Phase transition to super-rotating atmospheres in a simple planetary model for a nonrotating massive planet: Exact solution

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An energy-enstrophy model for the equilibrium statistical mechanics of barotropic flow on a massive nonrotating sphere is introduced and solved exactly for phase transitions to rotating solid-body atmospheres when the kinetic energy level is high. Unlike the Kraichnan theory which is a Gaussian model, we substitute a microcanonical enstrophy constraint for the usual canonical one, a step which is based on sound physical principles. This yields a spherical model with zero total circulation, microcanonical enstrophy constraint, and canonical constraint on energy, leaving angular momentum free as is required for any model whose objective is to predict super-rotation in planetary atmospheres. A closed-form solution of this spherical model, obtained by the Kac-Berlin method of steepest descent, provides critical temperatures and amplitudes of the symmetry-breaking rotating solid-body flows. The critical values depend linearly on the relative enstrophy, with proportionality constant derived from the spectrum of the Laplace-Beltrami operator on the sphere, as expected within an energy-enstrophy theory for macroscopic turbulent flows. This model and its results differ from previous solvable models for related phenomena in the sense that the model is not based on a mean-field assumption.

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I. INTRODUCTION

This paper offers results on the application of equilibrium statistical mechanics to complex geophysical and astrophysical flows [1–12]. In particular, we reexamine the enigmatic super-rotation of the Venusian atmosphere [13] from the point of view of self-organized emergence of domain-scale coherent structures in nearly inviscid quasi-two-dimensional (2D) flows. This well-known phenomenon was observed by the Venera and Magellan missions, and is the objective of more recent projects such as JAXA's Venus Climate Orbiter. They are confirmed in some numerical studies cited in [13]. Our results complement recent results on the statistical mechanics of atmospheric flows on rotating planets and related problems connected to the energy-enstrophy model [9,14].

By super-rotation, I mean an atmospheric flow where a significant part of a planet's or moon's atmosphere has a primary component or mode that rotates like a solid body at a spin rate greater than the planet's spin. In addition to this mode of highest energy in a super-rotating atmosphere, there could be smaller nontrivial amounts of flow energy in other spherical harmonics, as well as baroclinic (vertical) modes which are not treated in this model. By this definition, in the zero-planetary-spin case in this paper, any solid-body rotating flow state is considered super-rotating. Likewise, we refer to a nearly solid-body flow that rotates more slowly than the planet as subrotation or counter-rotation.

Venus's upper troposphere, which is about 20 km thick and reaches to 65 km from the surface [13], rotates like a solid body or top once in 4 Earth days with cloud-top wind speed of 100 m/s while Venus the planet spins clockwise very slowly once every 243 Earth days [13]—hence we have super-rotation. Observations of Venus's planetary spin rate show a slight change in the length of the Venusian day to compensate for the super-rotation of its heavy carbon dioxide atmosphere. We suggest the following reasons centered on the conservation of angular momentum in a system with no external torques. For simplicity, we assume that the mechanical system consisting of the solid planet and its enveloping atmosphere does not experience any significant external torques. While the planet Venus has a mass that is about 80% the Earth's mass, the Venusian atmosphere has a mass of 4×10^{20} kg which is about 90 times the mass of the Earth's total atmosphere; the surface density of the Venusian atmosphere is 67 $\mathrm{kg/m^3}$, about 7% that of liquid water. The significant energy of super-rotation comes from torqueless solar radiation which acts via gravitational instability at small scales. This convective mechanism—which is not treated in the barotropic model here—can transfer angular momentum from lower layers to upper layers of the atmosphere through generated waves. Since there is in principle zero net angular momentum in the solar radiation, the small eddies produced in the gravitational instability under the heating effect of the radiation are the likely mechanism for transferring the substantial angular momentum from the planet itself into the super-rotating atmosphere.

The main objective of our spherical model energy-enstrophy theory and its exact solution is to provide a simple plausible explanation for this enigmatic phenomenon [13] which remains an open problem for solar system astrophysics. For the practical purpose of applying the analytical results in this paper, we will take Venus's spin rate to be effectively zero as a first approximation. The only other related example in our solar system, to which may be applied an extension of our model in this paper to the case of rotating planets, is the moon Titan which has a faster spin rate than Venus and an atmosphere that super-rotates. As a disclaimer, at the outset we emphasize that the barotropic model on which our statistical mechanics analysis is based neglects all baroclinic effects such as the convective instability mechanism [15] by which solar radiation transfers its energy to the atmosphere, as well as bottom

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friction. We propose this nondivergent barotropic model as a first-step statistical mechanics study of an idealized situation that may be applicable to some actual planetary phenomena. In the future, more realistic flow models for the atmospheres of Venus and Titan, such as the shallow-water model, could be incorporated. Then the likelihood of an exact analytical solution is small but numerical solutions of these subsequent models are not without value, as depicted in some results for the gas giants [16–19].

For obvious technical reasons, experimental data on self-organized macroscopic angular momentum in 2D flows are available only for the planar geometry. Previous experimental and numerical results [3,6], in particular those concerning planar vortex flows in a box, have shown that domain-scale coherent vortices that spin up spontaneously from the small eddies are robust and sometimes intermittent.

In brief, we outline here the main physical ideas on which the formulation of the pherical models of energy-enstrophy theory is based. First, Stokes' theorem implies the conservation of zero total circulation of the flows, which then leads naturally to fixing it microanonically in the statistical mechanics model. Next, although enstrophy is conserved in ideal 2D flows but not in the barotropic flows with planetary torques in this paper, we need enstrophy to be conserved only on the time scales of equilibration. Assuming this, we impose a microcanonical constraint on the enstrophy or square norm of the vorticity. Last, the flow of kinetic energy in the atmosphere is generated by insolation and gravitational-thermal overturning. This suggests that as a first step, a plausible equilibrium statistical model here is one where the kinetic energy is constrained canonically, and hence a conjugate variable, namely, the temperature, while the angular momentum is left unconstrained. It is easy to see that the latter is correct for it is unphysical to constrain all three related quantities: kinetic energy, enstrophy, and angular momentum. Unlike other theories [4] which do not have simple closed-form solutions, these spherical models do not constrain any higher-vorticity moments.

The main objectives of this paper are thus (1) implementation of the microcanonical enstrophy constraint leading to an energy-enstrophy non-mean-field theory that is grounded in the physics of barotropic fluid flows, (2) exact solutions of the resulting spherical model with finite long-range interactions using the Kac-Berlin method [20] of steepest descent, and (3) physical justification of the microcanonical and canonical constraints. Further discussion of the use of an energy reservoir in this model will be given in the next section.

The main point is that the exact solutions of the resulting spherical model support high-energy phase transitions to super-rotation, which may be compared to the nearly-solid-body super-rotating upper troposphere of Venus.

Effective computer simulations and mean-field approximations of these energy-enstrophy models have been discussed elsewhere [9,10,16,21–25]. It is well known that in 2D, the mean-field approximation is exact for equilibrium statistical mechanics provided the interactions in the model are of infinite range. Indeed, rigorous results for the exactness of the mean field have been obtained for some specific flow geometries even when the interactions are of long but finite range. However, our present work establishes the exactness of the mean field for the barotropic vorticity model coupled by torques to a massive sphere, which will be shown below to have finite long-range interactions.

We note here the significant fact that, contrary to popular belief, we know of several spherical models with finite long-range interactions which support nontrivial phase transitions to ordered phases at sufficiently small but nonzero numerical values of the temperature [20,26]. We will calculate in closed form the critical temperature and the energy amplitudes in the super-rotation modes as functions of the fixed amount of relative enstrophy in the flow. It is not surprising that the critical temperature is linear in the fixed relative enstrophy in the case of nonrotating planets.

II. BACKGROUND: DYNAMICS AND STATISTICAL MECHANICS

A. Dynamics

To provide the background (and justification for excluding the case of rotating planets in this paper) for the spherical model energy-enstrophy theory, we summarize an earlier dynamical theory for barotropic flows on rotating planets [27]. The dynamical and variational stability theory predicts the following results for a rotating planet: (1) the super-rotating organized flow state occurs as the stable global energy maximum only when the kinetic energy of the flow is higher than a critical value that depends on the atmospheric enstrophy (square norm of the vorticity field) and the planetary spin rate, (2) below this critical value, the observable organized flow state is subrotation, which is a global energy minimum provided that, in addition, the planetary spin is large enough relative to atmospheric enstrophy; subrotation is a saddle point when the planetary spin is smaller than this critical value and should not be observable, (3) for all other combinations of energy, enstrophy, and planetary spin, the flow states are not organized ones and consist of multiple length scales.

The above results come from a variational dynamics theory that does not account for entropy effects, which are considered essential to a fuller theory. Such an improved theory—the chief aim here—is needed to predict phase transitions between the high-entropy unordered flow states and a small set of organized flows with extreme energies. The spherical model in this paper is a step further towards this aim, in the sense that it is not a mean-field theory, but because its partition function is solved exactly, it provides rigorous justification for the correctness of the mean-field approximation in the statistics of barotropic flow.

B. Statistical mechanics of 2D fluid flows

Fjortoft's and later, Kraichnan's [28] study of energy inverse cascades in nearly inviscid quasi-2D turbulence—a nonequilibrium result—renewed interest in Onsager's approach [29] to 2D turbulence which is based on equilibrium statistical mechanics [30]. The mean-field sinh-Poisson equations arising from the Lagrangian vortex gas methodology have yielded notable results on the emergence of domain-scale flows consisting of one or several large coherent vortices [29,30].

At the same time, the equilibrium statistical mechanics of the 2D Euler equations and the barotropic vorticity model in spectral and lattice forms has been studied extensively [1,4,12,28]. The classical theory in this field is Kraichnan's energy-enstrophy theory [1,28] which is based on the key assumption of existence of an inverse energy cascade that supports relatively short equilibration times during which the total flow energy and enstrophy are approximately fixed. Indeed before the successful work of Miller [4] and Robert and Sommeria [5], which stopped just short of producing closed-form solutions that support phase transitions involving the emergence of macroscopic angular momentum, Kraichnan's energy-enstrophy theory was considered the end of the line of thought begun by Onsager. Recent applications of the Miller-Robert-Sommeria theory include [9,10].

Although Onsager introduced the notion of negative temperatures to vortical flows in the 1940s with his seminal paper [29], Kraichnan and others solved the Gaussian energy-enstrophy theories for nearly inviscid 2D flows and showed that they *did not* support any interesting phase transitions to large-scale coherent flow structures even at negative temperatures. This inability of the classical energy-enstrophy theories to predict phase transitions to domain-scale flows is not due to the use of an incorrect energy functional, nor is it because of a fundamental shortcoming of the basic assumptions used to formulate these theories. The reasons for this inability are simple and lie in the incorrect choice of constraints in the partition function as discussed below. It will require the right number and correct choice of statistical mechanics constraints to produce an exactly solvable, predictive scientific theory for atmospheric super-rotation.

Due to the doubly canonical form of its Gibbs ensemble or partition function, the classical energy-enstrophy theories are exactly solvable Gaussian models which are not well defined at low numerical values of statistical temperatures [21]. However, most of the interesting physics of transitions to super-rotation and other domain-scale flows are expected to occur at sufficiently low numerical values of the relevant temperature. Note the important fact that the critical temperatures could be either positive or negative in the equilibrium statistical mechanics of many quasi-2D flows. In the latter case, the ordered phase usually coincides with negative temperatures that have numerical values less than the critical, which corresponds to flow states with extremely high kinetic energy. A non-Gaussian energy-enstrophy model is therefore the first requirement for a model that supports interesting transition physics in atmospheric flows.

What we offer here differs from previous works in three key ways: (A) formulation of a scientifically correct and solvable statistical mechanics theory of barotropic flows based on canonical constraint on energy, microcanonical constraint on enstrophy, and total circulation and nonconservation of angular momentum, (B) this theory is not a mean-field theory and its partition function is non-Gaussian, and (C) formulation of exact closed-form solutions of this model which predict qualitatively the phenomena of atmospheric super-rotation.

III. SIMPLE PLANETARY ATMOSPHERE MODEL

Consider the system consisting of a (possibly rotating) massive rigid sphere of radius R, enveloped by a thin shell of nondivergent barotropic fluid. The barotropic flow is assumed to be inviscid, and bottom friction is not included. It is assumed, however, that the fluid can exchange angular momentum with the infinitely massive solid sphere through an unmodeled torque mechanism. We also assume that the fluid is in radiation balance and there is no net energy gain or loss from insolation. This provides a simple model of the complex planet-atmosphere interactions, including the enigmatic torque mechanism responsible for the phenomenon of atmospheric super-rotation—one of the main applications motivating this work.

For a geophysical flow problem concerning super-rotation on a spherical surface there is little doubt that one of the key parameters is the angular momentum of the fluid. It is also clear that a 2D geophysical flow relaxation problem such as this one will also involve enstrophy. In nondivergent barotropic flows, there is no potential energy in the fluid because it has uniform thickness and density, and its upper surface is a rigid lid.

In principle, the total kinetic energy and angular momentum of the fluid and solid sphere are conserved quantities. By taking the sphere to have infinite mass, the active part of the model is just the fluid which exchanges angular momentum dynamically with the sphere and relaxes by exchanging kinetic energy with an infinite reservoir that crudely models the insolation plus gravitational instability mechanism at small scales.

A. Canonical energy constraint

As explained above, the microcanonical enstrophy constraint in the spherical model is introduced to derive a non-Gaussian model that remains exactly solvable; the microcanonical constraint on total circulation follows from the topology of vorticity fields on a sphere. The canonical constraint here is associated with a reservoir that exchanges energy with intermediate and smaller scales in the flow that are bounded below by and widely separated from molecular scales. The smallest scales where viscous dissipation acts are not treated in our model. This is a crude first attempt to model the realistic energy exchange mechanism of an atmosphere such as Venus's which is largely based on insolation coupled to gravitational-thermal instability. Our adoption here of a simple barotropic model precludes any meaningful modeling of these effects beyond precisely that of a canonical constraint in a statistical mechanics setting. The logarithmic Green's function in the barotropic model, having finite long-range interactions, is not ideally treated by a reservoir, but the alternative choice of a microcanonical constraint on energy yields a model that cannot be solved in closed form.

Nonetheless, the role of constraints has been debated in the context of mean-field models—in summary, there can be significant physical and mathematical differences between canonical versus microcanonical constraints on energy when the interactions are long range. For a review of the connections between several variational principles that have been introduced in statistical theories of 2D flows depending on the choice of the constraints, see [31].

Moreover, in this discussion of the physics of the canonical constraint on energy, it should be emphasized that the largest scales (near the domain length scale) in the barotropic flow are excluded by design from this exchange of energy with the reservoir—these scales comprise the long-range order from the phase transition and are treated in our application here of the steepest descent method as nonergodic scales separately from smaller so-called ergodic ones. These largest scales do interact and exchange energy and angular momentum with the solid planet (taken here to be infinitely massive and hence do not change its spin) through realistic mountain torque and other topographic stress for example (cf. also [3,6]).

B. Model equations

We will use spherical coordinates, $\cos\theta$ where θ is the colatitude and longitude ϕ. The total vorticity is given by

$$q(t;\cos\theta,\phi) = \Delta\psi + 2\Omega\cos\theta, \tag{1}$$

where $2\Omega\cos\theta$ is the planetary vorticity due to the spin rate Ω (which will be zero for the remainder of the paper), $w = \Delta\psi$ is the relative vorticity given in terms of a relative velocity stream function ψ, and Δ is the negative of the Laplace-Beltrami operator on the unit sphere S^2. Thus, a relative vorticity field, by Stokes' theorem, has the following expansion in terms of spherical harmonics:

$$w(x) = \sum_{l\geqslant 1,m} \alpha_{lm}\psi_{lm}(x). \tag{2}$$

A key property that will be established later is that the three spherical harmonics $\alpha_{1m}\psi_{1m}(x)$ contain all the angular momentum in the relative flow with respect to the frame rotating at the fixed angular velocity Ω of the sphere.

C. Physical quantities of the coupled barotropic vorticity model

The rest frame total kinetic energy of the fluid expressed in a frame that is rotating at the angular velocity of the solid sphere is

$$H_T[q] = \frac{1}{2}\int_{S^2} dx\big[(u_r + u_p)^2 + v_r^2\big]$$
$$= -\frac{1}{2}\int_{S^2} dx\,\psi q + \frac{1}{2}\int_{S^2} dx\,u_p^2,$$

where u_r and v_r are the zonal and meridional components of the relative velocity, u_p is the zonal component of the planetary velocity (the meridional component being zero since planetary vorticity is zonal), and ψ is the stream function for the relative velocity. Since the second term $\frac{1}{2}\int_{S^2} dx u_p^2$ is fixed for a given spin rate Ω, it is convenient to work with the pseudoenergy as the energy functional for the model,

$$H[w] = -\frac{1}{2}\int_{S^2} dx\,\psi q = -\frac{1}{2}\int dx\,\psi(x)\,[w(x) + 2\Omega\cos\theta]$$
$$= -\frac{1}{2}\int dx\,\psi(x)w(x) - \Omega\int dx\,\psi(x)\cos\theta.$$

The relative vorticity circulation in the model is fixed to be $\int w dx = 0$, which is a direct consequence of Stokes' theorem on a sphere. It is easy to see that the kinetic energy functional H is not well defined without the further requirement of a constraint on the size of its argument, the relative vorticity field $w(x)$. A natural constraint for this quantity is therefore its square norm or relative enstrophy, which gives a further justification for fixing microcanonically the relative enstrophy in the spherical models below.

The second term in the energy is equal to 4Ω times the variable angular momentum density of the relative fluid motion and has units of m^4/s. Incidentally, this further justifies our decision not to constrain the angular momentum in addition to the total kinetic energy, since it follows from this expression that the angular momentum is only one part of the total kinetic energy of the flow. The flow angular momentum in the case of a rotating planet is given by

$$\rho\int_{S^2} dx\,w\cos\theta = \rho\,\langle w,\cos\theta\rangle\,, \tag{3}$$

and implies that the only mode in the eigenfunction expansion of w that contributes to its net angular momentum is $\alpha_{10}\psi_{10}$, where $\psi_{10} = a\cos\theta$ is the first nontrivial spherical harmonic; it has the form of solid-body rotation vorticity. We shall see below that in the case of the massive nonrotating planet discussed here, the angular momentum in the super-rotating modes can be any combination of the three lowest spherical harmonics, which represents a symmetry-breaking phase transition in the sense that the turbulent barotropic flow has zero angular momentum in the disordered phase.

IV. SPHERICAL MODEL FOR ENERGY-ENSTROPHY THEORY

There is a natural vectorial formulation of the above physical quantities on a 2D mesh over the sphere that leads to a spherical model for barotropic flows on a massive sphere. We represent the normal vorticity at site j by the vector

$$\vec{s}_j = s_j\vec{n}_j,$$

where $\vec{n}_j$ denotes the outward unit normal to the sphere S^2 at x_j. Similarly, we represent the spin $\Omega \geqslant 0$ of the rotating frame by the vector

$$\vec{h} = \frac{2\pi}{N}\Omega\vec{n},$$

where $\vec{n}$ is the outward unit normal at the north pole of S^2. Denoting by γ_{jk} the angle subtended at the center of S^2 by the lattice sites x_j and x_k, we obtain the following energy functional for the total (fixed frame) kinetic energy of a barotropic flow in terms of a rotating frame at spin rate $\Omega \geqslant 0$:

$$H_N = -\frac{1}{2}\sum_{j\neq k}^{N} J_{jk}\vec{s}_j\cdot\vec{s}_k + \vec{h}\cdot\sum_{j=1}^{N}\vec{s}_j, \tag{4}$$

where the interaction matrix is now given by the *long- (finite-) range*

$$J_{jk} = \frac{16\pi^2}{N^2}\frac{\ln(1-\cos\gamma_{jk})}{\cos\gamma_{jk}},$$

where the center dot denotes the inner product in R^3 and $\vec{h}$ denotes a fixed external field arising from planetary spin if it is nonzero. Although long range in the sense that antipodal sites j and k on the planetary sphere have nonvanishing interaction J_{jk}, it is *finite* in the sense that as the size N of the mesh tends to infinity, the first term in H_N tends to the well-defined and finite limiting expression of the kinetic energy of relative flow, namely,

$$E^r_{\rm kin} = -\frac{1}{2}\int_{S^2} dx\, \psi(x) w(x) = -\frac{1}{2}\int_{S^2} dx' \int dx\, \Delta^{-1}(w(x'))w(x) < \infty.$$

The resulting spherical model consists of a canonical Gibbs ensemble with the following microcanonical constraints: (i) relative enstrophy

$$\frac{4\pi}{N}\sum_{j=1}^{N} \vec{s}_j \cdot \vec{s}_j = Q > 0,$$

and (ii) zero total circulation

$$\frac{4\pi}{N}\sum_{j=1}^{N} \vec{s}_j \cdot \vec{n}_j = 0,$$

which will be imposed by dropping the lowest spherical harmonic ψ_{00} from the eigenfunction representation of the vorticity. This will be explicit in the next section where the method of the saddle point will be used to solve this spherical model. It will be evident that the exact solutions will have a saddle point that *sticks at a nonzero critical inverse temperature* precisely because its long-range interactions have finite total energy.

Note that the vector

$$\Gamma = \frac{4\pi}{N}\sum_{j=1}^{N} \vec{s}_j$$

is the so-called magnetization, which turns out to be a natural order parameter in numerical simulations for the statistics of barotropic flows on a massive sphere.

V. SOLUTION OF THE SPHERICAL MODEL FOR $\Omega = 0$

The partition function of the above spherical models is calculated using Laplace's integral form for path integrals:

$$\begin{aligned} Z_N &\propto \int D(\vec{s}) \exp[-\beta H_N(\vec{s})]\delta\left(Q\frac{N}{4\pi} - \sum_{j=1}^{N}\vec{s}_j\cdot\vec{s}_j\right)\delta\left(\sum_{j=1}^{N}\vec{s}_j\cdot\vec{n}_j\right) \\ &= \int_{\delta(\sum_{j=1}^{N}\vec{s}_j\cdot\vec{n}_j)} D(\vec{s})\exp[-\beta H_N(\vec{s})]\left\{\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \exp\left[\eta\left(Q\frac{N}{4\pi} - \sum_{j=1}^{N}\vec{s}_j\cdot\vec{s}_j\right)\right]\right\} \\ &= \int_{\delta(\sum_{j=1}^{N}\vec{s}_j\cdot\vec{n}_j)} D(\vec{s})\exp\left(\frac{\beta}{2}\sum_{j\neq k}^{N} J_{jk}\vec{s}_j\cdot\vec{s}_k\right)\left\{\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \exp\left[\eta\left(N - \frac{4\pi}{Q}\sum_{j=1}^{N}\vec{s}_j\cdot\vec{s}_j\right)\right]\right\}, \end{aligned}$$

where $a > 0$ is chosen large enough. Thus,

$$\begin{aligned} Z_N &\propto \int_{\delta(\sum_{j=1}^{N}\vec{s}_j\cdot\vec{n}_j)} D(\vec{s})\exp\left(\frac{\beta}{2}\sum_{j\neq k}^{N} J_{jk}\vec{s}_j\cdot\vec{s}_k\right)\int_{a-i\infty}^{a+i\infty}\frac{d\eta}{2\pi i}\exp\left[\eta\left(N - \frac{4\pi}{Q}\sum_{j=1}^{N}\vec{s}_j\cdot\vec{s}_j\right)\right] \\ &= \int_{\delta(\sum_{j=1}^{N}\vec{s}_j\cdot\vec{n}_j)} D(\vec{s})\int_{a-i\infty}^{a+i\infty}\frac{d\eta}{2\pi i}\exp\left[N\left(\eta - \tfrac{4\pi}{QN}\eta\sum_{j=1}^{N}\vec{s}_j\cdot\vec{s}_j + \tfrac{\beta}{2N}\sum_{j\neq k}^{N} J_{jk}\vec{s}_j\cdot\vec{s}_k\right)\right] \\ &= \int_{\delta(\sum_{j=1}^{N}\vec{s}_j\cdot\vec{n}_j)} D(\vec{s})\int_{a-i\infty}^{a+i\infty}\frac{d\eta}{2\pi i}\exp\left[N\left(\eta - \frac{1}{N}\sum_{j\neq k}^{N} K_{jk}(Q,\beta,\eta)\,\vec{s}_j\cdot\vec{s}_k\right)\right], \end{aligned}$$

where

$$K_{jk}(Q,\beta,\eta) = \begin{cases} \frac{4\pi}{Q}\eta, & j = k, \\ -\frac{\beta}{2}J_{jk}, & j \neq k. \end{cases}.$$

To evaluate the Gaussian integrals in Z_N, we expand the relative vorticity vector field in terms of the spherical harmonics,

$$\vec{\omega}(x) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\alpha_{lm}\psi_{lm}(x)\vec{n}(x),$$

where $\vec{n}(x)$ is the outward unit normal to S^2 at x. We stress that this expansion need not include the harmonic $\psi_{00}(x)$ because of the zero-circulation condition on the vorticity $\vec{s}$.

Solution of the Gaussian integrals requires diagonalizing the interaction in H_N in terms of the spherical harmonics $\{\psi_{lm}\}_{l=1}^{\infty}$, which are natural Fourier modes for Laplacian eigenvalue problems on S^2 with zero circulation:

$$\vec{s}_j = \vec{n}_j\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\alpha_{lm}\psi_{lm}(x_j),\; -\frac{1}{2}\sum_{j\neq k}^{N} J_{jk}\,\vec{s}_j\cdot\vec{s}_k = \frac{1}{2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\lambda_{lm}\alpha_{lm}^2,$$

where the eigenvalues of the Green's function for the Laplace-Beltrami operator on S^2 are

$$\lambda_{lm} = \frac{1}{l(l+1)}, \quad l = 1, \dots, \sqrt{N}, \;\; m = -l, \dots, 0, \dots, l$$

and α_{lm} are the corresponding amplitudes. Thus,

$$\frac{1}{N}\sum_{j\neq k}^{N} K_{jk}(Q,\beta,\eta)\,\vec{s}_j\cdot\vec{s}_k = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(\frac{\beta}{2N}\lambda_{lm}+\frac{\eta}{Q}\right)\alpha_{lm}^2$$

and

$$\begin{aligned} Z_N &\propto \int_{\delta\left(\sum_{j=1}^N \vec{s}_j\cdot\vec{n}_j\right)} D(\vec{s})\int_{a-i\infty}^{a+i\infty}\frac{d\eta}{2\pi i}\exp\left[N\left(\eta-\frac{1}{N}\sum_{j\neq k}^{N}K_{jk}(Q,\beta,\eta)\,\vec{s}_j\cdot\vec{s}_k\right)\right] \\ &= \int\prod_{m=-1}^{1}d\alpha_{1m}\int D_{l\geqslant 2}(\alpha)\int_{a-i\infty}^{a+i\infty}\frac{d\eta}{2\pi i}\exp\left\{N\left[\eta-\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left(\frac{\beta}{2N}\lambda_{lm}+\frac{\eta}{Q}\right)\alpha_{lm}^2-\left(\frac{\beta}{4N}+\frac{\eta}{Q}\right)\sum_{m=-1}^{1}\alpha_{1m}^2\right]\right\}, \end{aligned}$$

where we have split off the integral over the three spherical harmonics ψ_{1m}, $m = -1,0,1$ which are the only modes with angular momentum. They represent the so-called nonergodic modes which motivate this split.

By choosing $\mathrm{Re}(\eta) = a > 0$ large enough, we interchange the order of integration of the term $\exp[-N\sum_{l=2}^{\infty}\sum_{m=-l}^{l}(\frac{\beta}{2N}\lambda_{lm}+\frac{\eta}{Q})\alpha_{lm}^2]$ to obtain

$$Z_N \propto \int\prod_{m=-1}^{1}d\alpha_{1m}\int_{a-i\infty}^{a+i\infty}\frac{d\eta}{2\pi i}\exp\left\{N\left[\eta-\left(\frac{\beta}{4N}+\frac{\eta}{Q}\right)\sum_{m=-1}^{1}\alpha_{1m}^2\right]\right\}\int D_{l\geqslant 2}(\alpha)\exp\left[-N\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left(\frac{\beta}{2N}\lambda_{lm}+\frac{\eta}{Q}\right)\alpha_{lm}^2\right].$$

A. Restricted partition function and nonergodic modes

Next we write the problem in terms of the restricted partition function $Z_N(\alpha_{10},\alpha_{1,\pm1};\beta,Q)$, that is,

$$\begin{aligned} Z_N(\beta,Q) &\propto \int\prod_{m=-1}^{1}d\alpha_{1m}\;Z_N(\alpha_{10},\alpha_{1,\pm1};\beta,Q) = \int\prod_{m=-1}^{1}d\alpha_{1m}\int_{a-i\infty}^{a+i\infty}\frac{d\eta}{2\pi i}\;\exp\left\{N\left[\eta-\left(\frac{\beta}{4N}+\frac{\eta}{Q}\right)\sum_{m=-1}^{1}\alpha_{1m}^2\right]\right\} \\ &\quad\times\int D_{l\geqslant 2}(\alpha)\exp\left[-N\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left(\frac{\beta}{2N}\lambda_{lm}+\frac{\eta}{Q}\right)\alpha_{lm}^2\right]. \end{aligned}$$

Due to nonergodicity of the ordered domain-scale nature of modes $\psi_{10},\psi_{1,\pm1}$, we do not integrate over these ordered modes whose amplitudes are denoted by α_{1m}. We also note that all the higher harmonics will turn out to have zero amplitudes in the ordered phase of this problem. The statistics of the problem are therefore completely determined by the restricted partition function $Z_N(\alpha_{10},\alpha_{1,\pm1};\beta,Q)$. The amplitudes $\alpha_{10},\alpha_{1,\pm1}$ of the ordered modes appear as parameters in this restricted partition function, and will have to be evaluated separately.

Standard Gaussian integration is used to evaluate the last integral, which yields, after scaling $\beta' N = \beta$,

$$\int_{l\geqslant 2} D(\alpha)\exp\left[-\sum_{l=2}\sum_{m=-l}^{l}\left(\frac{\beta' N\lambda_{lm}}{2}+\frac{N\eta}{Q}\right)\alpha_{lm}^2\right] = \prod_{l=2}^{\sqrt{N}}\prod_{m=-l}^{l}\left(\frac{\pi}{\frac{N\eta}{Q}+\frac{\beta' N}{2}\lambda_{lm}}\right)^{1/2},$$

provided the physically significant Gaussian conditions hold: for $l \geqslant 2$,

$$\frac{\beta'\lambda_{lm}}{2}+\frac{\eta}{Q} = \frac{\beta'}{2l(l+1)}+\frac{\eta}{Q} > 0. \tag{5}$$

Substituting these Gaussian expressions in the partition function gives

$$Z_N(\alpha_{10},\alpha_{1,\pm1};\beta,Q) \propto \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}d\eta\;\exp\left\{N\left[\eta-\left(\frac{\beta'}{4}+\frac{\eta}{Q}\right)\sum_{m=-1}^{1}\alpha_{1m}^2-\frac{1}{2N}\sum_{l=2}\sum_{m}\ln\left(\frac{N\eta}{Q}+\frac{\beta' N}{2}\lambda_{lm}\right)\right]\right\},$$

where the free energy per site evaluated at the most probable macrostate is $-\frac{1}{\beta'}F(\eta(\beta'),Q,\beta')$ with

$$F(\eta(\beta'),Q,\beta') = \eta(\beta')\left[1-\frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right] - \frac{\beta'}{4}\sum_{m=-1}^{1}\alpha_{1m}^2 - \frac{1}{2N}\sum_{l=2}\sum_{m}\ln\left(\frac{N\eta}{Q}+\frac{\beta' N}{2}\lambda_{lm}\right).$$

The partition function is now in the form where the saddle point method can be applied for N large.

B. Saddle points and energy threshold for the transition to super-rotation

Provided that the saddle point $\eta(\beta')$ can be determined at given inverse temperature β', the stable (most probable) macrostate is given by the extremum of the expression $F(\eta(\beta'),Q,\beta')$. The saddle point condition gives one equation for the determination of four variables η,α_{1m} in terms of the inverse temperature β' and relative enstrophy Q,

$$0 = \frac{\partial F}{\partial \eta} = \left(1-\frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) - \frac{1}{2NQ}\sum_{l=2}\sum_{m}\left(\frac{\eta(\beta')}{Q}+\frac{\beta'}{2}\lambda_{lm}\right)^{-1}, \qquad (6)$$

where $\eta = \eta(\beta')$ is taken to be the value of the saddle point.

We need three more conditions to determine the three amplitudes α_{1m} and the saddle point $\eta(\beta') > 0$. They are provided by the following equations of state representing the variational principle for free energy in stable states:

$$0 = \frac{\partial F}{\partial \alpha_{1m}} = -\left(\frac{2\eta(\beta')}{Q}+\frac{\beta'}{2}\right)\alpha_{1m}. \qquad (7)$$

Thus, a coupled system of four algebraic equations (6) and (7) determines four unknowns in terms of the given relative enstrophy $Q > 0$ and the scaled inverse temperature β'. The equations of state for α_{1m} implies that for $m = -1,0,1$ either

$$\alpha_{1m} = 0 \ \text{ or } \ \left(\frac{2\eta(\beta')}{Q}+\frac{\beta'}{2}\right) = 0.$$

Since the saddle point is on the line with real part $a > 0$, we deduce $\eta(\beta') > 0$, and with relative enstrophy $Q > 0$, we obtain

$$\beta' = -\frac{4\eta}{Q} < 0$$

when $\alpha_{1m} > 0$ for at least one value of m. This proves that in order to have positive energy in the super-rotating modes, the system's kinetic energy must be sufficiently high to make T' not only negative but numerically small (and the inverse temperature $\beta' < 0$ and numerically large). Thus, we conclude that the critical temperature $T_c' < 0$.

In addition, the Gaussian conditions (5) imply that for $l > 1$,

$$\frac{\beta'}{2l(l+1)}+\frac{\eta(\beta')}{Q} > 0,$$

which implies that the large-N limit in the right-hand side (RHS) of the saddle point condition is well defined and finite:

$$\left(1-\frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) = \lim_{N\to\infty}\frac{1}{2NQ}\sum_{l=2}\sum_{m}\left(\frac{\eta(\beta')}{Q}+\frac{\beta'}{2l(l+1)}\right)^{-1}. \qquad (8)$$

When either $T' > 0$ or $T' < 0$ and numerically large, there is no energy in the super-rotating modes, that is,

$$\left(1-\frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) = 1.$$

But for sufficiently high energy or hot enough negative temperatures, given by the threshold $T_c' < T' < 0$, the energy in the super-rotating modes increases until they contain all of the fixed enstrophy Q at absolute zero $T' = 0^-$:

$$\left(1-\frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) \searrow 0.$$

So to calculate $T_c'(Q) < 0$ we need to find the most negative value of the inverse temperature, denoted by β_c', for which

$$1 = \lim_{N\to\infty}\frac{1}{2NQ}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\frac{\eta(\beta')}{Q}+\frac{\beta'}{2l(l+1)}\right)^{-1}, \qquad (9)$$

and after which, for values of $\beta' < \beta_c' < 0$, $\lim_{N\to\infty}\frac{1}{2NQ}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}(\frac{\eta(\beta')}{Q}+\frac{\beta'}{2l(l+1)})^{-1} < 1$.

Inserting $\eta = -\frac{\beta' Q}{4}$ into the RHS of (8), we calculate β_c' to be given by

$$-\infty < \beta_c'(Q) = \lim_{N\to\infty}\frac{1}{QN}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\lambda_{lm}-\frac{1}{2}\right)^{-1} < 0,$$

and check that for $\beta' < \beta_c'(Q) < 0$,

$$\lim_{N\to\infty}\frac{1}{\beta' NQ}\sum_{l=2}\sum_{m}\left(-\frac{1}{2}+\lambda_{lm}\right)^{-1} < 1.$$

We note the significance of the critical temperature's linear dependence on the given relative enstrophy Q of he flow.

In other words, the extreme saddle point

$$\eta^* = -\frac{\beta' Q}{4}$$

is no longer adequate to solve (9) for $\beta' < \beta_c'(Q) < 0$, which is the so-called *sticking of the saddle point*at the critical point β_c'.

Moreover, the saddle point equation gives us a way to compute the equilibrium amplitudes of the super-rotating modes for temperatures hotter than the negative critical temperature T_c. For temperatures T such that $T_c < T < 0$,

$$\sum_{m=-1}^{1} \alpha_{1,m}^2(T) = Q\left(1 - \frac{T}{T_c}\right). \tag{10}$$

VI. CONCLUSION

In conclusion, the above exact solution by the saddle point method of the spherical model for barotropic energy-enstrophy theory shows that, for a nonrotating planet, there cannot be any energy in the solid-body rotating atmospheric modes at positive temperatures or low energy levels. Only at very high energy levels or negative temperatures smaller in numerical value than a critical threshold can there be the self-organization of barotropic energy into domain-scale coherent flows in the form of symmetry-breaking super-rotating atmospheric modes. These extremely high-energy modes carry a nonzero angular momentum that can be directed along an arbitrary axis since this problem is formulated in the inertial frame with planetary spin $\Omega = 0$. Moreover, it is clear from the analysis in this paper that only in the zero-spin case is the subsequent transition truly symmetry breaking in the Goldstone sense. We computed the threefold degeneracy in the super-rotation corresponding to precisely the first three spherical harmonics (those with azimuthal wave number equal to 1 and the only ones with nonzero angular momentum) that comprise a basis for the realized super-rotational axis [32].

In a future presentation we will extend this exact solution and spherical model to the harder and more interesting case of planetary spin $\Omega > 0$ where we expect to predict that the enigmatic phenomenon of subrotation not only occurs for sufficiently low energy levels but can only occur provided the planetary spin is fast enough relative to the atmospheric enstrophy.

A. Future work: Rotating planets

A previous dynamical or variational theory reported by this author in [27] suggests that a statistical mechanics theory for sub- and super-rotation should take into account both (1) the planets' spin rate and (2) the amount of atmospheric kinetic energy for fixed enstrophy. However, the former effect is complicated by a bifurcation where the subrotation state changes from a stable global energy minimizer to a saddle point when the planet's spin drops below a critical value that is proportional to the square root of the fixed enstrophy.

Additional complications in the rotating planet case arise in the fact that now we expect a positive critical temperature as well as the negative critical temperature already found in the nonrotating case. At the positive critical temperature which should exist provided planetary spin is large enough relative to the fixed enstrophy, the transition is from unordered flows to the extremely low-energy state of subrotation.

To summarize the expected theory for rotating planets, its main physical conclusion will be that a necessary and sufficient condition for subrotation in barotropic solid-body flows is that the planetary spin must be large enough relative to the atmospheric enstrophy and kinetic energy.

B. Other experimental considerations

Previous experimental and numerical results [3,6], in particular those concerning planar vortex flows in a box, have shown that domain-scale coherent vortices that spin up spontaneously from the small eddies are robust. This author would like to suggest an experiment where the box is fixed to a free and initially nonrotating turntable, to better observe the principle of angular momentum conservation which would cause the box and turntable to counter-rotate whenever a large vortex spins up in the flow. This experiment when performed would be the classical counterpart to the Einstein-Haas effect for electron spins in a ferromagnetic bar and the planar geometry analog of the spherical flow on the nonspinning planet in this paper.

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