. For a generic model these quantities provide the generalization of the saddle-point equations discussed in the previous section.

We proceed in full generality considering n replicas of a system of N spins s_i specified by a Hamiltonian $H_J(s)$ depending on some quenched parameter. Averages in the replicated system can be rewritten as

$$\langle \cdots \rangle \equiv \overline{\langle \cdots \rangle_J}, \tag{38}$$

where $\langle \cdots \rangle_J$ are thermal averages at fixed couplings J while the overline is the average over the couplings that must be performed, reweighting each disorder realization with the single system partition function to the power n,

$$\overline{O_J} = \frac{\int dP(J) O_J Z_J^n}{\int dP(J) Z_J^n}. \tag{39}$$

We define the following free-energy functional:

$$F(\lambda) \equiv -\frac{1}{N} \ln \langle e^{\sum_{(ab)} N \lambda_{ab} \delta \tilde{Q}_{ab}} \rangle, \tag{40}$$

where

$$\delta \tilde{Q}_{ab} \equiv \frac{1}{N} \sum_i s_i^a s_i^b - q \tag{41}$$

and q is the average value of the overlap in the absence of the replicated fields λ_{ab},

$$q \equiv \frac{1}{N} \sum_i \langle s_i^a s_i^b \rangle_{\lambda_{ab}=0} = \sum_i \overline{\langle s_i \rangle_J^2}. \tag{42}$$

The average value of the overlap is given by the derivative of the free energy with respect to the fields,

$$\langle \delta \tilde{Q}_{ab} \rangle = -\frac{\partial F}{\partial \lambda_{ab}}. \tag{43}$$

The replicated Gibbs free energy is defined as the Legendre transform of $F(\lambda)$,

$$G(\delta Q) \equiv F(\lambda) + \sum_{(ab)} \lambda_{ab} \delta Q_{ab}, \tag{44}$$

where λ is now a function of δQ_{ab} according to the following implicit equation:

$$\delta Q_{ab} = -\frac{\partial F}{\partial \lambda_{ab}}. \tag{45}$$

Correspondingly, the derivative of the Gibbs free energy yields the equation of state,

$$\lambda_{ab} = \frac{\partial G}{\partial \delta Q_{ab}}. \tag{46}$$

In a Ginzburg-Landau sense, the assumption that a given model undergoes a phase transition at some point corresponds to the assumption that the Gibbs free energy has a certain form near the critical point. The case of the preceding section corresponds to an equation of state of the form (setting $\lambda_{ab} = 0$)

$$0 = 2\tau \delta Q_{ab} + w_1 (\delta Q^2)_{ab} + O(Q^3). \tag{47}$$

We introduce a dynamical analog of the Gibbs free energy as the Legendre transform of a dynamical free energy defined as

$$F^{\mathrm{Dyn}}(\lambda) \equiv -\frac{1}{N} \ln \langle e^{N \int d1d2 \lambda(1,2) \delta \tilde{Q}(1,2)} \rangle, \tag{48}$$

where the angled bracket denotes the average in the superfield variables introduced in the preceding section. The dynamical analog of the overlap in the previous equation is the superoverlap,

$$\delta \tilde{Q}(1,2) \equiv \frac{1}{N} \sum_i [\phi_i(1)\phi_i(2) - \langle \phi_i(1)\phi_i(2) \rangle_{\lambda=0}^{\mathrm{FM}}]. \tag{49}$$

The average value of the superoverlap is given by the derivative of the dynamic Gibbs free energy with respect to the fields,

$$\langle \delta \tilde{Q}(1,2) \rangle = -\frac{\partial F^{\mathrm{Dyn}}}{\partial \lambda(1,2)}. \tag{50}$$

The dynamical Gibbs free energy is defined as the Legendre transform of $F^{\mathrm{Dyn}}(\lambda)$,

$$G^{\mathrm{Dyn}}(\delta Q) \equiv F(\lambda) + \int d1d2 \lambda(1,2) \delta Q(1,2), \tag{51}$$

where λ is now a function of δQ according to the following implicit equation,

$$\delta Q(1,2) = -\frac{\partial F}{\partial \lambda(1,2)}. \tag{52}$$

Correspondingly, the derivative of the dynamical Gibbs free energy yields the dynamical equation of state,

$$\lambda(1,2) = \frac{\partial G}{\partial \delta Q(1,2)}. \tag{53}$$

Now it is well known that the coefficients of the dynamical free energy in powers of $\lambda(1,2)$ can be expressed as cumulants of $\delta \tilde{Q}$ [37]. According to the discussion in the previous section it is clear that in the FM limit the structure of these dynamical cumulants is the same of the corresponding replicated objects. From this observation it follows that for a generic model if the replicated equation of state has a given structure, say Eq. (47) above, *the dynamical equation of state in the FM limit has precisely the same structure with the same coefficients*:

$$0 = 2\tau \delta Q(1,2) + w_1 \int \delta Q(1,3) \delta Q(3,2) + O(Q^3). \tag{54}$$

In the FM limit the above equation has the trivial solution $\delta Q(1,2) = 0$. Next we consider the first correction to the FM limit assuming a small but finite value of the parameter of the Langevin equation $1/\Gamma_0$. This will produce a small modification of the parameters of the dynamical equation of state, introducing additional terms proportional to the time derivatives of the correlations. The appearance of these time derivatives will have a dramatic effect at small time differences. Indeed, in the FM limit, the correlation $C(t_1 - t_2)$ jumps from $C(0) \neq 0$ at $t_2 = t_1$ to $C(t_2 - t_1) = 0$ instantaneously at $t_2 = t_1^+$ or $t_2 = t_1^-$ and the presence of time derivatives will have the effect of smoothing the correlation in time. As we have already noticed, as soon as we perturb the FM limit there is always a region of time differences of order $1/\Gamma_0$ where the dynamics cannot be obtained using the above perturbative expansion in $\delta Q(1,2)$ because $\delta Q(1,2)$ is large in this region. However, if we consider time differences large with respect to $1/\Gamma_0$, we can use the equation above looking for the nonzero solution $\delta Q(1,2)$ invariant under a rescaling of time and conclude that the critical dynamics is determined by Eqs. (10)–(12) with $\lambda = 0$. This determines the large-time behaviors up to an irrelevant numerical prefactor that depends on the precise form of the correlation in the small-time region.

The key point is that all these properties can be obtained solely under the assumption that the replicated equation of state admits the expansion (47) near the critical point. This remains true also in finite dimension provided we stay above the upper critical dimension.

In the following we will consider various types of SG transitions specified by the structure of the Gibbs free energy near the critical point and we will use static-dynamic mapping to determine the corresponding critical dynamics. The structure of the dynamical equations we will obtain is not at all new, although we will offer a concise and unified derivation. The new result is that, in all the cases we will consider below, the exponent parameter λ is determined by the ratio between the cubic coefficients of the static replica Gibbs free energy. In the next section we will further argue that the cubic coefficients of the replicated Gibbs free energy can be associated to appropriate cubic cumulants of the correlations bridging the dynamical exponent with physical observables.

B. Continuous transition in zero field

This type of transition is characterized by the following structure of the Gibbs free energy:

$$G(Q) = \frac{\tau}{2}\sum_{a,b} Q_{ab}^2 - \frac{w_1}{6}\text{Tr}\, Q^3 - \frac{w_2}{6}\sum_{a,b} Q_{ab}^3, \tag{55}$$

with $q_{aa} = 0$. Note the presence of the term $\sum_{a,b} Q_{ab}^3$, which was absent in the models considered earlier and which changes the coefficient of the term $C^2(t)$ in the dynamical equations (37). Such a structure describes, e.g., the continuous transition in the case of the Potts model [17]. Another instance is given by the continuous SG transition of the SK model in the presence of a small p-spin term with $p = 3$ in both the Ising and spherical cases. In order to study critical dynamics at large times we can just extend the results of the previous sections. The correlation at large time differences is described by

$$C(t) = \tau\, f(t/t^*) \quad t \gg 1, \quad t^* \propto \frac{1}{\tau^{\frac{1}{a}}}, \tag{56}$$

where the exponent a obeys

$$\frac{w_2}{w_1} = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)} \tag{57}$$

and the function $f(x)$ obeys the scale invariant equation

$$0 = f(x) + f^2(x)\left(1 - \frac{w_2}{w_1}\right) + \int_0^x (f(x-y) - f(x))\dot{f}(y)dy. \tag{58}$$

The solution of the above equation diverges as $1/x^a$ for $x \to 0$ and goes exponentially to zero for $x \to \infty$.

C. Continuous transition in a field

This transition is described by the vanishing of the replicon eigenvalue in a replica-symmetric (RS) Gibbs free energy with $n = 0$ [38]. In the SK model, the position of this transition in the temperature–magnetic-field plane defines the de Almeida Thouless line [36]. The replicated order parameter above the transition is replica symmetric and can be written as

$$q_{ab}^{\text{RS}} = \delta_{ab}(q_d - q_{\text{EA}}) + q_{\text{EA}}. \tag{59}$$

From this expression the corresponding dynamical order parameter in the FM limit is

$$\begin{gathered} Q_{\text{FM}}(1,2) = \delta(1,2)[C(0) - C(\infty)] + C(\infty), \\ C(0) = q_d, \quad C(\infty) = q_{\text{EA}}. \end{gathered} \tag{60}$$

Both q_d and q_{EA} are regular at the transition as a function of the external parameter because the longitudinal eigenvalue is regular [38]. The replicated equation of the order parameter expressed in terms of the deviation from the RS solution $\delta q_{ab} = q_{ab} - q_{ab}^{\text{RS}}$ is

$$\begin{aligned} 0 = {} & r\delta q_{ab} + m_2\left(\sum_c \delta q_{bc} + \delta q_{ac}\right) \\ & + m_3\sum_{cd}\delta q_{cd} + w_1(\delta q^2)_{ab} - w_2\delta q_{ab}^2, \end{aligned} \tag{61}$$

where r is the replicon eigenvalue that vanishes linearly while the relevant external parameter (temperature, field) approaches its critical value; instead, m_2, m_3, w_1, and w_2 remain finite and, as usual, can be assumed to be constant. The corresponding dynamical equation expressed in terms of the deviation from the FM solution $\delta Q(1,2) \equiv Q(1,2) - Q_{\text{FM}}(1,2)$ is

$$\begin{aligned} 0 = {} & r\,\delta Q(1,2) + m_2\left[\int d3\,\delta Q(1,3) + \delta Q(2,3)\right] \\ & + m_3\int d3d4\delta Q(3,4) + w_1\int d3\delta Q(1,3)\delta Q(3,2) \\ & - w_2\,\delta Q(1,2)^2. \end{aligned} \tag{62}$$

The above equation is similar to that corresponding to zero field except for the two terms proportional to m_2 and m_3. These two terms, however, give a vanishing contribution,

$$\begin{aligned} \int d2\delta Q(1,2) &= \int d2 Q(1,2) - \int d2 Q_{\text{FM}}(1,2) \\ &= \int d2 Q(1,2) - [C(0) - C(\infty)] = 0. \end{aligned} \tag{63}$$

Therefore, the critical behavior of the decay of the correlation to its infinite time limit is the same of the zero-field case,

$$C(t) - C(\infty) = r\, f(t/t^*) \quad t \gg 1, \quad t^* \propto \frac{1}{r^{\frac{1}{a}}}, \tag{64}$$

where the exponent a obeys

$$\frac{w_2}{w_1} = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)} \tag{65}$$

and the function $f(x)$ obeys the scale-invariant equation,

$$0 = f(x) + f^2(x)\left(1 - \frac{w_2}{w_1}\right) + \int_0^x [f(x-y) - f(x)]\dot{f}(y)dy. \tag{66}$$

The solution of the above equation diverge as $1/x^a$ for $x \to 0$ and goes exponentially to zero for $x \to \infty$.

D. Discontinuous transition

This transition can be described by a replica-symmetric Gibbs free energy with $n = 1$ replicas (see discussion at the end of Appendix B). The order parameter at the critical point is given by

$$q_{ab}^{\rm RS} = \delta_{ab}(q_d - q_{\rm EA}) + q_{\rm EA}. \tag{67}$$

The variational equation near the critical temperature is

$$0 = \tau + m_2\left(\sum_c \delta q_{bc} + \delta q_{ac}\right) + m_3 \sum_{cd} \delta q_{cd} + w_1(\delta q^2)_{ab} + w_2 \delta q_{ab}^2, \tag{68}$$

where τ vanishes linearly approaching the transition and δq_{ab} is the difference between order parameter q_{ab} and the solution at criticality corresponding to $r = 0$ and $\delta q_{ab} = 0$. Note that this definition differs from those we employed in the previous case where δq_{ab} was the difference between the order parameter and solution at the given value of r and, therefore, there was no constant term in the equation. At $n = 1$, both the replicon and longitudinal eigenvalues vanish [24,38]. This is connected with the fact that, although m_2 is finite at the transition, it gives a vanishing contribution at $n = 1$, i.e., for $\delta q_{ab} = \delta q$ we have

$$0 = \tau + 2m_2(n-1)\delta q + m_3 n(n-1)\delta q + (n-2)w_1 \delta q^2 + w_2 \delta q^2. \tag{69}$$

At $n = 1$, the linear term disappears and the equation becomes

$$0 = \tau + (w_1 - w_2)\delta q^2. \tag{70}$$

This reflects the fact that below T_d there is a static solution while above there is not. In order to recover the above results in a dynamical context, it is useful to consider the dynamics starting from an equilibrated initial configuration. A dynamical Gibbs free energy formally identically to the replicated one can be obtained introducing a superfield $\phi_i(1)$, where 1 is a coordinate that can either select the real replica that specifies the initial condition, in this case $\phi_i(1) = s_i^{\rm init}$, or take the value $(t_1,\theta_1,\overline{\theta}_1)$; in this case $\phi_i(1)$ is the standard superfield at time t_1. It can be checked that the corresponding structure is equivalent to the case of $n = 1$ replicas, in particular we have,

$$\int d1 = 1, \tag{71}$$

at variance with the case treated in the previous sections corresponding to $n = 0$, where we have $\int d1 = 0$. The corresponding FM dynamical solution can be written as

$$Q_{\rm FM}(1,2) = \delta(1,2)(C(0) - C_p) + C_p \tag{72}$$

and in any given model one can check that the corresponding equations for $C(0)$ and C_p (the plateau value) are precisely the same as those obtained in the replicated treatment for, respectively, q_d and $q_{\rm EA}$. In fully connected models, one can follow the derivation of the previous sections obtaining that the dynamics near the dynamical temperature can be described by the same equation (68) of the replicated treatment expressed in terms of

$$\delta Q(1,2) = Q(1,2) - Q_{\rm FM}(1,2). \tag{73}$$

The corresponding dynamical equation is, therefore,

$$0 = \tau + m_2 \int [\delta Q(1,3) + \delta Q(2,3)]d3 + m_3 \int d3d4\delta Q(3,4) + w_1 \int \delta Q(1,3)\delta Q(3,2)d3 + w_2 \delta Q(1,2)^2. \tag{74}$$

In order to study the above equation, we choose to evaluate it with 1 as the initial condition and $2 \equiv (t_2 = t, \theta_2 = 0, \overline{\theta}_2 = 0)$. We start noticing that the quadratic term can be written as

$$\int \delta Q(1,3)\delta Q(3,2)d3 = \int Q(1,3)Q(2,3)d3 - 2Q(1,2)[C(0) - C_p] + - C_p \int Q(2,3)d3 - C_p \int Q(1,3)d3 + 2C_p[C(0) - C_p] + C_p^2 \int d3. \tag{75}$$

Computing the various terms similarly to Appendix A, we obtain

$$\int Q(1,3)Q(2,3)d3 = C(0)C(t) - \int_0^t C(t-y)\dot{C}(y)dy, \tag{76}$$

$$\int Q(1,3)d3 = C(0), \tag{77}$$

$$\int Q(2,3)d3 = C(t) - \int_0^t \dot{C}(y)dy = C(0). \tag{78}$$

Putting everything together, we have

$$\int \delta Q(1,2)\delta Q(2,3)d2 = -\delta C^2(t) - \int_0^t [\delta C(t-y) - \delta C(t)] \times \delta\dot{C}(y)dy, \quad \delta C(t) \equiv C(t) - C_p, \tag{79}$$

the two linear terms proportional to m_2 and m_3 vanish as can be seen from Eqs. (77) and (78) and the the fact that $\int Q_{\rm FM}(1,3)d3 = C(0)$. Therefore, the dynamical equation reduces to the following equation for the correlation

$$0 = \tau - \left(1 - \frac{w_2}{w_1}\right)\delta C^2(t) - \int_0^t (\delta C(t-y) - \delta C(t))\delta\dot{C}(y)dy, \tag{80}$$

where the inessential rescaling $\tau \to \tau w_1$ was performed. The above equation is the same that obtained in schematic MCT theories [12]. According to this equation, the critical behavior of the correlation around the plateau value near the dynamical temperature T_d is given by

$$\delta C(t) \equiv C(t) - C_p = |\tau|^{1/2} f_\pm(t/t^*) \quad t \gg 1, \quad t^* \propto \frac{1}{|\tau|^{\frac{1}{2a}}}, \tag{81}$$

where the function f_- has to be chosen above the dynamical temperature ($\tau < 0$) while the function f_+ has to be choose below the dynamical temperature ($\tau > 0$). The exponent a obeys

$$\frac{w_2}{w_1} = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)}, \tag{82}$$

and the function $f_\pm(x)$ obeys the scale invariant equation

$$\pm 1 = f_\pm^2(x)\left(1 - \frac{w_2}{w_1}\right) + \int_0^x [f_\pm(x-y) - f_\pm(x)]\dot{f}_\pm(y)dy. \tag{83}$$

As we stated, this coincides with the equation of the critical regime of schematic MCT, i.e., Eq. (6.55a) in Ref. [12] with the exponent parameter λ given by the dimensionless ratio w_2/w_1. The solution of the above equation diverges as $1/x^a$ for $x \to 0$ for both f_+ and f_-. The behavior at large value of x instead differs completely. Above the dynamical temperature we have to choose f_- that goes to $-\infty$ as x^b for large x, where b is given by the well-known equation

$$\frac{w_2}{w_1} = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)}. \tag{84}$$

Below the dynamical temperature, we have to choose instead f_+ that decays exponentially to the constant $(1-\lambda)^{-1/2}$ for $x \to \infty$ [39]. We note that the mapping between statics and dynamics can be used only for $\tau > 0$ because the statics is defined only in the glassy phase. However, once the dynamical equation is obtained in the glassy phase from the static one, it is natural to argue the it can be continued to the liquid phase, simply assuming that $\tau < 0$ and that the parameter exponent $\lambda = w_2/w_1$ is the same above and below the dynamical transition temperature.

IV. REPLICATED GIBBS FREE ENERGY AND THE PHYSICAL OBSERVABLES

In the previous section we have considered various types of SG phase transitions characterized by the form of the replicated Gibbs free energy. We have argued that to each replicated Gibbs free energy corresponds a dynamical Gibbs free energy with the same coefficients in the FM limit. The dynamical Gibbs free energy in the FM limit determines the nature of critical slowing down at the transition, in particular the dynamical exponents are determined by a single nonuniversal exponent parameter λ which we identified with the ratio w_2/w_1 between the cubic coefficients.

In this section we will further explore the connection between statics and dynamics, building on the fact that the Gibbs free energy is defined as the Legendre transform of the free energy and, therefore, the coefficients w_1 and w_2 can be expressed in terms of cumulants of the order parameter. The details of the derivation will be given in Appendices B and C while in the following we will present and discuss the result.

The different types of SG transitions discussed in the previous section can all be associated to a replica-symmetric Gibbs free energy with either $n = 0$ or $n = 1$. For a general replicated spin-glass model the replica-symmetric Gibbs free energy reads

$$G(\delta Q) = \frac{1}{2}\sum_{(ab),(cd)} \delta Q_{ab} M_{ab,cd} \delta Q_{cd} - \frac{w_1}{6}\mathrm{Tr}\,\delta Q^3 - \frac{w_2}{6}\sum_{ab} \delta Q_{ab}^3. \tag{85}$$

As explained in Appendix B, the above expression is the Legendre transform of the free energy in the presence of an appropriate field and, therefore, the various coefficients can be associated to spin averages. In particular, $M_{ab,cd}$ is the inverse of the replica-symmetric dressed propagator $G_{ab,cd}$. Due to replica symmetry the dressed propagator has three possible values depending on the number of replica indexes that are equal: $G_{ab,ab} = G_1$, $G_{ab,ac} = G_2$, and $G_{ab,cd} = G_3$. The three propagators are associated to cumulants of four spins or, equivalently, to two-point functions of the overlap among different replicas,

$$G_1 \equiv N\langle \delta \tilde{Q}_{12}^2 \rangle = \frac{1}{N}\sum_{ij} (\overline{\langle s_i s_j \rangle^2} - q^2), \tag{86}$$

$$G_2 \equiv N\langle \delta \tilde{Q}_{12} \delta \tilde{Q}_{23} \rangle = \frac{1}{N}\sum_{ij} (\overline{\langle s_i s_j \rangle \langle s_i \rangle \langle s_j \rangle} - q^2), \tag{87}$$

$$G_3 \equiv N\langle \delta \tilde{Q}_{12} \delta \tilde{Q}_{34} \rangle = \frac{1}{N}\sum_{ij} (\overline{\langle s_i \rangle^2 \langle s_j \rangle^2} - q^2), \tag{88}$$

where we recall from the preceding section that

$$q \equiv \frac{1}{N}\sum_i \overline{\langle s_i \rangle^2} \tag{89}$$

is the Edwards-Anderson parameter and

$$\delta \tilde{Q}_{12} \equiv \frac{1}{N}\sum_i s_i^1 s_i^2 - q \tag{90}$$

is defined as the deviation of the overlap between two replicas with respect to its average value.

As usual, in the spin-glass context, the overline means average with respect to the disorder while the angled bracket means thermal average at fixed realization of the disorder. In a numerical simulation these objects can be computed studying the evolution of four different replicas with the same disorder but different thermal histories. The fluctuations of the various overlaps are related to G_1, G_2, and G_3 according to the above formulas. We stress that when the propagators are written in terms of fluctuations of the overlaps the thermal and disorder averages can be grouped in a single average that we also represent with an angle bracket with a slight abuse of notation. The above formulas are valid in the case $n = 0$; in the case $n = 1$ they have to be interpreted in a slightly different way. This point is discussed more extensively at the end of Appendix B; here we recall that the case $n = 1$

describes a glassy phase characterized by the fact that there is an exponential number of metastable states. In this case, we have to use the prescription that (see, e.g., Ref. [24]) *thermal averages in the right-hand side of the above expressions has to be performed inside the same metastable state and the overline must be interpreted as an average over both the disorder and the many states at given disorder*. In a numerical simulation one should then consider four replicas of the system with different thermal histories but with the *same* equilibrium initial condition. Indeed, the initial condition selects a given metastable state and different initial conditions correspond to different metastable states. We see that this framework can be easily extended to systems (notably structural glass models) with no quenched disorder but with self-induced disorder caused by the splitting of the equilibrium state into many amorphous components; in this case, the overline in the above expression means just average with respect to the different components.

The cubic coefficients of the Gibbs free energy turn out to be given by (see Appendix C),

$$w_1 = r^3\omega_1 \quad w_2 = r^3\omega_2, \tag{91}$$

where r is the replicon, i.e., the inverse of the spin-glass susceptibility that diverge at criticality,

$$r \equiv \frac{1}{G_1 - 2G_2 + G_3} = \left[\frac{1}{N}\sum_{ij}\overline{\langle s_i s_j\rangle_c^2}\right]^{-1}, \tag{92}$$

and ω_1, ω_2 are six-point functions given by

$$\omega_1 = \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j\rangle_c \langle s_j s_k\rangle_c \langle s_k s_i\rangle_c}, \tag{93}$$

$$\omega_2 = \frac{1}{2N}\sum_{ijk}\overline{\langle s_i s_j s_k\rangle_c^2}, \tag{94}$$

where the suffix c stands for connected correlation functions. The above formulas must be interpreted according to the aforementioned prescription in the case $n = 1$. From the above expressions we see that the ratio w_2/w_1 is precisely equal to the ratio ω_2/ω_1. We also note that since w_1 and w_2 are expected to be finite at the transition, it is natural to expect that ω_1 and ω_2 diverge as r^{-3} at criticality. The above expressions can be used to compute the parameter exponent when the Gibbs free energy cannot be directly computed as in the case of fully connected models. In particular, we devised a method to compute the above six-point functions for models defined on finite connectivity random lattices obtaining a prediction for the ratio w_1/w_2 that compares very well with the numerical simulations of the Bethe lattice SG [29,32]. The above expressions can be also used to evaluate the parameter exponent by measuring ω_1 and ω_2 in numerical simulations. In this context, it is convenient to consider the fluctuations of the overlaps between different replicas of the same system (additionally, they must have the same initial condition in the $n = 1$ case). As explained in Appendix B, one has to consider at least six different replicas in order to evaluate the following eight cubic overlaps,

$$\mathcal{W}_1 \equiv N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{23}\delta\tilde{Q}_{31}\rangle, \tag{95}$$

$$\mathcal{W}_2 \equiv N^2\langle\delta\tilde{Q}_{12}^3\rangle, \tag{96}$$

$$\mathcal{W}_3 \equiv N^2\langle\delta\tilde{Q}_{12}^2\delta\tilde{Q}_{13}\rangle, \tag{97}$$

$$\mathcal{W}_4 \equiv N^2\langle\delta\tilde{Q}_{12}^2\delta\tilde{Q}_{34}\rangle, \tag{98}$$

$$\mathcal{W}_5 \equiv N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{13}\delta\tilde{Q}_{24}\rangle, \tag{99}$$

$$\mathcal{W}_6 \equiv N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{13}\delta\tilde{Q}_{14}\rangle, \tag{100}$$

$$\mathcal{W}_7 \equiv N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{13}\delta\tilde{Q}_{45}\rangle, \tag{101}$$

$$\mathcal{W}_8 \equiv N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{34}\delta\tilde{Q}_{56}\rangle, \tag{102}$$

and then the six-point cumulants ω_1 and ω_2 can be obtained using the following formulas [38]:

$$\omega_1 = \mathcal{W}_1 - 3\mathcal{W}_5 + 3\mathcal{W}_7 - \mathcal{W}_8, \tag{103}$$

$$\omega_2 = \tfrac{1}{2}\mathcal{W}_2 - 3\mathcal{W}_3 + \tfrac{3}{2}\mathcal{W}_4 + 3\mathcal{W}_5 + 2\mathcal{W}_6 - 6\mathcal{W}_7 + 2\mathcal{W}_8. \tag{104}$$

As usual, in critical phase transitions, we expect cumulants of the order parameter to be divergent. More precisely we expect that in the case $n = 0$ the cumulants $\mathcal{W}$ diverge as $1/r^3$ on the base of the solution of mean-field models. It is to be expected that this feature complicates the numerical evaluation of ω_1 and ω_2; nevertheless, it appears still to be feasible according to preliminary results.

In order to simplify the numerical evaluation, one could take advantage of the fact that the eight cumulants defined above are not independent at criticality. This issue is discussed in Appendix D; in particular, in the case of the transition in a field, one can show that, at the leading $O(1/r^3)$, orders ω_1 and ω_2 can be expressed in terms of just $\mathcal{W}_1$ and $\mathcal{W}_2$, which can be evaluated considering only three real replicas:

$$\omega_1 = \tfrac{11}{30}\mathcal{W}_1 - \tfrac{2}{15}\mathcal{W}_2, \tag{105}$$

$$\omega_2 = \tfrac{4}{15}\mathcal{W}_1 - \tfrac{1}{15}\mathcal{W}_2, \tag{106}$$

The nature of the relationship between the eight cumulants, however, depends on the transition and, in particular, the above relations are only valid for the $n = 0$ case, provided that only the replicon eigenvalue is critical. This corresponds to the structure of Sec. III C but not to that of Sec. III B. In the case $n = 1$, corresponding to MCT, this kind of analysis will be performed in a separate publication; here we only mention that we expect the evaluation of the various cubic cumulants to be more difficult than in the $n = 0$ case. Indeed, based on a diagrammatic analysis similar to that of Ref. [24], it is expected that all the $\mathcal{W}_i$'s diverge as $1/r^5$ with the same prefactor. Linear combinations can be formed that diverge with powers $1/r^4$ and $1/r^3$. Therefore, the $O(1/r^5)$ and $O(1/r^4)$ contributions to ω_1 and ω_2 in Eqs. (103) and (104) should cancel exactly in order to give an $O(1/r^3)$ contribution according to Eq. (91) possibly leading to large finite-size effects. In addition to these problems, the $n = 1$ case is also more complicated because of metastability. Indeed, states are only defined in the glassy phase away from the dynamical transition; therefore, in order to evaluate the various $\mathcal{W}$'s at the critical point, one should proceed by extrapolation. Alternatively, a safer procedure is to perform a dynamical evaluation similar to what was done in Ref. [24] for the two-point functions G. In this case one should set the temperature at its critical value and study six replicas with different thermal histories and the same equilibrium condition, evaluating the $\mathcal{W}$'s (and correspondingly ω_1 and ω_2) as a function of time. A parametric plot in power of the average overlap could then be used in order to extract the critical value of the parameter exponent.

We conclude this section, noting that the expansion of the Gibbs free energy has actually eight types of cubic terms in δQ_{ab}. The form of these terms is the same of those of the free energy; see Eq. (B10) in Appendix B. However, in Eq. (85) we have displayed only those corresponding to $\sum_{ab} \delta Q_{ab}^3$ and $\mathrm{Tr}\, \delta Q^3$; this is because they are the only two terms relevant for the present discussion. An explicit computation shows that the remaining terms generate vanishing contributions to the dynamical equation of state in both the continuous and discontinuous cases. This is due to the fact that each of these terms turns out to be proportional to differences of one-time quantities computed respectively in the FM limit ($\Gamma_0 = \infty$) and at finite Γ_0. Since we are at equilibrium, one-time quantities are constant in time and, thus, independent of the value of Γ_0; therefore, these differences are strictly zero. This is also the reason why in Appendix C we do not report the full Legendre transform inversion but only the expression of w_1 and w_2.

V. CONCLUSIONS

We have established a connection between the parameter exponent and the replicated Gibbs free energy. In the case of the fully connected model considered at the beginning, this connection can be explicitly verified. In spite of its simplicity, the analysis of this simple mean-field model provides sufficient insight to argue that the result is more general. In order to do so we have proceeded *a la* Ginzburg-Landau, i.e., starting from the assumption that, at some point in parameter space, the given model displays a phase transition characterized by a certain form of the replicated Gibbs free energy. In particular, we have considered SG transitions that are governed by a RS theory with $n = 0$ replicas and dynamical transitions characterized by a RS theory with $n = 1$ replicas and the MCT phenomenology. Dynamics is described by the dynamical Gibbs free energy that contains much more information than the static one. However, we have argued that critical dynamics is governed by the dynamical Gibbs free energy in the so-called fast-motion limit. In this regime, one sees that the two-point correlation is described by a scaling function that obeys a universal equation determined qualitatively by the nature of the transition and quantitatively by a single nonuniversal parameter (the parameter exponent). As we saw in Sec. III, this parameter is given by the ratio of two cubic vertexes of the dynamical Gibbs free energy in the fast-motion limit. The key observation is that in the FM limit the proper vertexes of the dynamics Gibbs free energy are the same of the static replica theory as expected on the base of standard statistical mechanics. This last result is particularly evident within the context of the superfield formulation of Langevin dynamics. As we saw in Sec. III, going from statics to dynamics is straightforward in this context and the fact that the term $(\delta Q^2)_{ab}$ yields the appropriate memory term in the dynamics can be proved in a few lines (see Appendix A). In a similar way, it is straightforward to obtain the critical dynamical equation corresponding to a transition of a given type.

Let us comment that it is rather surprising that the dynamical exponents are completely determined by purely static quantities. It is also interesting to observe that while the phase transitions considered here are characterized by the divergence of four-point functions, the dynamical exponents are determined by six-point functions. As such there is a precise relationship between the dynamical exponents and the static six-point functions that can be checked directly in simulations and experiments although a major issue is that the six-point functions are also divergent at the critical point.

Within the context of the replica method Eq. (3) is of great technical importance. Indeed, it was often assumed in the past that the replica method, being essentially a static technique, would only be able to localize the critical temperature and the nonergodicity parameter but was intrinsically unable to characterize a strictly dynamical quantity like λ. This is not the case and our result Eq. (3) has been recently applied in Ref. [4] to compute λ within the hypernetted-chain approximation of cloned liquid theory. On the other hand, the universal nature of Eq. (1) and of the critical correlators Eqs. (81) and (83) is not a new result, although one may find it interesting to recover it by means of the replica method. The connection between the a and b exponents was first obtained within schematic MCT theories [11,12,40]. Later Götze [41] realized that it is valid under very general assumptions. He considered a generic mode-coupling functional $\hat{F}(f_k)$ and argued that the equations for the critical correlators are universal in the sense that they do not depend on the precise form of $\hat{F}(f_k)$. More precisely, they depend on it only through the parameter λ that can be expressed in terms of the second variation of $\hat{F}(f_k)$ at the critical point. Note that the derivation within the present paper is also very similar to Götze's: One introduces a rather general functional (the full mode-coupling functional $\hat{F}(f_k)$ in MCT or the replicated Gibbs free energy in the replica approach) and argues that there is universality solely under the assumption that there is a critical point. The parameter λ is then expressed in terms of the behavior of the general object near the critical point. The exact relationship between λ and static quantities represented by Eq. (4) is instead a novel physical prediction that was not obtained previously in the context of MCT. In order to avoid any confusion we recall that in standard MCT one can actually express λ in terms of the static structure factor of the liquid but the expression is only approximate. Indeed to compute λ within MCT one has to specify the mode-coupling functional $\hat{F}(f_k)$. The standard approximation [11] is to retain only the two-point vertex $V^{(2)}(q;k,p)$ and then to use Sjogren's approximation [42] that yields $V^{(2)}(q;k,p)$ as a function of the static structure factor of the liquid $S(k)$. As a result, the parameter λ is expressed in terms of $S(k)$; see, e.g., Eq. (18) in Ref. [43]. Due to the various approximations involved, the result, although often accurate, does not have the same status in MCT of Eq. (1) and Eqs. (81) and (83) that were shown to be exact in Ref. [41]. The most interesting aspect of Eq. (4) is that the quantities involved can be measured in an experiment or a numerical simulation. Therefore, in principle, Eq. (4) can be verified measuring independently both λ and the cumulants ω_2 and ω_1. For completeness we write explicitly the formulas of ω_1 and ω_2 for a supercooled liquid,

$$\omega_1 = \frac{1}{V}\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \times \overline{\langle\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\rangle_c\langle\rho(\mathbf{r}_2)\rho(\mathbf{r}_3)\rangle_c\langle\rho(\mathbf{r}_1)\rho(\mathbf{r}_3)\rangle_c}, \quad (107)$$

$$\omega_2 = \frac{1}{V}\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \overline{\langle\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\rho(\mathbf{r}_3)\rangle_c^2}, \quad (108)$$

where $\rho(\mathbf{r})$ is the density. The angled brackets denote thermal averages that have to be computed within the same glassy state. The overline denotes the average over the different metastable glassy states. The above quantities can, in principle, be measured both above or below the critical temperature. As explained in Sec. IV, in the latter case the thermal averages should be measured dynamically, starting from the same initial condition on the time scale of the β regime.

In Ref. [44] dynamical corrections to standard MCT were computed and it was shown that they lead to a renormalization of the parameter exponent but do not change the form of the critical dynamical equations, in particular, the relationship between the exponent a and b remains the same. This result led the authors to suggest that the critical MCT equation has some universal features that grant to it the status of a Landau theory of the glass transition. Our result shows that this is indeed the case, because the dynamical critical equation follows just from the structure of the replicated Gibbs free energy near the critical point. This means that all models that have mean-field behavior at the static level are described by the MCT critical equation. However, we stress that the assumption that the MCT transition in models of structural glass can be characterized by a replicated Gibbs free energy of the form considered here is a point that should be put on more solid ground by means of quantitative approximate computations, possibly along the lines of recent encouraging results [6]. In this respect, the present results could be also useful; indeed, the computation of the parameter exponent λ can be performed in a purely static framework, which is usually simpler than the dynamical one.

Below the upper critical dimension one would expect that the ratio w_2/w_1 should converge to a universal renormalized value. However, we do not expect that this ratio should still coincide with the parameter exponent. Indeed, the crucial assumption for the validity of the result is the fact that the Gibbs free energy admits an expansion like those of Sec. III around the critical point. This assumption fails below the upper critical dimension and the equation of state differs from those considered here. It is possible that there is still a close connection between the statical and dynamical equation of state but this is a completely open problem that we leave for future work.

These results can be developed in various directions. One can consider, for instance, off-equilibrium dynamics in the quasistatic regime. It is well known that aging dynamics below T_d in mean-field SG models is governed by the so-called threshold states that can be characterized through a 1RSB solution with a breaking parameter $x < 1$ fixed by the condition that the replicon vanishes [10,21]. In some models [45] one observes a β regime with two dynamical exponents that obey the following relationship:

$$\frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = x\frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \lambda, \quad (109)$$

and it would be interesting to check whether λ can be computed from an expansion of the Gibbs free energy in the static 1RSB solution. Another interesting case is when the parameter λ becomes equal to 1. This case has been studied extensively in the structural glass literature [14] and it would be interesting to understand if the connection between statics and dynamics holds, too. Work is under way in this direction and preliminary results show that it is indeed so and that the quartic coefficients in the replicated Gibbs free energy become relevant, corresponding to eight-point functions. We note that this point corresponds to the end point of a first-order glass transition line in the context of pinning [46].

ACKNOWLEDGMENTS

We thank F. Caltagirone, U. Ferrari, S. Franz, L. Leuzzi, F. Ricci-Tersenghi, and G. Szamel for useful discussions. The European Research Council has provided financial support through ERC Grant No. 247328.

APPENDIX A: THE TRACE TERM IN THE DYNAMICS

In this Appendix, we compute the boson-boson component of $\int \delta Q(1,3)\delta Q(3,2)d3$ (that can be obtained setting formally $\theta_1 = \theta_2 = \overline{\theta}_1 = \overline{\theta}_2 = 0$) and show that it is given by

$$\left.\int \delta Q(1,3)\delta Q(3,2)d3\right|_{\theta_1=\theta_2=\overline{\theta}_1=\overline{\theta}_2=0} = -C^2(t) - \int_0^t [C(t-y) - C(t)]\dot{C}(y)dy, \quad (A1)$$

where $t = t_1 - t_2$. Using the definition $\delta Q(1,2) = Q(1,2) - C(0)\delta(1,2)$ we write

$$\int \delta Q(1,3)\delta Q(3,2)d3 = \int Q(1,3)Q(3,2)d3 - 2C(0)Q(1,2). \quad (A2)$$

Note that the second term depends on $C(0)$, which is a model-dependent quantity of order $O(1)$. Therefore, it is disturbing for two reasons: (i) it is first order in $Q(1,2)$ and (ii) it depends on the nonuniversal quantity $C(0)$. However, we will see that it is canceled exactly by an opposite contribution from the first term.

From now on we set $\theta_1 = \theta_2 = \overline{\theta}_1 = \overline{\theta}_2 = 0$ and, thus, we can write

$$Q(1,3) = C(t_1,t_3) + T R(t_1,t_3)\theta_3\overline{\theta}_3, \quad Q(3,2) = Q(2,3) = C(t_2,t_3) + T R(t_2,t_3)\theta_3\overline{\theta}_3, \quad (A3)$$

where $R(t_1,t_2)$ is the equilibrium response function. The integration over the variable 3 selects only some of the terms,

$$\int Q(1,3)Q(3,2)d3 = -\int_{-\infty}^{t_1} C(t_2-t_3)\dot{C}(t_1-t_3)dt_3 - \int_{-\infty}^{t_2} C(t_1-t_3)\dot{C}(t_2-t_3)dt_3, \quad (A4)$$

where we have used time translational invariance and the fluctuation-dissipation theorem,

$$TR(t) = -\theta(t)\dot{C}(t). \tag{A5}$$

The above expression can be split into two parts,

$$\begin{aligned}\int Q(1,3)Q(3,2)d3 \\ = -\int_{t_2}^{t_1} C(t_2-t_3)\dot{C}(t_1-t_3)dt_3 \\ -\int_{-\infty}^{t_2}[C(t_1-t_3)\dot{C}(t_2-t_3)+\dot{C}(t_1-t_3)C(t_2-t_3)]dt_3.\end{aligned} \tag{A6}$$

Integrating by part and using $C(\infty)=0$ we obtain

$$\begin{aligned}\int Q(1,3)Q(3,2)d3 \\ = -C^2(t) - \int_0^t [C(t-y)-C(t)]\dot{C}(y)dy + 2C(t)C(0),\end{aligned} \tag{A7}$$

where $t \equiv t_1 - t_2$. Substituting the last equation in Eq. (A2) (evaluated at $\theta_1=\theta_2=\overline{\theta}_1=\overline{\theta}_2=0$) we obtain (A1). An explicit computation shows that the complete expression of $\int \delta Q(1,3)\delta Q(3,2)d3$ can be obtained by the application of the operator within curly brackets in expression (35) to the right-hand side of Eqs. (A1).

APPENDIX B: FREE ENERGY IN A FIELD

In this Appendix we express the coefficients of the free energy of a generic spin-glass model in terms of cumulants of the spin distribution. Indeed, the Gibbs free energy is defined as the Legendre transform of the free energy in presence of an appropriate replicated field and, therefore, in Appendix A we will obtain the relationship between the respective cubic coefficients. The following discussion applies to the continuous transition case ($n=0$) while at the end we will discuss the discontinuous case ($n=1$). We proceed in full generality considering n replicas of a system of N spins s_i specified by a Hamiltonian $H_J(s)$ depending on some quenched parameter. For the sake of readability, we will repeat some of the definitions already given in the body of the paper. Averages in the replicated system can be rewritten as

$$\langle\cdots\rangle \equiv \overline{\langle\cdots\rangle_J}, \tag{B1}$$

where $\langle\cdots\rangle_J$ are thermal averages at fixed couplings J while the overline is the average over the couplings that must be performed reweighting each disorder realization with the single system partition function to the power n,

$$\overline{O_J} = \frac{\int dP(J) O_J Z_J^n}{\int dP(J) Z_J^n}. \tag{B2}$$

Note that the thermal averages between different replicas factorize prior to the disorder averages. We define the following free-energy functional:

$$F(\lambda) \equiv -\frac{1}{N}\ln\left\langle e^{\sum_{(ab)} N\lambda_{ab}\delta\tilde{Q}_{ab}}\right\rangle, \tag{B3}$$

where

$$\delta\tilde{Q}_{ab} = \frac{1}{N}\sum_i s_i^a s_i^b - q \tag{B4}$$

and

$$q \equiv \frac{1}{N}\sum_i \langle s_i^a s_i^b\rangle = \sum_i \overline{\langle s_i\rangle_J^2}. \tag{B5}$$

We note that the above free-energy functional arises if we apply to each spin s_i^a of each replica a Gaussian distributed random field h_i^a with covariance matrix given by $\overline{h_i^a h_j^b} = \lambda_{ab}\delta_{ij}$. We expand $F(\lambda)$ in powers of λ at the third order assuming $\lambda_{aa}=0\ \forall a$,

$$\begin{aligned}F(\lambda) = -\frac{1}{2}\sum_{(ab),(cd)}\lambda_{ab}G_{ab,cd}\lambda_{cd} \\ -\frac{1}{6}\sum_{(ab),(cd),(ef)}\mathcal{W}_{ab,cd,ef}\lambda_{ab}\lambda_{cd}\lambda_{ef}.\end{aligned} \tag{B6}$$

The above expression is fully general; however, in a RS phase, we have only three possible values of G and eight possible values of W,

$$G_{ab,ab} = G_1, \quad G_{ab,ac} = G_2, \quad G_{ab,cd} = G_3, \tag{B7}$$

$$\begin{aligned}\mathcal{W}_{ab,bc,ca} = \mathcal{W}_1, \quad \mathcal{W}_{ab,ab,ab} = \mathcal{W}_2, \quad \mathcal{W}_{ab,ab,ac} = \mathcal{W}_3, \\ \mathcal{W}_{ab,ab,cd} = \mathcal{W}_4,\end{aligned} \tag{B8}$$

$$\begin{aligned}\mathcal{W}_{ab,ac,bd} = \mathcal{W}_5, \quad \mathcal{W}_{ab,ac,ad} = \mathcal{W}_6, \quad \mathcal{W}_{ac,bc,de} = \mathcal{W}_7, \\ \mathcal{W}_{ab,cd,ef} = \mathcal{W}_8.\end{aligned} \tag{B9}$$

The cubic part of the free energy can be recast in the following form [38]:

$$\begin{aligned}\sum_{(ab),(cd),(ef)}\mathcal{W}_{ab,cd,ef}\lambda_{ab}\lambda_{cd}\lambda_{ef} \\ = \omega_1\sum_{abc}\lambda_{ab}\lambda_{bc}\lambda_{ca} + \omega_2\sum_{ab}\lambda_{ab}^3 + \omega_3\sum_{abc}\lambda_{ab}^2\lambda_{ac} \\ + \omega_4\sum_{abcd}\lambda_{ab}^2\lambda_{cd} + \omega_5\sum_{abcd}\lambda_{ab}\lambda_{ac}\lambda_{bd} + \omega_6\sum_{abcd}\lambda_{ab}\lambda_{ac}\lambda_{ad} \\ + \omega_7\sum_{abcde}\lambda_{ac}\lambda_{bc}\lambda_{de} + \omega_8\sum_{abcdef}\lambda_{ab}\lambda_{cd}\lambda_{ef},\end{aligned} \tag{B10}$$

where the above identity leads to the following relationships between the ω's and the $\mathcal{W}$'s [38]:

$$\omega_1 = \mathcal{W}_1 - 3\mathcal{W}_5 + 3\mathcal{W}_7 - \mathcal{W}_8, \tag{B11}$$

$$\omega_2 = \tfrac{1}{2}\mathcal{W}_2 - 3\mathcal{W}_3 + \tfrac{3}{2}\mathcal{W}_4 + 3\mathcal{W}_5 + 2\mathcal{W}_6 - 6\mathcal{W}_7 + 2\mathcal{W}_8, \tag{B12}$$

$$\omega_3 = 3\mathcal{W}_3 - 3\mathcal{W}_4 - 6\mathcal{W}_5 - 3\mathcal{W}_6 + 15\mathcal{W}_7 - 6\mathcal{W}_8, \tag{B13}$$

$$\omega_4 = \tfrac{3}{4}(\mathcal{W}_4 - 2\mathcal{W}_7 + \mathcal{W}_8), \tag{B14}$$

$$\omega_5 = 3\mathcal{W}_5 - 6\mathcal{W}_7 + 3\mathcal{W}_8, \tag{B15}$$

$$\omega_6 = \mathcal{W}_6 - 3\mathcal{W}_7 + 2\mathcal{W}_8, \tag{B16}$$

$$\omega_7 = \tfrac{3}{2}\mathcal{W}_7 - \tfrac{3}{2}\mathcal{W}_8, \tag{B17}$$

$$\omega_8 = \tfrac{1}{8}\mathcal{W}_8. \tag{B18}$$

From the definition (B3) we easily see that the coefficients of $F(\lambda)$ can be related to spin averages; in particular, G is precisely the dressed propagator,

$$G_{(ab),(cd)} \equiv -\frac{\partial^2}{\partial\lambda_{ab}\partial\lambda_{cd}}F(\lambda) = N\langle\delta\tilde{Q}_{ab}\delta\tilde{Q}_{cd}\rangle. \quad \text{(B19)}$$

In the following and in the previous expression averages are always computed at $\lambda_{ab}=0$. Assuming that we are in a RS phase we obtain that $G_{(ab),(cd)}$ can take three possible values depending on whether there are two, three, or four different replica indexes. The corresponding values are

$$G_1 \equiv N\langle\delta\tilde{Q}_{12}^2\rangle = \frac{1}{N}\sum_{ij}(\overline{\langle s_i s_j\rangle^2} - q^2), \quad \text{(B20)}$$

$$G_2 \equiv N\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{13}\rangle = \frac{1}{N}\sum_{ij}(\overline{\langle s_i s_j\rangle\langle s_i\rangle\langle s_j\rangle} - q^2), \quad \text{(B21)}$$

$$G_3 \equiv N\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{34}\rangle = \frac{1}{N}\sum_{ij}(\overline{\langle s_i\rangle^2\langle s_j\rangle^2} - q^2). \quad \text{(B22)}$$

The cubic terms are given by the third derivative,

$$\begin{aligned}\mathcal{W}_{(ab),(cd),(ef)} &\equiv -\frac{\partial^3}{\partial\lambda_{ab}\partial\lambda_{cd}\partial\lambda_{ef}}F(\lambda)\\ &= N^2\langle\delta\tilde{Q}_{ab}\delta\tilde{Q}_{cd}\delta\tilde{Q}_{ef}\rangle_c\\ &= N^2\langle\delta\tilde{Q}_{ab}\delta\tilde{Q}_{cd}\delta\tilde{Q}_{ef}\rangle, \end{aligned}\quad \text{(B23)}$$

where the suffix c stands for connected functions with respect to the overlaps (not with respect to the spins) and the second equality follows from the fact that the average of δQ_{ab} is zero by definition. If some replica indexes are equal, the cubic cumulants can take eight possible values,

$$\mathcal{W}_1 = N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{23}\delta\tilde{Q}_{31}\rangle \quad \text{(B24)}$$

$$= \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j\rangle\langle s_j s_k\rangle\langle s_k s_i\rangle} - 3q\sum_{ij}\overline{\langle s_i s_j\rangle\langle s_i\rangle\langle s_j\rangle} + 2N^2q^3, \quad \text{(B25)}$$

$$\mathcal{W}_2 = N^2\langle\delta\tilde{Q}_{12}^3\rangle = \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j s_k\rangle^2} - 3q\sum_{ij}\overline{\langle s_i s_j\rangle^2} + 2N^2q^3, \quad \text{(B26)}$$

$$\mathcal{W}_3 = N^2\langle\delta\tilde{Q}_{12}^2\delta\tilde{Q}_{13}\rangle = \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j s_k\rangle\langle s_i s_j\rangle\langle s_k\rangle} - 2q\sum_{ij}\overline{\langle s_i s_j\rangle\langle s_i\rangle\langle s_j\rangle} - q\sum_{ij}\overline{\langle s_i s_j\rangle^2} + 2N^2q^3, \quad \text{(B27)}$$

$$\mathcal{W}_4 = N^2\langle\delta\tilde{Q}_{12}^2\delta\tilde{Q}_{34}\rangle = \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j\rangle^2\langle s_k\rangle^2} - 2q\sum_{ij}\overline{\langle s_i\rangle^2\langle s_j\rangle^2} - q\sum_{ij}\overline{\langle s_i s_j\rangle^2} + 2N^2q^3, \quad \text{(B28)}$$

$$\mathcal{W}_5 = N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{13}\delta\tilde{Q}_{24}\rangle = \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j\rangle\langle s_i s_k\rangle\langle s_k\rangle\langle s_j\rangle} - 2q\sum_{ij}\overline{\langle s_i s_j\rangle\langle s_i\rangle\langle s_j\rangle} - q\sum_{ij}\overline{\langle s_i\rangle^2\langle s_j\rangle^2} + 2N^2q^3, \quad \text{(B29)}$$

$$\mathcal{W}_6 = N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{13}\delta\tilde{Q}_{14}\rangle = \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j s_k\rangle\langle s_i\rangle\langle s_j\rangle\langle s_k\rangle} - 3q\sum_{ij}\overline{\langle s_i s_j\rangle\langle s_i\rangle\langle s_j\rangle} + 2N^2q^3, \quad \text{(B30)}$$

$$\mathcal{W}_7 = N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{13}\delta\tilde{Q}_{45}\rangle = \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j\rangle\langle s_k\rangle^2\langle s_i\rangle\langle s_j\rangle} - 2q\sum_{ij}\overline{\langle s_i\rangle^2\langle s_j\rangle^2} - q\sum_{ij}\overline{\langle s_i s_j\rangle\langle s_i\rangle\langle s_j\rangle} + 2N^2q^3, \quad \text{(B31)}$$

$$\mathcal{W}_8 = N^2\langle\delta\tilde{Q}_{12}\delta\tilde{Q}_{34}\delta\tilde{Q}_{56}\rangle = \frac{1}{N}\sum_{ijk}\overline{\langle s_i\rangle^2\langle s_j\rangle^2\langle s_k\rangle^2} - 3q\sum_{ij}\overline{\langle s_i\rangle^2\langle s_j\rangle^2} + 2N^2q^3. \quad \text{(B32)}$$

Substituting the above expressions into the relationship between the ω's and the $\mathcal{W}$ we obtain

$$\omega_1 = \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j\rangle_c\langle s_j s_k\rangle_c\langle s_k s_i\rangle_c}, \quad \text{(B33)}$$

$$\omega_2 = \frac{1}{2N}\sum_{ijk}\overline{\langle s_i s_j s_k\rangle_c^2}, \quad \text{(B34)}$$

$$\omega_3 = \frac{3}{N}\sum_{ijk}\overline{\langle s_i s_j s_k\rangle_c\langle s_i s_j\rangle_c\langle s_k\rangle}, \quad \text{(B35)}$$

$$\omega_4 = \frac{3}{4N}\sum_{ijk}\left[\overline{\langle s_i s_j\rangle_c^2\langle s_k\rangle^2} - \overline{\langle s_i s_j\rangle_c^2}\;\overline{\langle s_k\rangle^2}\right], \quad \text{(B36)}$$

$$\omega_5 = \frac{3}{N}\sum_{ijk}\overline{\langle s_i s_j\rangle_c\langle s_i s_k\rangle_c\langle s_k\rangle\langle s_j\rangle}, \quad \text{(B37)}$$

$$\omega_6 = \frac{1}{N}\sum_{ijk}\overline{\langle s_i s_j s_k\rangle_c\langle s_i\rangle\langle s_j\rangle\langle s_k\rangle}, \quad \text{(B38)}$$

$$\omega_7 = \frac{3}{2N}\sum_{ijk}[\overline{\langle s_i s_j\rangle_c\langle s_i\rangle\langle s_j\rangle\langle s_k\rangle^2} - \overline{\langle s_i s_j\rangle_c\langle s_i\rangle\langle s_j\rangle}\;\overline{\langle s_k\rangle^2}], \quad \text{(B39)}$$

$$\omega_8 = \frac{N^2}{8}\overline{(q_J - q)^3}. \quad \text{(B40)}$$

We remark that the above expression shows that on passing from the $\mathcal{W}$'s to the ω's there is an increase in symmetry; in particular, we see that, due to various cancellations, ω_1, ω_2, ω_3, ω_5, and ω_6 have a single disorder average; ω_4 and ω_7 have two disorder averages; and only ω_8 has three disorder averages.

The above discussion is valid in the case $n=0$; in the case of a discontinuous transition the final expressions are the same but have to be interpreted in a different way. As we have already said in Sec. IV, below the dynamical transition temperature there is an exponential number of metastable states. Correspondingly, *the thermal averages in the above expressions have to be performed within the same metastable state, while the overline stands for both the summation over all possible states and the standard disorder average*. In the following, we discuss more extensively this issue and also the fact that the relevant Gibbs free energy of the problem is RS with $n = 1$ replicas. After all, the starting point in the replica framework is the same as that of the continuous transition; that is, the computation of the free energy of a system is replicated n times with $n \to 0$ and one may wonder why we end up considering $n \to 1$ instead.

One way to characterize the presence of many metastable states below the dynamical transition is to select an equilibrium configuration of the system (the reference configuration) and then study the effect of a small field pointing in the direction of the reference configuration [47]. When the field goes to zero the overlap between a configuration of the constrained system and the reference configuration should be equal to the equilibrium value q_0 of the overlap between two independent copies. Instead, below the dynamical transition temperature, the overlap with the reference configuration goes to a higher value q_1 because the system remains stuck in the metastable state to which the reference configuration belongs. The smaller value q_0 corresponds to the overlap between configurations in different states, while q_1 corresponds to the overlap between configurations in the same state. However, the overlap between two independent replicas remains q_0 because the probability of extracting two configurations from the same state vanishes since there is an exponential number of states with essentially equal (and therefore infinitesimal) probability.

It is well known [22,47,48] that in the replica framework this feature is encoded by spontaneous replica-symmetry breaking. More precisely, above the transition the equation of state of the order parameter (the $n \times n$ overlap matrix q_{ab}) has only the solution $q_{ab} = q_0$ while below it a new solution appears. This solution is characterized by the fact that there is a subgroup of size m of the n replicas such that the overlap is q_1 if and only if both replicas are inside the subgroup while it is q_0 otherwise. Analytical continuation has to be taken with $m \to 1$ and $n \to 0$.

As we said in Sec. III, in order to study critical dynamics we may consider $n' \to 0$ static replicas of the system in order to select the (equilibrium) reference configuration and then we may consider its dynamical evolution. In this mixed dynamical-replica framework, the system of the initial configuration and of its evolution at later times corresponds in the FM limit to a set of $m = 1 + 0$ replicas. The key point is that the overlap between the initial configuration and the remaining $n' - 1$ replicas is equal to q_0 and does not change considering the time evolution of the initial equilibrium configuration; therefore, at all orders in the expansion around the FM limit, we can set $\delta Q(a,b) = 0$ whenever one of the two indexes corresponds to one of the $n' - 1$ replicas. This means that in order to study the correction to the FM limit in the dynamics we only need to take into account deviations of the order parameter inside the block with $q_{ab} = q_1$ and this leads to a replica-symmetric Gibbs free energy with $n = 1$. Going back to Eq. (B3) we must consider nonzero values of λ_{ab} only inside a $m \times m$ block with $m = 1$. As a consequence, the replica indexes in Eqs. (B19) and (B23) are all from the same block and, therefore, the thermal averages in the right-hand side of Eqs. (B20)–(B22) and Eqs. (B25)–(B32) have to be computed with different thermal histories but within the *same* metastable state. More explicitly, in the case of the discontinuous transition, we should rewrite Eqs. (B33) and (B34) as

$$\omega_1 = \frac{1}{N} \sum_{ijk} \overline{\sum_{\alpha} P_\alpha \langle s_i s_j \rangle_c^\alpha \langle s_j s_k \rangle_c^\alpha \langle s_k s_i \rangle_c^\alpha}, \tag{B41}$$

$$\omega_2 = \frac{1}{2N} \sum_{ijk} \overline{\sum_{\alpha} P_\alpha \big(\langle s_i s_j s_k \rangle_c^\alpha\big)^2}, \tag{B42}$$

where the overline denotes the usual disorder average, α denotes the different metastable states, $\langle \cdots \rangle^\alpha$ is thermal average inside metastable state α, and P_α is the thermodynamic weight of the state α. This expression holds also in the case of a model with self-induced disorder where there is no overline but only the summation over different components.

We note that in Refs. [1–3] the discontinuous transition in various fully connected models were studied. These models have the peculiar property that $q_0 = 0$; as a consequence, it is almost immediate to see the connection of the problem with a RS free energy with $n = 1$. In order to avoid a misinterpretation of these results, we stress that, according to the above discussion, the connection is more general and does not at all require $q_0 = 0$.

APPENDIX C: THE INVERSION OF THE LEGENDRE TRANSFORM

The Gibbs free energy is defined as the Legendre transform of the free energy $F(\lambda)$,

$$G(\delta Q) \equiv F(\lambda) + \sum_{(ab)} \lambda_{ab} \delta Q_{ab}, \tag{C1}$$

where λ is a function of δQ_{ab} according to the following implicit equation:

$$\delta Q_{ab} = -\frac{\partial F}{\partial \lambda_{ab}}. \tag{C2}$$

On the other hand, the free energy is the Legendre transform of the Gibbs free energy and we have

$$\lambda_{ab} = \frac{\partial G}{\partial \delta Q_{ab}}. \tag{C3}$$

We consider the expansion of the free energy at third order, taking into account only the term $\mathrm{Tr}\, \lambda^3$ and $\sum_{ab} \lambda^3$ because the other terms are not relevant in the present context, see the end of Sec. IV,

$$F(\lambda) = -\frac{1}{2} \sum_{(ab),(cd)} \lambda_{ab} G_{ab,cd} \lambda_{cd} - \frac{\omega_1}{6} \mathrm{Tr}\, \lambda^3 - \frac{\omega_2}{6} \sum_{ab} \lambda_{ab}^3. \tag{C4}$$

Differentiating the free energy with respect to λ_{ab} we obtain (using Einstein's sum convention)

$$\delta Q_{ab} = G_{(ab),(cd)} \lambda_{cd} + \omega_1 (\lambda^2)_{ab} + \omega_2 \lambda_{ab}^2. \tag{C5}$$

From this we obtain

$$\lambda_{ab} = M_{(ab),(cd)} \delta Q_{cd} - \omega_1 M_{(ab),(cd)} (\lambda^2)_{cd} - \omega_2 M_{(ab),(cd)} \lambda_{cd}^2, \tag{C6}$$

where M is the inverse propagator that can be rewritten in the form

$$M_{ab,cd} = M_3 + r(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + (M_2 - M_3)(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}) \quad \text{(C7)}$$

and $r \equiv (G_1 - 2G_2 + G_3)^{-1} = M_1 - 2M_2 + M_3$ is the replicon eigenvalue. At linear order we have

$$\lambda_{ab} = r\delta Q_{ab} + (M_2 - M_3)\sum_c(\delta Q_{ac} + \delta Q_{bc}) + M_3 \sum_{(cd)} \delta Q_{cd} \quad \text{(C8)}$$

and we can substitute the above expression in Eq. (C6) in order to determine λ at second order in δQ. Since in the end we are only interested in terms the forms $(\delta Q)^2_{ab}$ and δQ^2_{ab}, in which both indexes a and b appear, we can replace as follows:

$$\omega_1 M_{(ab),(cd)}(\lambda^2)_{cd} \longrightarrow r^3\omega_1(\delta Q)^2_{ab}, \quad \text{(C9)}$$

$$\omega_2 M_{(ab),(cd)}\lambda^2_{cd} \longrightarrow r^3\omega_2\delta Q^2_{ab}. \quad \text{(C10)}$$

We define the coefficients of the expansion of the Gibbs free energy according to

$$G(\delta Q) = \frac{1}{2}\sum_{(ab),(cd)} \delta Q_{ab}M_{ab,cd}\delta Q_{cd} - \frac{w_1}{6}\mathrm{Tr}\,\delta Q^3 - \frac{w_2}{6}\sum_{ab}\delta Q^3_{ab}. \quad \text{(C11)}$$

With the above definition we obtain

$$w_1 = r^3\omega_1 \quad w_2 = r^3\omega_2. \quad \text{(C12)}$$

APPENDIX D: CRITICAL BEHAVIOR OF THE CUBIC CUMULANTS

The cubic cumulants $\mathcal{W}_i$ defined previously are all divergent at criticality. Nevertheless, depending on the nature of the transition, one can obtain some relationship between their singular parts. We will consider the case of a $n = 0$ RS critical point where the replicon vanishes corresponding, e.g., to the dAT line, leaving the study of the $n = 1$ RS critical point where the replicon and longitudinal eigenvalues vanish to a separate publication.

We start recalling the relationships that hold at criticality among the quadratic cumulants G_1, G_2, and G_3 in the $n = 0$ case,

$$G_1 - 2G_2 = 0\,, \quad G_1 - 3G_3 = 0. \quad \text{(D1)}$$

In the above relationship it is intended that we are considering the critical part; indeed, each G_i diverges as $1/r$ approaches the critical point but the above differences remain finite, i.e., their singular part is zero. The above relationships can be obtained in various ways. For instance, we can use the following relationship:

$$\sum_a \langle O\delta Q_{ab}\rangle = 0, \quad \text{(D2)}$$

where O is any observable. It follows from the fact that summing over a replica index we are removing the singular replicon component from δQ_{ab} and we are left with only the regular anomalous and longitudinal component.

Another possible way of obtaining the above result is to consider the expression of the longitudinal and anomalous eigenvalues in terms of G_1, G_2, and G_3 (see Ref. [38]) and impose that, contrary to the replicon, they remains finite, i.e., the singular part of their inverse is zero. However, in this case, one must be careful in taking the limit $n \to 0$ because the longitudinal and anomalous eigenvalue are degenerate in this limit and one remains with a single equation. In order to obtain the second equation, one must consider also their difference divided by n and impose that it remains finite in the limit $n \to 0$.

The above procedures can be used to obtain similar relationship between the cubic cumulants [49]. The resulting system of equations is

$$2\mathcal{W}_3 = \mathcal{W}_2, \quad \text{(D3)}$$

$$3\mathcal{W}_4 = \mathcal{W}_2, \quad \text{(D4)}$$

$$6\mathcal{W}_5 = -\mathcal{W}_2 + 15\mathcal{W}_8, \quad \text{(D5)}$$

$$3\mathcal{W}_6 = \mathcal{W}_2, \quad \text{(D6)}$$

$$4\mathcal{W}_7 = 5\mathcal{W}_8, \quad \text{(D7)}$$

$$2\mathcal{W}_1 = -2\mathcal{W}_2 + 15\mathcal{W}_8, \quad \text{(D8)}$$

In principle, the expression of the ω's in terms of the $\mathcal{W}_i$ requires all eight $\mathcal{W}_i$ (see the previous section); instead, the above formulas can be used to obtain an expression that depends on just two of them. For instance, if we express everything in terms of three replica cumulants (i.e., $\mathcal{W}_1$ and $\mathcal{W}_2$) we have

$$\omega_1 = \tfrac{11}{30}\mathcal{W}_1 - \tfrac{2}{15}\mathcal{W}_2, \quad \text{(D9)}$$

$$\omega_2 = \tfrac{4}{15}\mathcal{W}_1 - \tfrac{1}{15}\mathcal{W}_2. \quad \text{(D10)}$$

Note that these relationships hold on the dAT line but cannot hold on its zero-field end point. Indeed, in the zero field, we must have $\mathcal{W}_2 = \omega_2 = 0$ while $\mathcal{W}_1$ remains finite. This apparent contradiction is solved and we note that at the end point the longitudinal and anomalous eigenvalues also vanish, producing divergent corrections to the above formulas.

[1] F. Caltagirone, G. Parisi, and T. Rizzo, Phys. Rev. E **85**, 051504 (2012).

[2] U. Ferrari, L. Leuzzi, G. Parisi, and T. Rizzo, Phys. Rev. B **86**, 014204 (2012).

[3] F. Caltagirone, U. Ferrari, L. Leuzzi, G. Parisi, and T. Rizzo, Phys. Rev. B **86**, 064204 (2012).

[4] S. Franz, H. Jacquin, G. Parisi, P. Urbani, and F. Zamponi, PNAS **109**, 18725 (2012).

[5] T. Rizzo, arXiv:1209.5578.
[6] G. Szamel, Europhys. Lett. **91**, 56004 (2010).
[7] H. Sompolinsky, Phys. Rev. Lett. **47**, 935 (1981).
[8] H. Sompolinsky and A. Zippelius, Phys. Rev. B **25**, 6860 (1982).
[9] T. R. Kirkpatrick and D. Thirumalai, Phys. Rev. Lett. **58**, 2091 (1987); Phys. Rev. B **36**, 5388 (1987); T. R. Kirkpatrick, D. Thirumalai, and P. G. Wolynes, Phys. Rev. A **40**, 1045 (1989).
[10] A. Crisanti, H. Horner, and H. Sommers, Z. Phys. B **92**, 257 (1993).
[11] U. Bengtzelius, W. Götze, and A. Sjölander, J. Phys. C **17**, 5915 (1984).
[12] W. Götze, Z. Phys. B **56**, 139 (1984).
[13] W. Götze, in *Les Houches Session 1989*, edited by J. Hansen, D. Levesque, and J. Zinn-Justin (North Holland, Amsterdam, 1991).
[14] W. Götze, *Complex Dynamics of Glass-Forming Liquids: A Mode-Coupling Theory* (Oxford University Press, Oxford, UK, 2009).
[15] J.-P. Bouchaud, L. Cugliandolo, J. Kurchan, and M. Mézard, Physica A **226**, 243 (1996).
[16] J. M. Carlson *et al.*, J. Stat. Phys. **61**, 1069 (1990).
[17] D. J. Gross, I. Kanter, and H. Sompolinsky, Phys. Rev. Lett. **55**, 304 (1985).
[18] Y. Y. Goldschmidt, Europhys. Lett. **6**, 7 (1988).
[19] A. Crisanti and H. Sommers, Z. Phys. B **87**, 341 (1992).
[20] A. Cavagna and T. Castellani, J. Stat. Mech. (2005) P05012.
[21] L. F. Cugliandolo and J. Kurchan, Phys. Rev. Lett. **71**, 173 (1993).
[22] R. Monasson, Phys. Rev. Lett. **75**, 2847 (1995).
[23] S. Franz and G. Parisi, Physica A **261**, 317 (1998).
[24] S. Franz, G. Parisi, F. Ricci-Tersenghi, and T. Rizzo, Eur. Phys. J. E **34**, 102 (2011).
[25] S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).
[26] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
[27] G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979).
[28] A. Crisanti, Nucl. Phys. B **796**, 425 (2008).
[29] F. Caltagirone, U. Ferrari, L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and T. Rizzo, Phys. Rev. Lett. **108**, 085702 (2012).
[30] C. Brangian, W. Kob, and K. Binder, Europhys. Lett. **59**, 546 (2002).
[31] T. Sarlat, A. Billoire, G. Biroli, and J.-P. Bouchaud, JSTAT P08014 (2009).
[32] G. Parisi, F. Ricci-Tersenghi, and T. Rizzo (to be published).
[33] H.-J. Sommers and K. H. Fischer, Z Phys. B **58**, 125 (1984).
[34] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Oxford University Press, Oxford, UK, 2002), Chaps. 16 and 17.
[35] T. Sarlat, A. Billoire, G. Biroli, and J.-P. Bouchaud, J. Stat. Mech. (2009) P08014.
[36] M. Mézard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
[37] This result will be used in the following section, see also the detailed derivation in Appendix B.
[38] T. Temesvari, C. De Dominicis, and I. R. Pimentel, Eur. Phys. J. B **25**, 361 (2002).
[39] We note that the critical equations of the $n = 0$ case can be reduced through a constant shift to the equation for f_+.
[40] E. Leutheusser, Phys. Rev. A **29**, 2765 (1984).
[41] W. Götze, Z. Phys. B **60**, 195 (1985).
[42] L. Sjogren, J. Phys. C **13**, 705 (1980); Phys. Rev. A **22**, 2883 (1980).
[43] M. Nauroth and W. Kob, Phys. Rev. E **55**, 657 (1997).
[44] A. Andreanov, G. Biroli, and J.-P. Bouchaud, Europhys. Lett. **88**, 16001 (2009).
[45] L. F. Cugliandolo, J. Kurchan, and P. Le Doussal, Phys. Rev. Lett. **76**, 2390 (1996).
[46] C. Cammarota and G. Biroli, PNAS **109**, 8850 (2012).
[47] S. Franz and G. Parisi, J. Phys. I (France) **5**, 1401 (1995).
[48] M. Mezard and G. Parisi, J. Chem. Phys. **111**, 1076 (1999).
[49] Concerning the second procedure, we note that from formulas (48) in Ref. [38], since only the replicon eigenvalue is critical, we have $g_3 = g_4 = g_5 = g_6 = g_7 = g_8 = 0$, which can be translated into six equations for the $\mathcal{W}_i$'s using Eqs. (49) in Ref. [38] and the equations below (21) connecting the ω_i's and the $\mathcal{W}_i$'s. However, again because of the degeneracy between the longitudinal and anomalous eigenvalues, it is convenient to consider the following combinations: $g_3 = 0$, $(g_3 - g_4)/n = 0$, $g_6 = 0$, $(g_7 - g_6)/n = 0$, $[g_7 - g_6 - 1/3(g_8 - g_6)]/n^2 = 0$ which remain independent in the $n \to 0$ limit.