

Semiclassical approach to the quantum Loschmidt echo in deep quantum regions: From validity to breakdown

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Semiclassical results are usually expected to be valid in the semiclassical regime. An interesting question is, in models in which appropriate effective Planck constants can be introduced, to what extent will a semiclassical prediction stay valid when the effective Planck constant is increased? In this paper, we numerically study this problem, focusing on semiclassical predictions for the decay of the quantum Loschmidt echo in deep quantum regions. Our numerical simulations, carried out in the chaotic regime in the sawtooth model and in the kicked rotator model and also in the critical region of a one-dimensional Ising chain in transverse field, show that the semiclassical predictions may work even in deep quantum regions, particularly for perturbation strength in the so-called Fermi-Golden-Rule regime.

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I. INTRODUCTION

The semiclassical theory is powerful in dealing with many problems in various fields of physics [1,2]. It is usually expected to work in the semiclassical regime, in which the (effective) Planck constant is sufficiently small in a certain relative sense. An interesting question is, how deep in the quantum regime could the semiclassical predictions remain valid? Obviously, the answer should depend on the physical quantity of interest.

In this paper, we study a quantity for which the semiclassical approach has recently been found to be quite successful in the semiclassical regime. It is the so-called quantum Loschmidt echo (LE) [3–5], which is given by the overlap of the time evolution of the same initial state under two slightly different Hamiltonians,

$$M(t) = |m(t)|^2$$

$$\text{with } m(t) = \langle \Psi_0 | \exp(iH_1 t/\hbar) \exp(-iH_0 t/\hbar) | \Psi_0 \rangle, \quad (1)$$

where $H_1 = H_0 + \epsilon V$, with ϵ a small quantity and V a generic perturbation. The quantity $m(t)$ is usually called the amplitude of the LE. The LE gives a measure to the stability of quantum motion under small perturbation.

The LE has quite rich behaviors, depending on the nature of the dynamics of the underlying classical system, as well as on the perturbation strength. Usually, the LE has a quadratic decay within a certain initial time interval, as predicted by the first-order perturbation theory [6]. Beyond the initial time interval, in a chaotic system, loosely speaking, the LE has a Gaussian decay [5,7–10] below a perturbative border and has an exponential decay in the so-called Fermi-Golden-Rule (FGR) regime above the perturbative border with intermediate perturbation strength [9–15]. With further increase of the perturbation strength, in the so-called Lyapunov regime with relatively strong perturbation, the LE usually has a perturbation-independent decay [12,15–20]; but, in certain cases, a perturbation-dependent oscillation in the decay rate may also appear [15,19–22].

On the other hand, in regular systems, in the case of one degree of freedom, the LE has a Gaussian decay [11], followed by a power-law decay [23,24]. Meanwhile, in the case of many degrees of freedom, the LE may have an exponential decay for times much shorter than the recurrence time of the LE [25].

The above-discussed semiclassical predictions for the LE decay have been tested numerically in the deep semiclassical regime in some models, in which effective Planck constants can be suitably introduced. In this regime, the effective Planck constants are sufficiently small, such that the stationary phase approximation is applicable in the derivation of the semiclassical propagator from Feynman's path integral theory. Here, we are interested in the extent to which the predictions may remain valid in the opposite deep-quantum regime. In this regime, the effective Planck constants are not very small, such that the validity of the above-mentioned stationary phase approximation becomes questionable. To study this problem, it is necessary to rely mainly on numerical simulations in concrete models. Our numerical results obtained in the sawtooth model and in the kicked rotor model show that the semiclassical predictions may work well even in the deep quantum regime.

We also study the LE decay in the vicinity of the quantum phase transition (QPT) in a one-dimensional (1D) Ising chain in transverse field. At a QPT, at which the ground level has level crossing with other level(s), the ground state has drastic change(s) in its fundamental properties [26]. Quantities borrowed from the quantum information field have been found useful in characterizing QPT, e.g., the fidelity as the overlap of ground states [27–29] and the LE [30–33]. As shown in Ref. [25], in the neighborhood of the critical point of the Ising chain, an effective Planck constant can be introduced and the semiclassical theory is useful in predicting the decaying behavior of the survival probability, which is a special case of the LE. In this paper, we study the validity of the semiclassical prediction when the effective Planck constant is increased.

The paper has the following structure. In Sec. II, we recall the semiclassical approach to the LE decay. In Sec. III, we study the LE decay in the deep-quantum region in the sawtooth model and in the kicked rotator model. Section IV is devoted to a study of the LE decay in the vicinity of the QPT of the

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1D Ising chain in a transverse field. Finally, conclusions and discussions are given in Sec. V.

II. SEMICLASSICAL APPROACH TO THE LE

Before presenting our results, let us first recall semiclassical predictions for the decay of LE. As is well known, the quantum transition amplitude from a point \mathbf{r}_0 to a point \mathbf{r} in a d -dimensional configuration space within a time period t can be expressed in terms of Feynman's path integral [34,35]. In the semiclassical limit, one may use the stationary phase approximation to approximately compute the transition amplitude. Contributions from paths close to classical trajectories give the following well-known semiclassical evolution, in terms of Van Vleck–Gutzwiller propagator $K_{sc}(\mathbf{r}; \mathbf{r}_0; t)$:

$$\Psi_{sc}(\mathbf{r}; t) = \int d\mathbf{r}_0 K_{sc}(\mathbf{r}; \mathbf{r}_0; t) \Psi_0(\mathbf{r}_0), \quad (2)$$

where $K_{sc}(\mathbf{r}; \mathbf{r}_0; t) = \sum_s K_s(\mathbf{r}; \mathbf{r}_0; t)$ and

$$K_s(\mathbf{r}; \mathbf{r}_0; t) = \frac{C_s^{1/2}}{(2\pi i\hbar)^{d/2}} \exp\left[\frac{i}{\hbar} S_s(\mathbf{r}; \mathbf{r}_0; t) - \frac{i\pi}{2} \mu_s\right]. \quad (3)$$

Here, the subscript s indicates classical trajectories, $C_s^{1/2} = |\det(\partial^2 S_s / \partial r_{i0} \partial r_j)|$, μ_s is the Maslov index counting conjugate points, and $S_s(\mathbf{r}; \mathbf{r}_0; t)$ is the action, i.e., the time integral of the Lagrangian along the trajectory s , $S_s(\mathbf{r}; \mathbf{r}_0; t) = \int_0^t dt' \mathcal{L}$.

Let us consider an initial narrow Gaussian wave packet,

$$\Psi_0(\mathbf{r}_0) = \left(\frac{1}{\pi\xi^2}\right)^{d/4} \exp\left[\frac{i}{\hbar} \tilde{\mathbf{p}}_0 \cdot \mathbf{r}_0 - \frac{(\mathbf{r}_0 - \tilde{\mathbf{r}}_0)^2}{2\xi^2}\right], \quad (4)$$

where $(\tilde{\mathbf{r}}_0, \tilde{\mathbf{p}}_0)$ indicates the packet center and ξ is the dispersion. Semiclassically, the LE is written as $M_{sc}(t) = |m_{sc}(t)|^2$, where

$$m_{sc}(t) = \int d\mathbf{r} [\Psi_{sc}^{H_1}(\mathbf{r}; t)]^* \Psi_{sc}^{H_0}(\mathbf{r}; t). \quad (5)$$

As shown in Refs. [16,36], the amplitude $m_{sc}(t)$ has the following explicit expression:

$$m_{sc}(t) \simeq \left(\frac{\xi^2}{\pi\hbar^2}\right)^{\frac{d}{2}} \int d\mathbf{p}_0 \exp\left[\frac{i}{\hbar} \Delta S(\mathbf{p}_0, \tilde{\mathbf{r}}_0; t) - \frac{(\mathbf{p}_0 - \tilde{\mathbf{p}}_0)^2}{(\hbar/\xi)^2}\right], \quad (6)$$

where $\Delta S(\mathbf{p}_0, \tilde{\mathbf{r}}_0; t)$ is the action difference along two nearby trajectories in two systems H_1 and H_0 . In the first-order classical perturbation theory, with the difference between the two trajectories neglected, one has

$$\Delta S(\mathbf{p}_0, \tilde{\mathbf{r}}_0; t) \simeq \epsilon \int_0^t dt' V[\mathbf{p}_0(t')]. \quad (7)$$

The LE amplitude in Eq. (6) can be written as an integration over ΔS . As a result, the LE is written as

$$M(t) = \left| \int d\Delta S e^{i\Delta S/\hbar} P(\Delta S) \right|^2, \quad (8)$$

where $P(\Delta S)$ is the distribution of the action difference defined by

$$P(\Delta S) \simeq \left(\frac{\xi^2}{\pi\hbar^2}\right)^{\frac{d}{2}} \int d\mathbf{p}_0 \delta[\Delta S - \Delta S(\mathbf{p}_0, \tilde{\mathbf{r}}_0; t)] \times \exp\left[-\frac{(\mathbf{p}_0 - \tilde{\mathbf{p}}_0)^2}{(\hbar/\xi)^2}\right]. \quad (9)$$

Let us first discuss the LE decay in chaotic systems. In such a system, with an average performed over initial states, the distribution $P(\Delta S)$ is usually not far from a Gaussian distribution. When the perturbation is not strong, in the so-called FGR regime, deviation of $P(\Delta S)$ from the Gaussian distribution can be neglected. In this case, the semiclassical theory predicts the following FGR decay for the LE [9]:

$$M_{sc}(t) \simeq e^{-2\sigma^2 R(E)t}, \quad (10)$$

where $\sigma = \epsilon/\hbar$ and $R(E)$ is the classical action diffusion constant,

$$R(E) = \int_0^\infty dt \{ \langle V[r(t)] - \langle V \rangle \} \{ V[r(0)] - \langle V \rangle \}, \quad (11)$$

with $\langle \cdot \rangle$ indicating the average over the primitive periodic orbits of a very long period. Consistently, similar results for the FGR decay can also be obtained in the approach of random matrix theory [7,37,38].

With increasing perturbation strength, deviation of the distribution $P(\Delta S)$ from the Gaussian form cannot be neglected, and one enters into the so-called Lyapunov regime. In this regime, due to the quadratic dependence of the FGR rate on the perturbation strength, the part of the LE having the FGR decay decreases quite fast; beyond a time scale at which this part of the LE reduces to a negligible value [39], the LE will be dominated by the contribution from the above-mentioned deviation of the distribution $P(\Delta S)$ from the Gaussian form [15]. It has been found that the latter contribution is mainly given by ΔS close to its stationary points with respect to initial momentum, and the stationary phase approximation predicts the following perturbation-independent decay [19,20]:

$$M_{sc}(t) \propto \exp[-\Lambda_1(t)t], \quad (12)$$

where

$$\Lambda_1(t) = -\frac{1}{t} \lim_{\delta x(0) \rightarrow 0} \ln \left| \frac{\delta x(t)}{\delta x(0)} \right|^{-1}, \quad (13)$$

with the average taken over initial states. One should note that $\Lambda_1(t)$ is usually not equal to the Lyapunov exponent λ_L ,

$$\lambda_L = \lim_{t \rightarrow \infty} \frac{1}{t} \lim_{\delta x(0) \rightarrow 0} \ln \left| \frac{\delta x(t)}{\delta x(0)} \right|, \quad (14)$$

due to local fluctuations. When the time t is sufficiently long such that $\Lambda_1(t)$ becomes close to its long-time limit, the LE has a decay determined by the long-time limit of $\Lambda_1(t)$ discussed in Ref. [17]. In a system with a homogeneous phase space, i.e., with a constant local Lyapunov exponent, $\Lambda_1(t)$ is given by the Lyapunov exponent and the LE has the Lyapunov decay [16].

Next, we discuss the integrable case. In a 1D regular system with periodic classical motion, the LE has the following

semiclassical expression up to a second-order perturbation contribution [24]:

$$M_{\text{sc}}(t) \simeq c_0(1 + \xi^2 t^2)^{-1/2} e^{-\Gamma t^2/(1 + \xi^2 t^2)}, \quad (15)$$

where $c_0 \sim 1$ and

$$\Gamma = \frac{1}{2} \left(\frac{\varepsilon w_p}{\hbar} \frac{\partial U}{\partial p_0} \right)^2, \quad \xi = \left| \frac{\varepsilon w_p^2}{2\hbar} \frac{\partial^2 U}{\partial p_0^2} \right|, \quad (16)$$

with the derivatives evaluated at the center of the initial Gaussian wave packet. Here, w_p is the width of the initial Gaussian wave packet in the momentum space, $U = \frac{1}{T_p} \int_0^{T_p} V dt$, and T_p is the period of the classical motion. Equation (15) shows that the LE has an initial Gaussian decay predicted in Refs. [10,11] and a long-time power-law decay $1/t$.

In the opposite case of a regular system with many degrees of freedom and many different frequencies, when the time is not long such that the classical motion does not show any sign of quasiperiodicity, the classical motion looks like a chaotic one. In this case, the LE also has an initial Gaussian decay [10,11], but it is followed by a FGR-type decay in Eq. (10) [25], with

$$R(E) = \frac{1}{2t} \left(\left\langle \left[\int_0^t V(t') dt' \right]^2 \right\rangle - \left\langle \int_0^t V(t') dt' \right\rangle^2 \right). \quad (17)$$

Finally, we note that Eq. (8) is a general expression, not restricted to the case of the semiclassical limit. To show this point, one may use Feynman's path-integral formulation. For brevity, let us write Feynman's propagator as

$$K_F(\mathbf{r}, \mathbf{r}_0; t) = \mathcal{N} \sum_{\alpha} \exp \{ i S_{\alpha}(\mathbf{r}, \mathbf{r}_0; t) / \hbar \}, \quad (18)$$

where α indicates possible paths going from \mathbf{r}_0 to \mathbf{r} within a time interval t , and \mathcal{N} is the normalization coefficient. Using this propagator, the exact time evolution of the wave function $\Psi(\mathbf{r}, t)$ can be written in a form similar to that in Eq. (2), with K_{sc} replaced by K_F . Then, substituting the expression obtained into the definition of $m(t)$ in Eq. (1), one obtains

$$m(t) = \mathcal{N} \mathcal{N}' \int d\mathbf{r}_0 \mathbf{r}'_0 \sum_{\alpha\alpha'} \exp(i \Delta S_e / \hbar) \Psi_0(\mathbf{r}_0) \Psi_0^*(\mathbf{r}'_0), \quad (19)$$

where $\Delta S_e = S_{\alpha}^{H_0} - S_{\alpha'}^{H_1}$. It is seen that the LE amplitude $m(t)$ can always be written as an integration over the exact action difference ΔS_e , with the distribution $P(\Delta S)$ defined accordingly. As a result, the LE can always be written in the form of Eq. (8).

III. LE DECAY IN THE DEEP QUANTUM REGION OF TWO KICKED SYSTEMS

In studying the validity of semiclassical predictions in the deep quantum region, it would be convenient to employ models in which effective Planck constants can be suitably introduced. In such a model, the value of the effective Planck constant gives a natural measure to the quantum "deepness."

A. Two kicked models

We employ the sawtooth model and the kicked rotator model, whose Hamiltonians have the following form:

$$H = \frac{1}{2} p^2 + V(r) \sum_{n=0}^{\infty} \delta(t - nT), \quad (20)$$

where $V(r) = -k(r - \pi)^2/2$ for the sawtooth model and $V(r) = k \cos r$ for the kicked rotator model. Here, for simplicity in the discussion, we consider their dimensionless form. Hereafter, we take the unit Planck constant, $\hbar = 1$.

The classical dynamics in the kicked rotator model generates the standard map,

$$\begin{aligned} \tilde{p}_{n+1} &= \tilde{p}_n + K \sin(r_n) \pmod{2\pi}, \\ r_{n+1} &= r_n + \tilde{p}_{n+1} \pmod{2\pi}, \end{aligned} \quad (21)$$

where $\tilde{p}_n = T p_n$, $K = kT$.

The classical motion is regular for sufficiently small K and is almost chaotic for K larger than 6 or so. In the sawtooth model, one has the following classical mapping:

$$\begin{aligned} \tilde{p}_{n+1} &= \tilde{p}_n + K(r_n - \pi) \pmod{2\pi}, \\ r_{n+1} &= r_n + \tilde{p}_{n+1} \pmod{2\pi}. \end{aligned} \quad (22)$$

Equation (22) can be written in the matrix form

$$\begin{pmatrix} \tilde{p}_{n+1} \\ r_{n+1} - \pi \end{pmatrix} = \begin{pmatrix} 1 & K \\ 1 & K + 1 \end{pmatrix} \begin{pmatrix} \tilde{p}_n \\ r_n - \pi \end{pmatrix}. \quad (23)$$

The constant matrix in the above equation possesses two eigenvalues $1 + (K \pm \sqrt{(K)^2 + 4K})/2$. The motion of the classical sawtooth model is chaotic for $K > 0$, with the Lyapunov exponent

$$\lambda_L = \ln \{ (2 + K + [(2 + K)^2 - 4]^{1/2}) / 2 \}, \quad (24)$$

given by the larger eigenvalue of the constant matrix.

We utilize the method of quantization on a torus to get the quantum versions of the above two classical systems, with periodic boundary conditions for the coordinate and momentum variables, $0 \leq r < r_m$, $0 \leq p < p_m$ [1,40–42]. For a Hilbert space with dimension N , an effective Planck constant can be introduced, denoted by h_{eff} , with $h_{\text{eff}} = r_m p_m / N$. In the specific choice of $r_m = p_m = 2\pi$, which will be taken in what follows, one has $h_{\text{eff}} = (2\pi)^2 / N$, hence $\hbar_{\text{eff}} = 2\pi / N$. The value of h_{eff} gives a measure to the "deepness" in the quantum region. The evolution operator for one period of time T , with $T = 2\pi / N = \hbar_{\text{eff}}$, is written as

$$U = \exp \left[-\frac{i}{2\hbar_{\text{eff}}} \tilde{p}^2 \right] \exp \left[-\frac{i}{\hbar_{\text{eff}}} \tilde{V}(r) \right], \quad (25)$$

where $\tilde{V}(r) = TV(r)$.

In the two kicked models discussed above, the quantity $R(E)$ appearing in the FGR decay in Eq. (10) has the following expression [43,44]:

$$R(E) = \frac{1}{2} C(0) + \sum_{l=1}^{\infty} C(l), \quad (26)$$

where $C(l) = \{ \langle V[r(l)] \rangle - \langle V \rangle \} \{ \langle V[r(0)] \rangle - \langle V \rangle \}$. In the sawtooth model with an integer K , $C(0) = \pi^4/45$ and $C(l) = 0$

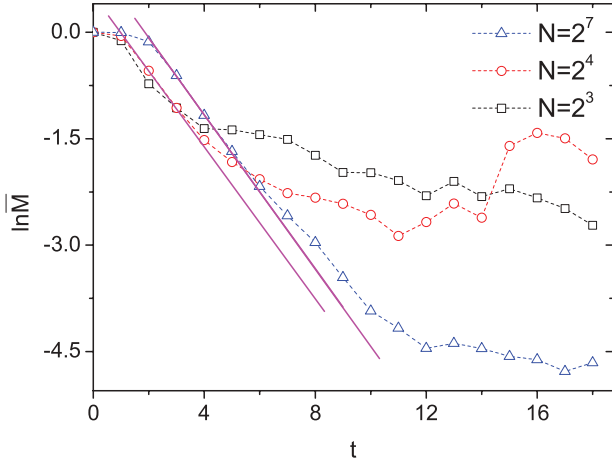


FIG. 1. (Color online) LE decay in the FGR regime in the sawtooth model, for different values N of the dimension of the Hilbert space. Parameters: $K = 2.0$ and $\sigma = 0.5$. The LE (averaged over initial states) has two stages of decay. In the first stage, it follows the semiclassically predicted FGR decay (solid straight lines), namely $e^{-\sigma^2 \pi^4 t / 45}$ in Eq. (27); in the second stage, it decays with slower rates.

for $l \neq 0$, hence

$$M_{\text{sc}}(t) \simeq e^{-\pi^4 \sigma^2 t / 45}. \quad (27)$$

Meanwhile, in the kicked rotator model, $R(E)$ is a function of the parameter K and does not have an explicit analytical expression.

B. Numerical results in the sawtooth model

In this subsection, we discuss our numerical simulations obtained in the sawtooth model, with Gaussian wave packets as the initial states. In the FGR regime, when N is not large, it was found that, beyond some initial times, the LE has two stages of decay (see Fig. 1): In the first stage, the LE follows the semiclassically predicted FGR decay, while, in the second stage, it is somewhat slower than the FGR decay. After these two stages of decay, the LE oscillates around its saturation value, which is on average approximately equal to $1/N$ [11]. Interestingly, the first-stage decay of the LE exists even for small dimension N of the Hilbert space, in other words, for values of the effective Planck constant not much smaller than its upper border $\hbar_{\text{eff}}^{\text{ub}} = 2\pi$. Hence, it exists in the deep quantum regime.

Let us use t_d to indicate the transition time of the above-discussed two stages of decay of the LE, i.e., the time at which an obvious deviation from the FGR decay appears. As seen in Fig. 2, t_d increases with increasing N . In addition, we observe that, when N is increased, the second-stage decay of the LE approaches the FGR decay, that is, the difference between the decay rates of the two stages decreases.

To get a further understanding in the above-discussed first and second stages of decay of the LE, let us reconsider the expression of the LE in Eq. (8). As discussed previously, Eq. (8) is not just a semiclassical expression, but is an exact expression, if the distribution $P(\Delta S)$ is appropriately defined in terms of contributions from Feynman paths. The distribution $P(\Delta S)$ always has some deviation from its Gaussian approximation,

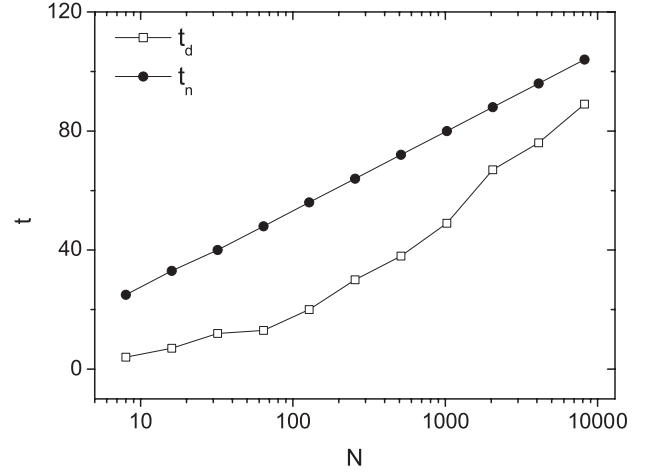


FIG. 2. Variation of t_d (empty squares) with N in the sawtooth model, where t_d is the time at which the second-stage decay of the LE in the FGR regime appears (see Fig. 1). For comparison, we also plot t_n (solid circles), the time at which the FGR decay is expected to reach the saturation value $1/N$, i.e., $t_n = [45 \ln(N)] / (\sigma^2 \pi^4)$. Parameters: $K = 2.0$, $\sigma = 0.2$.

which we denote by P_G , with G standing for Gaussian, that is,

$$P(\Delta S) = P_G + \Delta P. \quad (28)$$

The above-discussed numerical results imply that the deviation ΔP is not sufficiently large for times shorter than t_d (beyond some initial times). As a result, the LE still follows the FGR decay.

However, for times beyond t_d , the deviation cannot be neglected. In fact, the deviation ΔP has mainly two sources: One comes from contributions not included in the stationary phase approximation, which has been used when deriving the semiclassical propagator from Feynman's path integral formulation. The other is related to the fact that the right-hand side of Eq. (9) for classical trajectories does not have an exact Gaussian form. The second-stage non-FGR decay of the LE appears for quite small values of the dimension N , which correspond to values of the effective Planck constant not much smaller than the upper border $\hbar_{\text{eff}}^{\text{ub}}$. This implies that its deviation from the FGR decay might have a nonsemiclassical origin, i.e., the above-mentioned first factor might play the major role here.

Next, we discuss the Lyapunov regime in the sawtooth model. In this regime, we did not observe a two-stage decay similar to that discussed above in the FGR regime. For large N , as shown in previous work [15], the LE has approximately the semiclassically predicted Lyapunov decay, with the decaying rate given by the Lyapunov exponent. (The sawtooth model has a homogeneous phase space.) With decreasing N , as shown in Fig. 3, the decay of the LE gradually deviates from the Lyapunov decay.

To quantitatively characterize the above-discussed deviation of the exact LE decay from the semiclassical prediction in the Lyapunov regime, we have studied the standard deviation of $x_n \equiv |\ln \overline{M}_e(t = n) - \ln M_{\text{sc}}(t = n)|$, where $\overline{M}_e(t)$ denotes the exact numerical result. That is, we have studied the

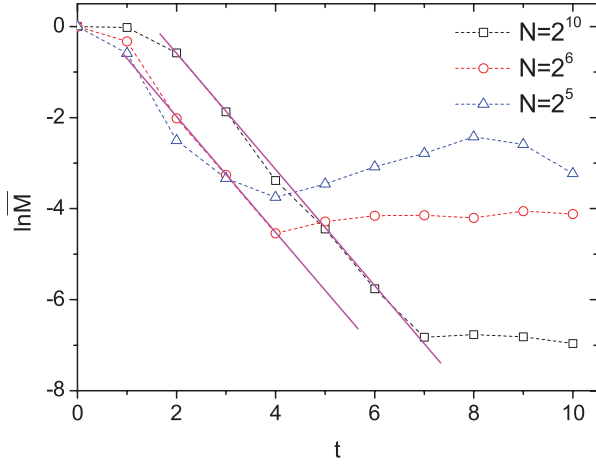


FIG. 3. (Color online) Similar to Fig. 1, but for $\sigma = 3.0$ in the Lyapunov regime of the sawtooth model. The LE has a decay close to the semiclassically predicted Lyapunov decay, namely $e^{-\lambda_L t}$ (solid lines), for $N \geq 2^6$.

quantity D ,

$$D \equiv \sqrt{\frac{1}{M} \sum_{n=1}^M (x_n - \bar{x})^2}, \quad (29)$$

where $\bar{x} = \frac{1}{M} \sum_{n=1}^M x_n$ is the average value of x_n . Our numerical simulations show that the value of D remains small for large N and becomes not small when N is below some value, which we denote by N_c (see Fig. 4). That is, the semiclassical prediction for the LE decay works well for N above N_c , but not well for N below N_c .

The value of N_c was found to be dependent on the parameter σ , as shown in Fig. 5. On average, N_c increases with increasing σ . This dependence may be related to a requirement used in the derivation of the above-mentioned semiclassical predictions for the LE decay, namely ϵ being small. Indeed, due to the

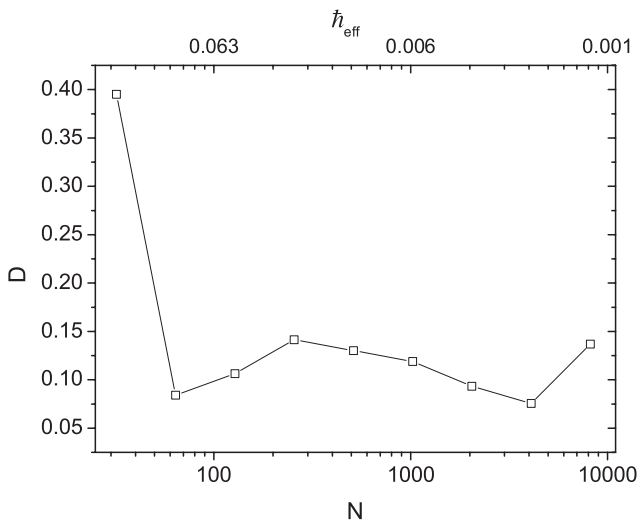


FIG. 4. Variation of the deviation D in Eq. (29) with N in the Lyapunov regime of the sawtooth model, with parameters $K = 2.0$ and $\sigma = 3.0$. The value of D remains small for $N \geq N_c = 64$ and becomes large when N is smaller than N_c .

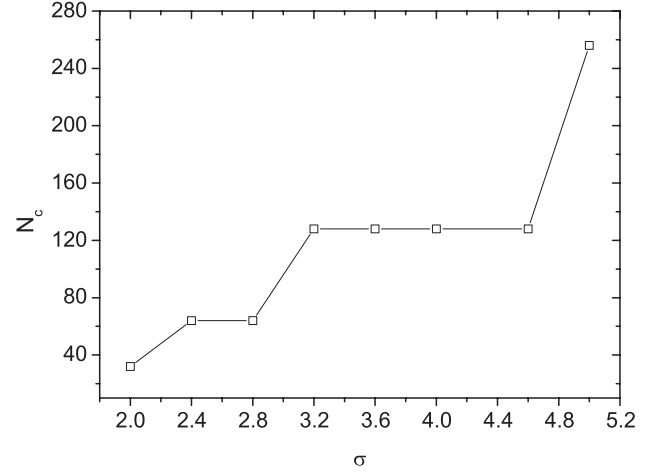


FIG. 5. Variation of N_c with σ in the Lyapunov regime of the sawtooth model with parameter $K = 1.0$.

relation $\epsilon = \sigma \hbar_{\text{eff}} = 2\pi\sigma/N$, to keep ϵ at a fixed small value, N should be proportional to σ .

C. Numerical results in the kicked rotator model

In the kicked rotator model, numerically we found the behaviors of the LE to be more or less similar to those in the sawtooth model discussed in the previous subsection, also with Gaussian wave packets as the initial states. Specifically, in the FGR regime, when N is not large, we also observed a two-stage decay of the LE. But, in this model, the second-stage decay is faster than the first-stage FGR decay. See Fig. 6 for some examples, where the value of $R(E)$ in the FGR decay was computed numerically, making use of Eq. (26).

The kicked rotator does not have a homogeneous phase space. Hence in the Lyapunov regime, the semiclassical prediction for the LE decay is not given by the Lyapunov exponent of the underlying classical dynamics, but is given by Eq. (12). As expected, only for large N did we numerically find agreement between the prediction of Eq. (12) and the exact LE

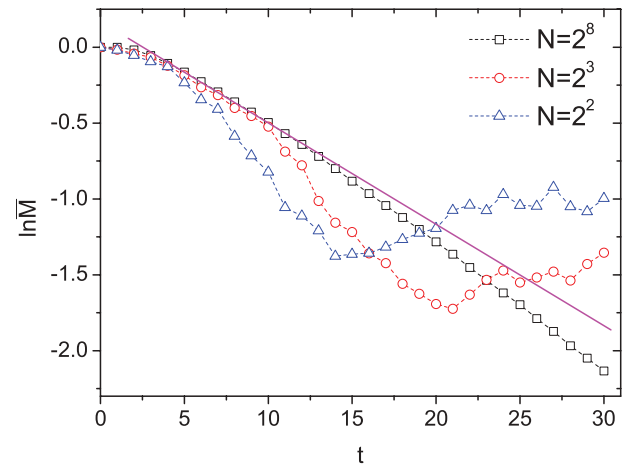


FIG. 6. (Color online) Same as in Fig. 1 for the kicked rotator model with parameters $K = 11.0$, $\sigma = 0.3$ in the FGR regime. The semiclassical prediction of the FGR decay, $e^{-2\sigma^2 R(E)t}$ with $R(E) = 0.375$, is indicated by a solid line.

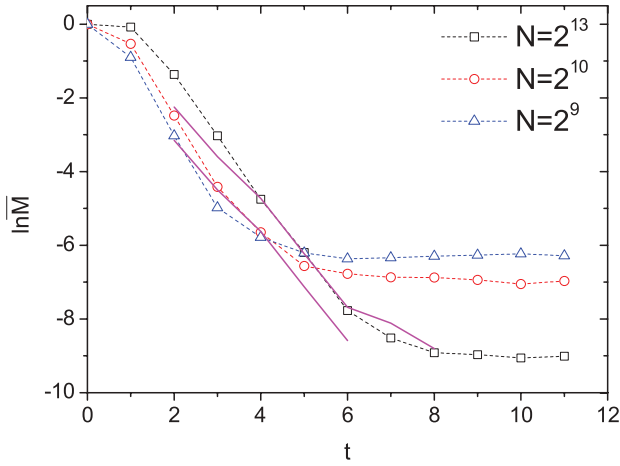


FIG. 7. (Color online) Similar to Fig. 6, but for the Lyapunov regime with parameters $K = 15.0$ and $\sigma = 20.5$ in the kicked rotator model. The semiclassical prediction, given by Eq. (12), is indicated by solid lines.

decay beyond some initial time. Some examples are given in Fig. 7, where it is seen that the agreement is good for $N = 2^{13}$ with $t \in [4, 8]$ while the agreement is not good for $N \leq 2^9$. In fact, for $N = 2^9$, the LE approaches its saturation value before the semiclassically predicted decay in Eq. (12) can be seen.

IV. LE DECAY IN THE VICINITY OF A QUANTUM PHASE TRANSITION

In this section, we discuss the LE decay in the vicinity of a critical point of a 1D Ising chain in a transverse field, with N_p spins. As shown in Ref. [25], the semiclassical theory is useful in predicting the LE decay in the close neighborhood of those QPTs whose ground levels are infinitely degenerate at the critical points. The closer the controlling parameter λ is to the critical point λ_c , the better the semiclassical theory might work. For this Ising chain, the semiclassical theory predicts an exponential decay for relatively long times.

The Ising chain undergoes a QPT at the critical point in the thermodynamic limit $N_p \rightarrow \infty$. As will be discussed below, an effective Planck constant can be introduced in the low-energy region in this model, which is inversely proportional to N_p . We will be studying the extent to which the above-mentioned semiclassical prediction for an exponential decay of the LE may remain valid when the value of N_p is decreased.

The dimensionless Hamiltonian of the 1D Ising chain is written as

$$H(\lambda) = - \sum_{i=1}^{N_p} (\sigma_i^z \sigma_{i+1}^z + \lambda \sigma_i^x). \quad (30)$$

The spin-spin interaction intends to force the spins to polarize along the z direction, while the transverse field intends to polarize them along the x direction. Competition between the two interactions results in two critical points, $\lambda_c = \pm 1$, with the ferromagnetic phase for $-1 < \lambda < 1$ and the paramagnetic phase for $|\lambda| > 1$. Without a loss of generality, we consider the critical point $\lambda_c = 1$.

The above Ising Hamiltonian can be diagonalized by utilizing the Jordan-Wigner and Bogoliubov transformations,

giving [26,45,46]

$$H(\lambda) = \sum_k e_k (b_k^\dagger b_k - 1/2), \quad (31)$$

where b_k^\dagger and b_k are fermionic creation and annihilation operators, e_k is the corresponding single quasiparticle energy,

$$e_k = 2\sqrt{1 + \lambda^2 - 2\lambda \cos(ka)}, \quad (32)$$

and $k = 2\pi m/aN_p$, with $m = -M, -M+1, \dots, M$. Here, a is the lattice spacing and $M = (N_p - 1)/2$.

As discussed in Ref. [25], in the very neighborhood of the critical point with λ sufficiently close to λ_c and for sufficiently large N_p , the low-lying states have single-particle energies $e_k \approx (4\pi|m|)/N_p$ and can be mapped to bosonic modes by the method of bosonization [26]. A bosonic mode, labeled by α , has a single-particle energy $e_\alpha^b \approx n_\alpha \delta E$, where $n_\alpha = 1, 2, \dots$ and $\delta E = 4\pi/N_p$. This expression of the single-particle energy e_α^b suggests that an effective Planck constant \hbar_{eff} may be introduced,

$$\hbar_{\text{eff}} = \delta E = 4\pi/N_p, \quad (33)$$

which gives $e_\alpha^b = \hbar_{\text{eff}} \omega_\alpha$, with $\omega_\alpha \approx n_\alpha$. In the case in which the frequencies ω_α are sufficiently incommensurable, the classical counterpart has a motion like a chaotic one when the time is not long. Then, as discussed in Sec. II, the semiclassical theory predicts the exponential decay in Eq. (10) with $R(E)$ given by Eq. (17).

In computing the LE, H_0 in its definition in Eq. (1) is taken as $H(\lambda_0)$ and H_1 as $H(\lambda)$. The initial state $|\Psi_0\rangle$ is chosen as the ground state of $H(\lambda_0)$; in this case, the LE is in fact a survival probability. Numerically, the LE was found to have an initial Gaussian decay, as predicted in Ref. [30]. For large values of N_p , the semiclassically predicted exponential decay was also observed for relatively long times, i.e., beyond the initial Gaussian decay and before the revival time [25]. However, when N_p is decreased to some value, denoted by N_d , an obvious deviation from the exponential decay has been observed (see Fig. 8).

Figure 8 shows that for large N_p , the LE has a good scaling behavior, $\ln M(t) \sim N_p$. To understand this phenomenon, we note that here the perturbation ϵV in the definition of the LE takes the form of $\epsilon = (\lambda_0 - \lambda)$ and

$$V = \sum_{i=1}^{N_p} \sigma_i^x. \quad (34)$$

Then, according to Eq. (17), the quantity $R(E)$ is given by the square of the summation of N_p terms with mean zero, each of which is a time integration of $\epsilon(\sigma_i^x - \overline{\sigma_i^x})$. As discussed above, the classical counterpart has a motion like a chaotic one, hence the time integrations mentioned above can usually be regarded as being uncorrelated. As a result, for large N_p , the quantity $R(E)$ is approximately proportional to N_p , hence $\ln M(t) \sim N_p$.

To see more clearly the process of the above-discussed deviation of the LE from the semiclassically predicted exponential decay, we have calculated the deviation D in Eq. (29) for $x_t = |\ln \overline{M}_c(t) - \ln M_{\text{sc}}(t)|/N_p$ (see Fig. 9). In our computation, $(\ln M_{\text{sc}})/N_p$ was computed in the large- N_p

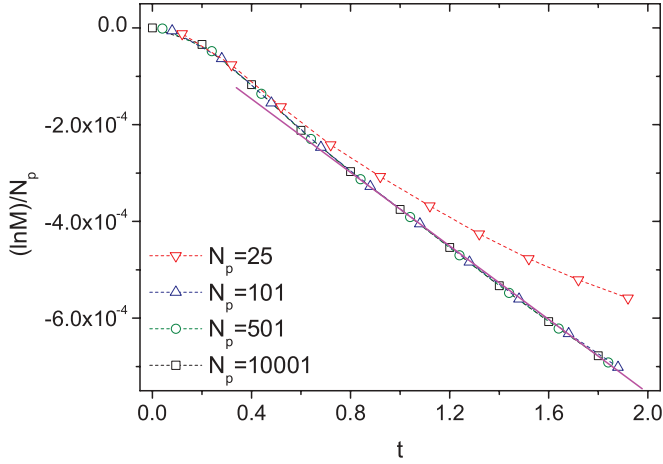


FIG. 8. (Color online) LE decay of the Ising chain for different values of N_p , with parameters $\lambda_0 = \lambda_c - 4 \times 10^{-2}$, $\lambda = \lambda_c - 10^{-2}$, and $\lambda_c = 1.0$. For large N_p and relatively long times, the LE has an exponential decay as predicted by the semiclassical theory. (A solid straight line is drawn to guide the eyes.) Note also that the LE has a good scaling behavior of $\ln M \propto -N_p t$ for large N_p . For relatively small N_p ($N_p = 25$), the LE has neither the exponential decay nor the scaling behavior.

limit. It is seen in Fig. 9 that an obvious deviation from the semiclassically predicted exponential decay appears at $N_d \approx 100$.

Furthermore, we found that the value of N_d has a strong dependence on $\delta\lambda = \lambda - \lambda_c$, as shown in Fig. 10. Specifically, N_d is almost inversely proportional to $\delta\lambda$. Therefore, the value of N_d can be not large for $\delta\lambda$ not very small. However, for quite small $\delta\lambda$, N_d can be very large. Since a large value of N_d implies “deep” in the semiclassical regime, it is reasonable to expect that this deviation from the semiclassically predicted exponential decay may be due to the invalidity of some approximation used in the semiclassical derivation.

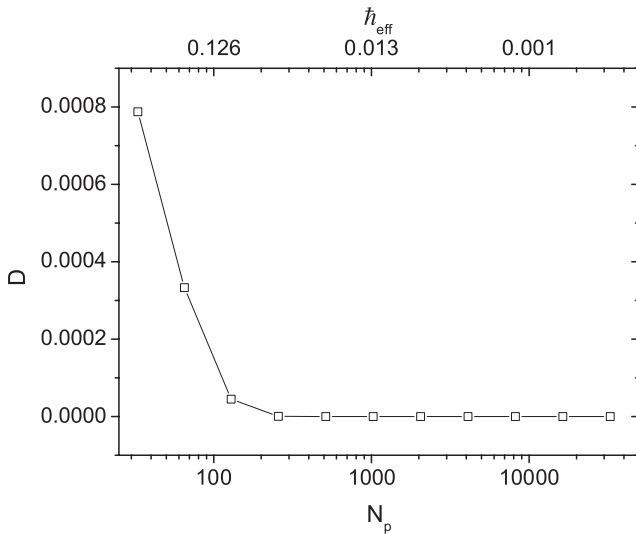


FIG. 9. Variation of the deviation D with the spin number N_p in the Ising model, with parameters $\lambda_0 = \lambda_c - 4 \times 10^{-2}$ and $\lambda = \lambda_c - 10^{-2}$. D is large for $N_p < N_d \approx 100$.

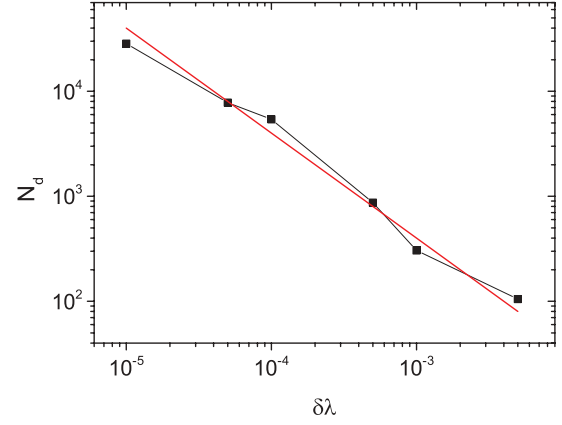


FIG. 10. (Color online) Variation of N_d with the distance $\delta\lambda$ in the Ising model, with parameters $\lambda_0 = \lambda_c - \delta\lambda$ and $\lambda = \lambda_0 - \delta\lambda$. The solid line represents $N_d = 2/(5\delta\lambda)$.

Indeed, as shown below, the above-mentioned deviation can be explained by approximate commensurability of the frequencies ω_α , which may invalidate the derivation for the exponential decay. Let us go back to the single-particle energy e_k in Eq. (32) and get its approximate expression for large N_p and small $|m|$, with the λ dependence written explicitly,

$$e_k \simeq \frac{4\pi}{N_p} |m| \sqrt{\lambda + G_\lambda^2}, \quad (35)$$

where

$$G_\lambda = \frac{N_p \delta\lambda}{2\pi m}. \quad (36)$$

Note that for $\lambda = \lambda_c = 1$, this expression gives the approximation used previously, namely $e_k \approx (4\pi|m|)/N_p$. When the term G_λ^2 is small compared with 1, one can argue that the low-lying states of the model can still be mapped to bosonic modes. For λ close to λ_c , the frequencies of the bosonic modes are written as

$$\omega_\alpha \simeq n_\alpha \sqrt{1 + G_\lambda^2}. \quad (37)$$

For the LE to have FGR-type exponential decay, ω_α should be sufficiently incommensurable. Hence the term G_λ^2 cannot be very small, i.e., G_λ should be larger than some small quantity. Obviously, the breakdown dimension N_d estimated in this way is inversely proportional to $\delta\lambda$, in agreement with numerical results given in Fig. 10.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have studied the change from validity to breakdown of some semiclassical predictions for the LE decay in several models, when the effective Planck constants are increased and the systems move from the semiclassical region to the deep quantum region. Our numerical results show that some semiclassical predictions for the LE decay work well even in the deep quantum region.

In particular, in the FGR regime with intermediate perturbation strength in the two quantum chaotic systems studied, there is always some time interval within which the LE follows the FGR decay; the length of this time interval decreases when the effective Planck constant is increased. Making use of an exact

expression of the LE, which is obtained resorting to Feynman's path integral formulation of quantum mechanics, it is argued that this phenomenon should be universal for quantum chaotic systems. This is in agreement with the fact that the same FGR decay can also be derived by other methods, namely by the random matrix theory [7] and by a linear response theory [10,11]. Still in the FGR regime, beyond the time interval discussed above, deviation of the LE from the FGR decay has been observed in the two chaotic systems in the deep quantum region. This deviation is expected to be induced by nonsemiclassical contributions and may also appear in other chaotic models.

In the Lyapunov regime with relatively strong perturbation, a different situation has been found. In particular, the semiclassical prediction has been found to be invalid in a sufficiently deep quantum region. This difference from the FGR regime is understandable, since the mechanism for the LE decay is different in the two regimes.

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- [1] F. Haake, *Quantum Signatures of Chaos*, 2nd ed. (Springer-Verlag, Berlin, 2001).
- [2] M. Brack and R. K. Bhaduri, *Semiclassical Physics* (Addison-Wesley, Boston, 1997).
- [3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
- [4] G. Benenti, G. Casati, and G. Strini, *Principles of Quantum Computation and Information* (World Scientific, Singapore, 2004).
- [5] A. Peres, *Phys. Rev. A* **30**, 1610 (1984).
- [6] D. A. Wisniacki, *Phys. Rev. E* **67**, 016205 (2003).
- [7] Ph. Jacquod, P. G. Silvestrov, and C. W. J. Beenakker, *Phys. Rev. E* **64**, 055203(R) (2001).
- [8] T. Prosen, T. H. Seligman, and M. Žnidarič, *Prog. Theor. Phys.* **150**, 200 (2003).
- [9] N. R. Cerruti and S. Tomsovic, *Phys. Rev. Lett.* **88**, 054103 (2002); *J. Phys. A* **36**, 3451 (2003).
- [10] T. Prosen, *Phys. Rev. E* **65**, 036208 (2002).
- [11] T. Prosen and M. Žnidarič, *J. Phys. A* **34**, L681 (2001); **35**, 1455 (2002).
- [12] B. Gutkin, D. Waltner, M. Gutierrez, J. Kuipers, and K. Richter, *Phys. Rev. E* **81**, 036222 (2010).
- [13] D. A. Wisniacki, E. G. Vergini, H. M. Pastawski, and F. M. Cucchietti, *Phys. Rev. E* **65**, 055206 (2002).
- [14] D. A. Wisniacki and D. Cohen, *Phys. Rev. E* **66**, 046209 (2002).
- [15] W.-G. Wang, G. Casati, and B. Li, *Phys. Rev. E* **69**, 025201(R) (2004).
- [16] R. A. Jalabert and H. M. Pastawski, *Phys. Rev. Lett.* **86**, 2490 (2001).
- [17] P. G. Silvestrov, J. Tworzydło, and C. W. J. Beenakker, *Phys. Rev. E* **67**, 025204(R) (2003).
- [18] F. M. Cucchietti, H. M. Pastawski, and R. A. Jalabert, *Phys. Rev. B* **70**, 035311 (2004).
- [19] W.-G. Wang, G. Casati, B. Li, and T. Prosen, *Phys. Rev. E* **71**, 037202 (2005).
- [20] W.-G. Wang and B. Li, *Phys. Rev. E* **71**, 066203 (2005).
- [21] N. Ares and D. A. Wisniacki, *Phys. Rev. E* **80**, 046216 (2009).
- [22] I. García-Mata and D. A. Wisniacki, *J. Phys. A* **44**, 315101 (2011).
- [23] Ph. Jacquod, I. Adagideli, and C. W. J. Beenakker, *Europhys. Lett.* **61**, 729 (2003).
- [24] W.-G. Wang, G. Casati, and B. Li, *Phys. Rev. E* **75**, 016201 (2007).
- [25] W.-G. Wang, P.-Q. Qin, L. He, and P. Wang, *Phys. Rev. E* **81**, 016214 (2010).
- [26] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).
- [27] P. A. Lee, N. Nagaosa, and X.-G. Wen, *Rev. Mod. Phys.* **78**, 17 (2006).
- [28] P. Zanardi and N. Paunkovic, *Phys. Rev. E* **74**, 031123 (2006).
- [29] P. Zanardi, P. Giorda, and M. Cozzini, *Phys. Rev. Lett.* **99**, 100603 (2007).
- [30] H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, *Phys. Rev. Lett.* **96**, 140604 (2006).
- [31] Z. G. Yuan, P. Zhang, and S. S. Li, *Phys. Rev. A* **75**, 012102 (2007).
- [32] Y. C. Li and S. S. Li, *Phys. Rev. A* **76**, 032117 (2007).
- [33] D. Rossini, T. Calarco, V. Giovannetti, S. Montangero, and R. Fazio, *Phys. Rev. A* **75**, 032333 (2007).
- [34] R. P. Feynman, *Rev. Mod. Phys.* **20**, 367 (1948).
- [35] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integral* (McGraw-Hill, New York, 1965).
- [36] J. Vaníček and E. J. Heller, *Phys. Rev. E* **68**, 056208 (2003).
- [37] Ph. Jacquod, I. Adagideli, and C. W. J. Beenakker, *Phys. Rev. Lett.* **89**, 154103 (2002).
- [38] F. M. Cucchietti, C. H. Lewenkopf, E. R. Mucciolo, H. M. Pastawski, and R. O. Vallejos, *Phys. Rev. E* **65**, 046209 (2002).
- [39] In the case in which this time scale is not very short, the LE may show a clear FGR decay before the perturbation-independent decay appears. This phenomenon has been observed numerically in Ref. [23].
- [40] J. H. Hannay and M. V. Berry, *Physica D* **1**, 267 (1980).
- [41] J. Ford, G. Mantica, and G. H. Ristow, *Physica D* **50**, 493 (1991).
- [42] J. Wilkie and P. Brumer, *Phys. Rev. E* **49**, 1968 (1994).
- [43] A. Lakshminarayanan, N. R. Cerruti, and S. Tomsovic, *Phys. Rev. E* **60**, 3992 (1999).
- [44] N. R. Cerruti, A. Lakshminarayanan, J. H. Lefebvre, and S. Tomsovic, *Phys. Rev. E* **63**, 016208 (2000).
- [45] E. Lieb, T. Schultz, and D. Mattis, *Ann. Phys. (NY)* **16**, 407 (1961).
- [46] P. Pfeuty, *Ann. Phys. (NY)* **57**, 79 (1970).