Bounds of percolation thresholds on hyperbolic lattices

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We analytically study bond percolation on hyperbolic lattices obtained by tiling a hyperbolic plane with constant negative Gaussian curvature. The quantity of our main concern is p_{c2}, the value of occupation probability where a unique unbounded cluster begins to emerge. By applying the substitution method to known bounds of the order-5 pentagonal tiling, we show that $p_{c2} \geqslant 0.382\,508$ for the order-5 square tiling, $p_{c2} \geqslant 0.472\,043$ for its dual, and $p_{c2} \geqslant 0.275\,768$ for the order-5-4 rhombille tiling.

I. INTRODUCTION

Hyperbolic geometry is an important model of non-Euclidean geometry where mathematicians have devoted a great deal of efforts since Carl Friedrich Gauss [1]. In the context of statistical physics, hyperbolic geometry has served as a conceptual setting to understand geometric frustrations in glassy materials [2]. It also has a nontrivial connection to the two-dimensional conformal field theory [3] and there have been attempts to identify it with the underlying geometry of complex networks [4]. For this reason, basic understanding of physical processes on this geometry is expected to be relevant in a wider context of statistical physics as well. If we are to discretize a surface by means of regular tiling to study physical systems defined on a lattice, in particular, hyperbolic geometry provides infinitely more possibilities than the Euclidean geometry: Let $\{p,q\}$ denote tiling where q regular p-gons meet at each vertex. This bracket representation is called the Schläfli symbol. It is easy to see that a flat plane admits only three possibilities: $\{3,6\}$ (triangular), $\{4,4\}$ (square), and $\{6,3\}$ (honeycomb), while every $\{p,q\}$ such that $(p-2)(q-2) > 4$ describes a hyperbolic plane with constant negative Gaussian curvature. In other words, each pair of such $\{p,q\}$ defines a hyperbolic lattice that can completely cover the infinite hyperbolic plane with translational symmetry. The most important physical property of a hyperbolic plane is that the area of a circle on it is an exponential function of the radius, which means that the circumference increases exponentially, too. Therefore, choosing any finite domain on a hyperbolic lattice, we find that the vertices at the boundary always occupy a finite portion of the whole number of vertices inside the domain even if the domain is very large. This property is called nonamenable in literature [5] and makes essential differences in many physical systems from their planar counterparts.

Percolation is a simple yet most interesting problem of fully geometric nature, asking the possibility of a global connection out of local connections [6]. Let us introduce the bond percolation problem, which will be studied in this work: For a given structure of sites and bonds linking them, suppose that each bond is open with probability p and closed with $1 - p$, where the parameter p is called occupation probability. One fundamental question in percolation is to find a critical

value $p = p_c$ where an unbounded cluster of open bonds begins to form. This question has been already answered for the three regular ways of tiling a flat plane [7–9] and also for more general ones provided that they allow a generalized cell–dual-cell transformation [10]. On these flat lattices, there exists a unique p_c above which the largest cluster occupies a finite fraction of the system, and the length scale of this cluster becomes unbounded at this point. On a hyperbolic plane, on the other hand, studies of percolation started about one decade ago [5,11]. The most remarkable prediction here is that there generally exist two different percolation thresholds p_{c1} and p_{c2} with $p_{c1} < p_{c2}$, so that an unbounded cluster begins to appear at p_{c1} while a *unique* unbounded cluster is observed only when p reaches a higher value, p_{c2}. Note that this is a consequence of the nonamenable property [6] and that these two thresholds coalesce on a flat plane by $p_{c1} = p_{c2} = p_c$. Numerical calculations have qualitatively supported this mathematical prediction [12], but a direct numerical estimate of p_{c2} is usually a difficult task since the system size increases exponentially as the length scale grows. This has led to a debate about p_{c2} on some hyperbolic structures [13,14].

Recently, nontrivial upper bounds of p_{c1} for self-dual tiling $\{m,m\}$ were derived by a combinatorial argument [15,16]. If $m = 5$, for example, $p_{c1}^{\{5,5\}}$ is bounded as $1/4 \leqslant p_{c1}^{\{5,5\}} \leqslant 0.381\,296$. The upper bound is a solution of the following polynomial equation:

$$-2 + 3p + 3p^2 - 567p^3 + 6721p^4 - 35\,655p^5$$
$$+ 115\,505p^6 - 257\,495p^7 + 41\,8210p^8 - 509\,100p^9$$
$$+ 469\,900p^{10} - 328\,480p^{11} + 171\,560p^{12} - 65\,000p^{13}$$
$$+ 16\,900p^{14} - 2700p^{15} + 200p^{16} = 0,$$

and the lower bound originates from the simple fact that

$$p_{c1} \geqslant 1/(n-1) \tag{1}$$

for a lattice $\{m,n\}$ with coordination number n. This work is important in two aspects: First, it showed the possibility of rigorous analytic bounds free from any numerical ambiguities. Second, it dealt with the problem from a new point of view, that is, in terms of the capacity of a quantum erasure channel. In this Brief Report, we point out that the nontrivial bounds in Ref. [16] also imply nontrivial bounds of other hyperbolic lattices. Specifically, it is made possible by using the substitution method [17], and the lattices considered here

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will be endowed with transitivity to remove any undesirable boundary effects, which allows us to exploit duality properties among the lattices, too. In the next section, we will briefly explain the substitution method and then show how to apply it to hyperbolic lattices as well as the results in Sec. III. This work is summarized in Sec. IV.

II. SUBSTITUTION METHOD

The easiest way to explain the substitution method is to begin with the star-triangle transformation [6,17,18]. Then, we proceed to other cases such as star-square and star-pentagon transformations, which will be used in our problem. Consider a lattice $\mathcal{L}$ with congruent n-gons as its basic building blocks. By drawing an n-star with n-bonds inside every n-gon, we obtain another lattice $\mathcal{L}'$, where the occupation probability is denoted as q in order to avoid confusion with p of $\mathcal{L}$.

A. Triangle

If $n = 3$, $\mathcal{L}$ is the triangular lattice [Fig. 1(a)], whereas if $n = 4$, $\mathcal{L}$ is the square lattice [Fig. 1(b)]. To explain the substitution method in a simple manner, we consider the $n = 3$ case. Suppose that we happen to know the percolation threshold q_c of $\mathcal{L}'$. We wish to find bounds of p_c on $\mathcal{L}$ from the knowledge of q_c. Consider one of the triangles T in $\mathcal{L}$ and its corresponding star T' in $\mathcal{L}'$ so that T and T' share the boundary vertices, A, B, and C. We look at all the possible cases of connection among the boundary vertices A, B, and C on T and T', respectively. Suppose that $\mathcal{L}$ is in the supercritical state and $\mathcal{L}'$ is at the critical state (percolating phase). Then, we can make the following qualitative statement: it is probable that T has more connectivity among its boundary vertices than T'. To transform this qualitative statement into a quantitative expression, we introduce some combinatorial concepts. Let $\mathcal{S}$ be a set of all the possible partitions of boundary vertices A, B, and C: If A and B are connected by open bonds and C is separated from them, the representation is partition $AB|C$. A set of connected vertices in a partition will be called a block. For partition $AB|C$, AB and C are two distinct blocks. And for n boundary vertices in general, there are nth order Bell numbers of elements in $\mathcal{S}$ [18]. The set $\mathcal{S}$ can be a partially ordered set if we define an order: For two partitions π and σ of $\mathcal{L}$, we define $\pi \leqslant \sigma$ if and only if for a block b_π of π, there exists a block b_σ containing b_π on σ. And we say that π is more refined than σ, or that π is a refinement of σ. For example, $A|B|C \leqslant AB|C$ and $AB|C \leqslant ABC$. But there is no order between $AB|CD$ and $ABC|D$ because there is no block in $AB|CD$ which covers ABC of $ABC|D$ and vice versa. A subset $\mathcal{U}$ of $\mathcal{S}$ is an up-set if and only if for $\pi_1, \pi_2 \in \mathcal{S}$, $\pi_1 \in \mathcal{U}$ and $\pi_1 \leqslant \pi_2$, then $\pi_2 \in \mathcal{U}$. By this definition, we can generate

nine up-sets as follows:

$$\mathcal{U}_0 = \mathcal{S}, \quad \mathcal{U}_1 = \{ABC\}, \quad \mathcal{U}_2 = \{ABC, A|BC\},$$
$$\mathcal{U}_3 = \{ABC, AB|C\}, \quad \mathcal{U}_4 = \{ABC, B|AC\},$$
$$\mathcal{U}_5 = \{ABC, A|BC, AB|C\}, \quad \mathcal{U}_6 = \{ABC, AB|C, B|AC\},$$
$$\mathcal{U}_7 = \{ABC, A|BC, B|AC\},$$
$$\mathcal{U}_8 = \{ABC, A|BC, AB|C, B|AC\}.$$

Let $P_p(\mathcal{U})$ and $Q_q(\mathcal{U})$ be probabilities that T and T' form an element partition of up-set $\mathcal{U}$, respectively. Then, we rewrite the qualitative statement as such $P_p(\mathcal{U}) \geqslant Q_{q_c}(\mathcal{U})$ for every up-set $\mathcal{U}$ of $\mathcal{S}$ with probability 1. We can solve this inequality with respect to p and get a solid interval of $p_m \leqslant p \leqslant 1$, for $P_p(\mathcal{U})$ is known as a monotone increasing polynomial function of p with $P_{p=0}(\mathcal{U}) = 0$ and $P_{p=1}(\mathcal{U}) = 1$ for every up-set $\mathcal{U} \neq \mathcal{S}$ while $Q_{q_c}(\mathcal{U})$ is a constant. Here, p_m is a lower bound of the percolation threshold of $\mathcal{L}$ since if p is smaller than p_m, the above inequality cannot hold and $\mathcal{L}$ cannot be in the supercritical state. On the contrary, suppose that $\mathcal{L}$ is in the subcritical state and $\mathcal{L}'$ is at the critical state. Similarly, we can conclude $P_p(\mathcal{U}) \leqslant Q_{q_c}(\mathcal{U})$. By solving this inequality again, we get another interval, $0 \leqslant p \leqslant p_M$, where p_M is an upper bound of the percolation threshold of $\mathcal{L}$. By taking the intersection of the two intervals, we obtain $p_m \leqslant p \leqslant p_M$. This is the interval in which the percolation threshold of $\mathcal{L}$ can exist. In $n = 3$ case, the resulting set of inequalities with respect to all its up-sets is found as $P_\mathcal{L}[\mathcal{U}_i] \leqslant P_{\mathcal{L}'}[\mathcal{U}_i]$ with $i = 0, \ldots, 8$. In fact, by symmetry and triviality, seven of them turn out to be redundant or trivial so we are left with

$$3p^2(1 - p) + p^3 \leqslant q_c^3,$$
$$3p(1 - p)^2 + 3p^2(1 - p) + p^3 \leqslant 3q_c^2(1 - q_c) + q_c^3,$$

with $0 \leqslant p \leqslant 1$. The largest value satisfying all these inequalities for given q_c is a lower bound of p_c. In order to find an upper bound for p_c, we need to revert both the inequalities above and the smallest p satisfying the reversed inequalities gives us an upper bound of p_c. The results are shown in Fig. 2(a). There is one point where the upper and lower bounds coalesce, that is, $(q_c, p_c) = (1 - 2\sin\frac{\pi}{18}, 2\sin\frac{\pi}{18})$. Here, the star-triangle transformation yields the exact percolation threshold $p_c = 2\sin(\pi/18)$ for the triangular lattice {3,6} and $q_c = 1 - 2\sin(\pi/18)$ for the honeycomb lattice {6,3} [9]. But generally, the substitution method gives us an interval in which the percolation threshold can exist.

B. Square

Let us now turn our attention to the star-square case shown in Fig. 2(b). By enumerating all the possible 345 up-sets, we find 53 different inequalities. Many of them are redundant, however, and we need to consider only the following inequalities:

$$4p^2 - 4p^3 + p^4 \leqslant 2q_c^2 - q_c^4, \quad 2p^2 - p^4 \leqslant q_c^4,$$
$$4p - 6p^2 + 4p^3 - p^4 \leqslant 4q_c^2 - 4q_c^3 + q_c^4,$$

with $0 < p < 1$ in order to find a lower bound of p_c for given q_c. The first inequality is for an up-set $\{ABCD, A|BCD, B|ACD, C|ABD, D|ABC, AC|BD, A|C|BD, B|D|AC\}$, generated by $\{A|C|BD, B|D|AC\}$. The second
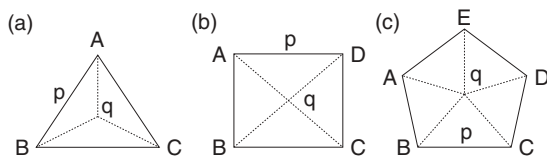

FIG. 1. Substitution regions of the (a) star-triangle, (b) star-square, and (c) star-pentagon transformations. The solid lines are occupied with probability p and the dashed lines are occupied with q.
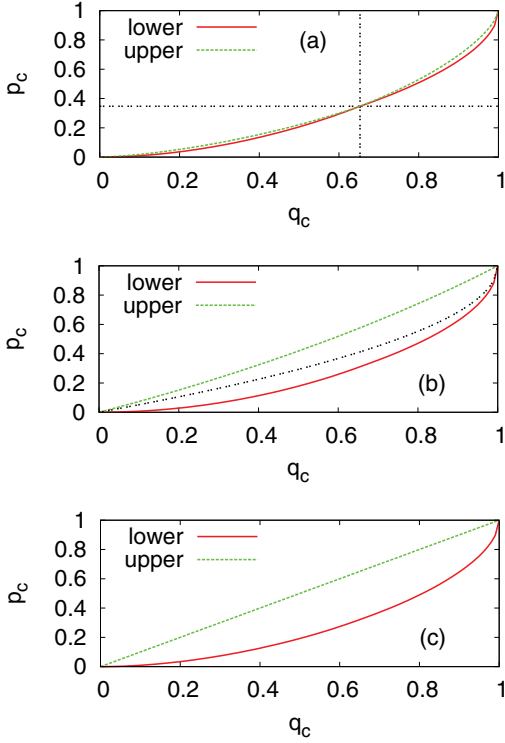
FIG. 2. (Color online) Solutions of the inequalities in the (a) star-triangle, (b) star-square, and (c) star-pentagon transformations, respectively. In (a), the horizontal dashed line represents $p_c = 2\sin(\pi/18) \approx 0.347\,296$ and the vertical dashed line represents $q_c = 1 - 2\sin(\pi/18) \approx 0.652\,704$. In (b), the dashed curve means $p_c = 1 - \sqrt{1 - q_c}$ to check the square lattice {4,4} (see text).

inequality is for $\{ABCD, AB|CD, AD|BC\}$ generated by $\{AB|CD, AD|BC\}$, and the third is for an up-set generated by $\{A|B|CD, B|C|AD, C|D|AB, A|D|BC\}$, respectively. As above, reverting all the three inequalities will yield an upper bound for p_c, and the results are plotted in Fig. 2(b).

It is interesting to consider the square lattice {4,4} since the star-square transformation transforms a square lattice with double bonds to another square lattice, rotated by angle $\pi/4$ from the original lattice. Then, p_c and q_c should be related by $p_c = 1 - \sqrt{1 - q_c}$. Our upper and lower bounds include this relationship over the whole region of q_c. If this happened only within a limited region of q_c, we could obtain nontrivial bounds for p_c directly from this plot. Although this is not the case, this example shows that the substitution method indeed yields correct results.

C. Pentagon

For the star-pentagon case, the number of possible up-sets is 161 166, from which 1237 different inequalities are found. Once again, most of them are redundant, and the set of inequalities to solve turns out to be

$$5p - 10p^2 + 10p^3 - 5p^4 + p^5 \leqslant 5q^2 - 5q^3 + q^5,$$
$$5p^3 - 5p^4 + p^5 \leqslant q^5,$$
$$5p^2 - 5p^3 + p^5 \leqslant 5q_c^2 - 5q_c^3 + q_c^5,$$

with $0 \leqslant p \leqslant 1$ when we are to find a lower bound. The first inequality is for an up-set generated by $\{AB|C|D|E, AE|B|C|D, BC|A|D|E, CD|A|B|E, DE|A|B|C\}$. The second inequality is for an up-set generated by $\{AB|CDE, AE|BCD, BC|ADE, CD|ABE, DE|ABC\}$. Finally, the third inequality is for an up-set generated by $\{A|B|CDE, A|E|BCD, B|C|ADE, C|D|ABE, D|E|ABC\}$ and essentially the same as $p \leqslant q_c$. Finding an upper bound is also straightforward. The results are shown in Fig. 2(c).

III. RESULTS

A. Order-5 square tiling

By applying the star-pentagon transformation to the order-5 pentagonal tiling {5,5} with double bonds, we find the order-5 square tiling {4,5} [Fig. 3(a)]. Note that we need double bonds in order to distribute five bonds to every pentagonal face. Recall that the threshold p_{c1} of {5,5} is bounded as $1/4 \leqslant p_{c1}^{\{5,5\}} \leqslant 0.381\,296$ [15], which automatically implies $0.618\,704 \leqslant p_{c2}^{\{5,5\}} \leqslant 3/4$ by self-duality since the duality implies

$$p_{c1}^{\{m,n\}} + p_{c2}^{\{n,m\}} = 1, \quad p_{c2}^{\{m,n\}} + p_{c1}^{\{n,m\}} = 1, \tag{2}$$

for lattices represented by $\{m,n\}$ and $\{n,m\}$ [11]. If every neighboring pair of vertices in {5,5} are connected by double bonds with occupation probability p', the corresponding bounds of the critical threshold are located by the simple relation $p_c' = 1 - \sqrt{1 - p_c}$ as

$$0.133\,975 \leqslant p_{c1}'^{\{5,5\}} \leqslant 0.213\,423,$$
$$0.382\,508 \leqslant p_{c2}'^{\{5,5\}} \leqslant 0.5.$$

Then, solving the inequalities for the star-pentagon case, we obtain bounds for {4,5}. The detailed procedure is given as follows: Suppose that $p_{c1}'^{\{5,5\}} = 0.133\,975$. The corresponding bounds are $0.133\,975 \leqslant p_{c1}^{\{4,5\}} \leqslant 0.413\,131$, whereas if $p_{c1}'^{\{5,5\}} = 0.213\,423$, the bounds are $0.213\,423 \leqslant p_{c1}^{\{4,5\}} \leqslant 0.527\,957$. Therefore, the resulting bounds should be $0.133\,975 \leqslant p_{c1}^{\{4,5\}} \leqslant 0.527\,957$ in total, and the same reasoning yields $0.382\,508 \leqslant p_{c2}^{\{4,5\}} \leqslant 0.807\,697$. However, Eq. (1) further constrains $p_{c1}^{\{4,5\}}$ as larger than or equal to $1/4$.
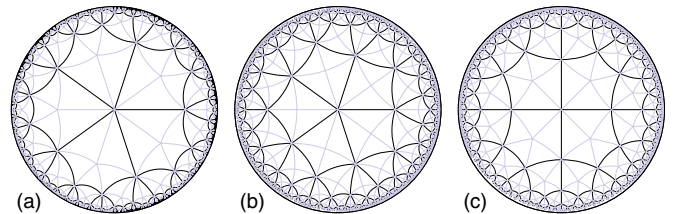

FIG. 3. (Color online) Lattice structures depicted on the Poincaré disk [19]. (a) The star-pentagon transformation of the order-5 pentagonal tiling {5,5} (black) leads to the order-5 square tiling {4,5} (gray). (b) The star-square transformation of {4,5} (black) leads to the order-5-4 rhombille tiling (gray). (c) The star-pentagon transformation relates the order-4 pentagonal tiling {5,4} (black) to the order-5-4 rhombille tiling (gray). If shifted by one lattice spacing, the order-5-4 rhombille tiling in (c) looks the same as that in (b).

TABLE I. Analytic bounds of bond percolation thresholds on hyperbolic lattices.

Tiling	Lower threshold	Upper threshold	Method
Order-5 pentagon	$1/4 \leqslant p_{c1} \leqslant 0.381\,296$	$0.618\,704 \leqslant p_{c2} \leqslant 3/4$	Ref. [16]
Order-5 square	$1/4 \leqslant p_{c1} \leqslant 0.527\,957$	$0.382\,508 \leqslant p_{c2} \leqslant 2/3$	substitution
Order-4 pentagon	$1/3 \leqslant p_{c1} \leqslant 0.617\,492$	$0.472\,043 \leqslant p_{c2} \leqslant 3/4$	duality
Order-5-4 rhombille	$1/4 \leqslant p_{c1} \leqslant 0.656\,963$	$0.275\,768 \leqslant p_{c2} \leqslant 2/3$	substitution

Likewise, we see that $p_{c1}^{\{5,4\}} \geqslant 1/3$, which implies $p_{c2}^{\{5,4\}} \leqslant 2/3$ from the duality [Eq. (2)]. We thus conclude that

$$1/4 \leqslant p_{c1}^{\{4,5\}} \leqslant 0.527\,957, \quad 0.382\,508 \leqslant p_{c2}^{\{4,5\}} \leqslant 2/3.$$

B. Order-5-4 rhombille tiling

The same procedure can be repeated on the order-5 square tiling $\{4,5\}$. The star-square transformation changes it to the order-5 rhombille tiling, whose face configuration can be denoted by $V4.5.4.5$ [Fig. 3(b)]. The face configuration means the numbers of faces at each of vertices around a face. The computation is similar to the above one: By putting double bonds between every pair of vertices in $\{4,5\}$, we see that the critical probabilities are bounded as

$$0.133\,975 \leqslant p_{c1}^{\prime\{4,5\}} \leqslant 0.312\,946,$$
$$0.214\,194 \leqslant p_{c2}^{\prime\{4,5\}} \leqslant 0.422\,65.$$

When the star-square transformation is applied, it is straightforward to obtain

$$0.178\,197 \leqslant p_{c1}^{V5.4.5.4} \leqslant 0.656\,963,$$
$$0.275\,768 \leqslant p_{c2}^{V5.4.5.4} \leqslant 0.760\,854,$$

but some are no better than trivial since $p_{c1}^{V5.4.5.4} \geqslant 1/4$ for coordination number $n \leqslant 5$, and the dual of $V5.4.5.4$, called the tetrapentagonal tiling, has $p_{c1} \geqslant 1/3$ with coordination number 4. The bounds for this order-5-4 rhombille tiling are therefore found to be

$$1/4 \leqslant p_{c1}^{V5.4.5.4} \leqslant 0.656\,963, \quad 0.275\,768 \leqslant p_{c2}^{V5.4.5.4} \leqslant 2/3.$$

C. Order-4 pentagonal tiling

It is notable that the order-4 pentagonal tiling $\{5,4\}$ is also related to the order-5-4 rhombille tiling by the star-

pentagon transformation [Fig. 3(c)]. Since the duality [Eq. (2)] also imposes conditions for thresholds in $\{5,4\}$ and $\{4,5\}$ as

$$p_{c1}^{\{4,5\}} + p_{c2}^{\{5,4\}} = 1, \quad p_{c2}^{\{4,5\}} + p_{c1}^{\{5,4\}} = 1,$$

one could expect sharper bounds by exploiting both the relations, i.e., the star-pentagon transformation and the duality transformation. Unfortunately, since the star-pentagon transformation yields too large bounds [see Fig. 2(c)], it adds no information to the duality results, which are expressed as

$$1/3 \leqslant p_{c1}^{\{5,4\}} \leqslant 0.617\,492, \quad 0.472\,043 \leqslant p_{c2}^{\{5,4\}} \leqslant 3/4,$$

where $1/3$ is a trivial bound from the coordination number $n = 4$ and $p_{c2}^{\{5,4\}} \leqslant 3/4$ is a direct consequence of $p_{c1}^{\{4,5\}} \geqslant 1/4$.

IV. SUMMARY

In summary, we have obtained analytic bounds of percolation thresholds on three hyperbolic lattices by applying the substitution method to the known bounds for the order-5 pentagonal tiling $\{5,5\}$. Our results are summarized in Table I. The obtained bounds are admittedly too broad to be very informative. But our approach illustrates how analytic bounds for one lattice can be made useful in estimating those for other lattices tiling a hyperbolic plane. Precise knowledge of p_{c2} is still greatly needed in studies of percolation on hyperbolic lattices in general, and we hope that this approach can make further progress in future studies.

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