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Reductive renormalization of the phase-field crystal equation

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It has been known for some time that singular perturbation and reductive perturbation can be unified from the renormalization-group theoretical point of view: Reductive extraction of space-time global behavior is the essence of singular perturbation methods. Reductive renormalization was proposed to make this unification practically accessible; actually, this reductive perturbation is far simpler than most reduction methods, such as the rather standard scaling expansion. However, a rather cryptic exposition of the method seems to have been the cause of some trouble. Here, an explicit demonstration of the consistency of the reductive renormalization-group procedure is given for partial differentiation equations (of a certain type, including time-evolution semigroup type equations). Then, the procedure is applied to the reduction of a phase-field crystal equation to illustrate the streamlined reduction method. We conjecture that if the original system is structurally stable, the reductive renormalization-group result and that of the original equation are diffeomorphic.

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I. INTRODUCTION

Many models of pattern formation and phase transition dynamics are described in terms of partial differential equations (PDEs) that are generally analytically intractable and even numerically nontrivial. Thus, reducing them to simpler equations while preserving the key features of the original system describing particular situations of interest should be desirable. The so-called reductive perturbation theory [1] and other techniques have been devised and utilized, but from a practical point of view, the convenient method for system reduction is to use the proto-renormalization-group (proto-RG) equation approach [2,3]. However, the method has not been understood correctly [4] and some controversies remain [5].

The purpose of this paper devoted to reductive renormalization is twofold. Lingering controversy and misunderstanding seem to be caused largely by a rather cryptic writing of the theory part of Ref. [2]. Therefore, in this paper a formal demonstration of the consistency of the method to all orders is given explicitly. It is implicit in Ref. [2], and now at least for ordinary differential equations (ODEs) mathematical proofs of the consistency of the renormalization-group (RG) approach to singular perturbation problems are available [6,7], but the original demonstration behind Ref. [2] adapted to PDEs should be more accessible for physicists, and its resultant recipe should be practically useful. The second aim of this paper is to illustrate the usefulness of the practical rule made explicit in the theoretical exposition.

This paper is organized as follows. In Sec. II we present a formal demonstration of the RG theoretical reduction result for PDEs. It will be seen that, formally, the procedure is quite parallel to the one for the ODE case [8]. The proto-RG equation is then obtained (Sec. III). Subsequently, in Sec. IV the result is applied to the reduction of the phase-field crystal equation [9]. This equation describes the evolution of the conserved field. It is the conserved analog of the Swift-Hohenberg equation [10], which constitutes a paradigm of spatially periodic pattern formation. In Sec. V we make general remarks about the proto-RG equations and the reductive RG results. The Appendix contains the demonstration of the consistency (to all orders) of the RG condition that is employed in the proto-RG approach. The demonstration is technically tedious, but do not confuse its complication with the practical simplicity of the reductive RG method. Those who are only interested in using the method may read only the first paragraphs of Secs. II and III and the last section.

II. RENORMALIZATION PROCEDURE

Since the main purpose of this paper is to show explicitly the consistency of the reductive RG approach, many details and higher-order general formulas appear. Those who simply wish to reduce Eq. (1) below should look at the unperturbed result (2) and formal perturbative result (4) (often to the lowest nontrivial order), then use the procedure outlined at the beginning of Sec. III. Those who wish to understand the renormalization philosophy (background) and how it straightforwardly leads to the reductive renormalization idea should read Chap. 3 of Ref. [8]. Also there is an early expository review [11].

We study the following equation whose independent variables are collectively denoted as x, which is often time and space $(t,\boldsymbol{r})$:

$$L(\partial_x)\psi = \epsilon N[\psi], \tag{1}$$

where L is a constant coefficient linear differential operator [12] with distinct eigenvalues and N is a certain nonlinear operator. A small positive constant ϵ represents the smallness of the nonlinearity. ∂_x implies a set of (partial) differential operators with respect to the independent variables. The solution to the zeroth order equation

$$L\psi_0 = 0 \tag{2}$$

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may be written as

$$\psi_0(x) = \sum_k A_k e_k(x), \tag{3}$$

where $e_k(x)$ is the basis functions of the null space of L [13]. Thus, needless to say, our formalism does not cover all the partial differential equations but still covers evolution equations that describe pattern formations and phase transition dynamics. We write the formal expansion of the solution to Eq. (1) as

$$\psi(x,A) = \sum_k A_k e_k(x) + \epsilon\psi_1(x,A) + \epsilon^2\psi_2(x,A) + \cdots, \tag{4}$$

where the A_k dependence of the higher order terms is explicitly written; A denotes $\{A_k\}$ collectively. Since each ψ_n ($n = 1,2,\ldots$) has its own secular terms Y_n, let us write $\psi_n = \eta_n + Y_n$, where η_n denotes the nonsecular terms in ψ_n. Now, we wish to renormalize this bare perturbation series Eq. (4) by replacing A_k with $A_k = Z^k A_{Rk}$ (no summation convention implied) by choosing renormalization constants Z^k appropriately. Splitting all the monomials x^m in the secular prefactors of $e_k(x)$ in Y_n as $[x^m - \chi^m] + \chi^m$ [as, e.g., $x^3\exp(ikx) \to \{(x^3-\chi^3)+\chi^3\}\exp(ikx)$], we try to absorb all these "squeezed-out" χ factors into Z^k. Let us write the x monomials in the secular prefactors in $f(x,A)$ explicitly as the second variable to denote $f(x,A)$ as $f(x,\{x^m\},A)$ [i.e., $f(x,A) = f(x,\{x^m\},A)$, but the x monomials in the secular prefactors are represented collectively by $\{x^m\}$ and are explicitly written as the second variable]. Then, we do the following transformation: $\psi(x,\{x^m\},A) = \psi(x,\{x^m\},Z(\chi)A_R(\chi)) = \psi(x,\{x^m-\chi^m\},A_R(\chi))$, where $Z(\chi)A_R(\chi)$ collectively denotes $\{Z^k(\chi)A_{Rk}(\chi)\}$ (a simple concrete example may be found in Sec. II of Ref. [2]). Eventually, we set $\chi = x$ to obtain $\psi(x,\{x^m\},A) = \psi(x,\{0\},A_R(x))$. Notice that $\psi(x,\{\chi^m\},A) = \psi(x,\{0\},A_R(\chi))$.

To be systematic, let us expand the renormalization constant in powers of ϵ:

$$Z^k = 1 + \epsilon Z_1^k + \epsilon^2 Z_2^k + \cdots. \tag{5}$$

Also we expand $\psi(x,\{x^m\},A)$ in powers of ϵ:

$$\psi(x,\{x^m\},A) = \sum_k A_k e_k(x) + \epsilon\psi_1(x,\{x^m\},A) + \epsilon^2\psi_2(x,\{x^m\},A) + \cdots. \tag{6}$$

Then, we replace A_k in Eq. (6) with

$$A_k = A_{Rk}(\chi) + \epsilon Z_1^k(\chi)A_{Rk}(\chi) + \epsilon^2 Z_2^k(\chi)A_{Rk}(\chi) + \cdots. \tag{7}$$

Using A in the form of Eq. (7), we wish to rewrite Eq. (6) in terms of A_R, which collectively denotes $\{A_{Rk}\}$. Assuming needed differentiabilities, we have

$$\begin{aligned}\psi(x,\{x^m\},A) = {} & \sum_k \left[1 + \epsilon Z_1^k(\chi) + \epsilon^2 Z_2^k(\chi) + \cdots\right]A_{Rk}(\chi)e_k(x) + \epsilon\psi_1(x,\{x^m\},A_R(\chi)) \\ & + \epsilon^2\left[\psi_2(x,\{x^m\},A_R(\chi)) + \sum_k \partial_k\psi_1(x,\{x^m\},A_R(\chi))Z_1^k(\chi)A_{Rk}(\chi)\right] \\ & + \epsilon^3\left[\psi_3(x,\{x^m\},A_R(\chi)) + \sum_k\left\{\partial_k\psi_2(x,\{x^m\},A_R(\chi))Z_1^k(\chi)A_{Rk}(\chi) + \partial_k\psi_1(x,\{x^m\},A_R(\chi))Z_2^k(\chi)A_{Rk}(\chi)\right\}\right. \\ & \left. + \frac{1}{2}\sum_{k,l}\partial_k\partial_l\psi_1(x,\{x^m\},A_R(\chi))Z_1^k(\chi)A_{Rk}(\chi)Z_1^l(\chi)A_{Rl}(\chi)\right] + \cdots,\end{aligned} \tag{8}$$

where ∂_k implies the partial derivative with respect to A_k.

The renormalization condition of order ϵ reads

$$\sum_k A_{Rk}(\chi)Z_1^k(\chi)e_k(x) + Y_1(x,\{\chi^m\},A_R(\chi)) = 0. \tag{9}$$

This is obtained just by honestly following the procedure described between Eqs. (4) and (5): we must remove the secular terms from $\sum_k A_{Rk}(\chi)Z_1^k(\chi)e_k(x) + \psi_1(x,\{\chi^m\},A_R(\chi))$, so we separate out the secular term Y_1 from ψ_1 as $\psi_1 = \eta_1 + Y_1$. Then, as noted above, we separate all the monomials x^m in the secular prefactors of $e_k(x)$ in Y_1 as $[x^m - \chi^m] + \chi^m$, and the separated $Y_1(x,\{\chi^m\},A_R(\chi))$ should be absorbed into Z. This is just the above condition Eq. (9). Note that at this stage A_R are (formally) understood as constants. $Z_2^k(\chi)$ must be determined to replace all the second variables $\{x^m\}$ as $\{\chi^m\}$ from the order ϵ^2 term:

$$\sum_k A_{Rk}(\chi)Z_2^k(\chi)e_k(x) + Y_2(x,\{\chi^m\},A_R(\chi)) + \sum_k \partial_k\psi_1(x,\{\chi^m\},A_R(\chi))Z_1^k(\chi)A_{Rk}(\chi) = 0. \tag{10}$$

Here, notice that in the third term on the left-hand side even the regular (i.e., nonsecular) contribution $\partial_k\eta_1(x,A_R)$ cannot be ignored (for now) because it is multiplied by a χ-dependent factor $Z_1^k(\chi)$. The third order renormalization condition requires

$$\begin{aligned}& \sum_k A_{Rk}(\chi)Z_3^k(\chi)e_k(x) + Y_3(x,\{\chi^m\},A_R(\chi)) + \sum_k\left[\partial_k\psi_2(x,\{\chi^m\}A_R(\chi))Z_1^k(\chi)A_{Rk}(\chi) + \partial_k\psi_1(x,\{\chi^m\},A_R(\chi))Z_2^k(\chi)A_{Rk}(\chi)\right] \\ & + \frac{1}{2}\sum_{k,l}\partial_k\partial_j\psi_1(x,\{\chi^m\},A_R(\chi))Z_1^k(\chi)A_{Rk}(\chi)Z_1^l(\chi)A_{Rl}(\chi) = 0.\end{aligned} \tag{11}$$

Thus, we see that the renormalization condition such as Eq. (11) has the following structure (symbolically, since the third term is not explicitly written out) for order ϵ^n:

$$\begin{aligned}&\sum_k A_{Rk}(\chi)Z_n^k(\chi)e_k(x) + Y_n(x,\{\chi^m\},A_R(\chi))\\ &\quad + \sum [\text{derivatives with respect to } A_R(\chi) \text{ of } \psi_s(x,\{\chi^m\},A_R(\chi))] \prod Z_m^j(\chi)A_{Rj}(\chi) = 0,\end{aligned} \tag{12}$$

where ψ_s $(s < n)$ in the third term are the results of the lower order calculations and Z_m^j $(m < n)$ are the terms already determined by the lower order renormalization conditions.

We have not yet considered whether Z can be determined to satisfy these requirements, although it is trivial to see that the $O(\epsilon)$ requirement can be satisfied. In this section, first we assume that Z can be consistently determined and simplify these renormalization conditions. Then, we look for the condition for simplification to be consistent with the original renormalization condition. That this consistency condition is indeed satisfied is demonstrated in the Appendix. In Ref. [2] the simplified condition was explicitly written. The reason why the simplification is legitimate was alluded to, but no explicit demonstration was published. This in part explains why the proto-RG was regarded by some as a new RG scheme; as is explicitly written in this paper, our scheme is nothing but the quite standard RG scheme.

In order to study in more detail the structure of the renormalization condition the nth order $(n > 1)$ result ψ_n is dissected as

$$\psi_n(x,\{x^m\},A) = Y_n^{[r]}(x,\{x^m\},A) + Y_n^{[nr]}(x,\{x^m\},A) + \eta_n(x,A). \tag{13}$$

Here, the first term on the right-hand side (RHS), $Y_n^{[r]}(x,\{x^m\},A)$, denotes the resonant secular terms, that is, a linear combination of $e_k(x)$'s with polynomial coefficients of x without a constant term (which may be absorbed in the unperturbed solution). $Y_n^{[nr]}(x,\{x^m\},A)$ is the nonresonant secular terms, that is, all other secular terms not included in $Y_n^{[r]}(x,\{x^m\},A)$. This is generally a linear combination of the products of $e_k(x)$'s [that cannot be reduced to a single $e_j(x)$] with polynomial coefficients of x without constant terms. $\eta_n(x,A)$ is the nonsecular terms, which is a linear combination of the products of $e_k(x)$'s with constant coefficients. η_n does not contain any linear combination of $e_k(x)$'s (which can be absorbed into the zeroth order terms by the redefinition of A_k's).

We assume that the perturbation solutions belong to a ring generated by finitely many products of $e_j(x)$'s (with polynomial coefficients; note that all the examples illustrated subsequently in this paper satisfy this condition). Then, Eq. (12) must hold at least for the individual coefficients of $e_j(x)$'s. Namely, we find that

$$\begin{aligned}&\sum A_{Rk}(\chi)Z_n^k(\chi)e_k(x) + Y_n^{[r]}(x,\{\chi^m\},A_R(\chi))\\ &\quad + \sum \left[\text{derivatives with respect to } A_R(\chi) \text{ of } Y_s^{[r]}(x,\{\chi^m\},A_R(\chi))\right] \prod Z_m^j(\chi)A_{Rj}(\chi) = 0.\end{aligned} \tag{14}$$

Thus, condition (12) may be split into two parts, the resonant portion Eq. (14) and the rest:

$$\begin{aligned}&Y_n^{[nr]}(x,\{\chi^m\},A_R(\chi)) + \sum \left\{\text{derivatives with respect to } A_R(\chi) \text{ of } \left[Y_s^{[nr]}(x,\{\chi^m\},A_R(\chi)) + \eta(x,A_R(\chi))\right]\right\}\\ &\quad \times \prod Z_m^j(\chi)A_{Rj}(\chi) = 0.\end{aligned} \tag{15}$$

As could be seen from the calculation of concrete examples and as was noted in Ref. [2], all the nonresonant secular terms of order n come from secular terms of order less than n [i.e., the secularity of $Y_n^{[nr]}(x,\{x^m\},A)$ is solely due to the lower order secular terms appearing in the perturbation series]. Thus, an order-by-order renormalization implies Eq. (15) since the singularities in $Y^{[nr]}$ are already renormalized away. This implies the key observation of the proto-RG approach that Eq. (14) is the sole renormalization condition. An explicit demonstration of the claim that the renormalization constants determined by the resonant singular terms according to Eq. (14) indeed remove all the singularities is given in the Appendix.

Adding all the orders of Eq. (14) [recall that the third term in Eq. (14) is from the Taylor expansion of ψ around A_R], we see that the renormalization condition must satisfy

$$\sum_k [A_k - A_{Rk}]e_k(x) + Y^{[r]}(x,\{\chi^m\},A) = 0, \tag{16}$$

where $Y^{[r]} = \sum_{n=1}^{\infty} Y_n^{[r]}$. When this is used to determine $Z^k(\chi)$, A_k's are interpreted as Eq. (7) and A_{Rk}'s are regarded constant. After determining $Z^k(\chi)$ we can determine the χ dependence of A_{Rk} by $Z^k(\chi)A_{Rk}(\chi) = A_k$, which is constant, and rewrite Eq. (16) as

$$\sum_k [A_k - A_{Rk}(\chi)]e_k(x) + Y^{[r]}(x,\{\chi^m\},A) = 0. \tag{17}$$

Combining all these equations determining the renormalization constants order by order, we obtain

$$\psi(x,\{\chi^m\},A) = \sum_k A_{Rk}(\chi)e_k(x) + \eta(x,A_R(\chi)). \tag{18}$$

As expected, the singular terms in the perturbation series are completely removed and are absorbed in the renormalized coefficients A_{Rk}. If we set $\chi = x$, $\psi(x,\{\chi^m\},A)$ is just $\psi(x,A)$, so we have obtained the renormalized perturbation result.

III. PROTO-RG APPROACH

Before going to give a general method with justifications, the resultant practical procedure is outlined for practitioners. To reduce Eq. (1) to order ϵ, which is usually the most important application of the reduction methods, we first introduce the operator $\mathcal{L}$ as defined by (19) [see also Eq. (20)] and then arrive at Eq. (27). As is illustrated in Sec. IV, this is the proto-RG equation (to the lowest nontrivial order) for A_{Rk}, i.e., the renormalized A_k. As noted in Sec. V, *the derivation of the proto-RG equation is not the final goal of system reduction*. We must further demand that the obtained equation describes a certain intermediate asymptotic process allowed by the system under study. Read the final remarks in Sec. V to finish the reduction.

Let us introduce a linear operator $\mathcal{L}$ as

$$\mathcal{L}(\partial_x,\partial_\chi) = L(\partial_x+\partial_\chi) - L(\partial_x), \tag{19}$$

where $\partial_x = \partial/\partial x$ and $\partial_\chi = \partial/\partial\chi$. Notice that if it is applied to a function that does not contain χ, it gives zero. For an eigenfunction $e_k(x)$ of L with its characteristic root λ_k we define a new operator $\mathcal{L}_k(\partial_\chi)$ by

$$\begin{aligned}\mathcal{L}[f(\chi)e_k(x)] &= \{[L(\lambda_k+\partial_\chi) - L(\lambda_k)]f(\chi)\}e_k(x)\\ &= [\mathcal{L}_k(\partial_\chi)f(\chi)]e_k(x).\end{aligned} \tag{20}$$

Applying $\mathcal{L}$ to Eq. (17), we have

$$0 = -\sum_{k=1}^{n}[\mathcal{L}_k(\partial_\chi)A_{Rk}(\chi)]e_k(x) + \mathcal{L}Y^{[r]}(x,\{\chi^m\},A). \tag{21}$$

Since $Y^{[r]}(x,\{\chi^m\},A)$ has the structure

$$\begin{aligned}Y^{[r]}(x,\{\chi^m\},A) &= \epsilon\sum_k P_1^k(\chi,A)e_k(x)\\ &\quad + \epsilon^2\sum_k P_2^k(\chi,A)e_k(x) + \cdots,\end{aligned} \tag{22}$$

Eq. (21) implies

$$\begin{aligned}&\sum_k\left[\mathcal{L}_k(\partial_\chi)\left\{-A_{Rk}(\chi) + \epsilon P_1^k(\chi,A)\right.\right.\\ &\quad\left.\left. + \epsilon^2 P_2^k(\chi,A) + \cdots\right\}\right]e_k(x) = 0,\end{aligned} \tag{23}$$

or

$$\mathcal{L}_k(\partial_\chi)A_{Rk}(\chi) = \mathcal{L}_k(\partial_\chi)P^k(\chi,A), \tag{24}$$

where $P^k(\chi,A) = \sum_n \epsilon^n P_n^k(\chi,A)$. Notice that $A_k = Z^k(\chi)A_{Rk}(\chi)$ on the RHS of Eq. (24) are constants (χ independent), so when $\mathcal{L}_k$ is applied, they are intact. After this procedure, we replace A_k with $A_{Rk}(\chi)Z^k(\chi)$. This replacement should have eliminated all the secular terms, so there must not be any χ left. Therefore, after this replacement there must not be any χ on the RHS; A_R depends on χ, but the equation governing A_R does not explicitly depend on χ. Thus, we can simply set $\chi = 0$ and $Z^k = 1$ [14]; we have reached the proto-RG equation:

$$\mathcal{L}_k(\partial_\chi)A_{Rk}(\chi) = \mathcal{L}_k(\partial_\chi)P^k(\chi,A_R)|_{\chi=0}. \tag{25}$$

Here, on the RHS the differential operator $\mathcal{L}_k$ applies only to χ (not A_R), and after differentiation χ (if any left [15]) is set to zero. The reason why it is called the proto-RG equation will become clear from the discussion in Sec. V. Incidentally, Eq. (25) reminds us of the method of variation of coefficients to solve inhomogeneous equations. Notice, however, the inhomogeneous terms on the RHS of Eq. (25) contain only secular terms in the present case.

The RHS of Eq. (25) can be calculated from the perturbation equation. For example, the first order proto-RG equation can be read off immediately from the first order perturbation equation $L\psi_1 = N[\psi_0]$, which is itself rewritten as

$$L(\partial_x+\partial_\chi)Y_1^{[r]}(x,\{\chi^m\},A)|_{\chi=x} = \mathcal{P}N[\psi_0], \tag{26}$$

where $\mathcal{P}$ is the projection to the linear space spanned by $\{e_k\}$. From this equation, comparing the coefficients of $e_k(x)$, we arrive at

$$\mathcal{L}_k(\partial_\chi)A_{Rk}(\chi)|_{\chi=x} = \mathcal{P}_k N[\psi_0]. \tag{27}$$

Here, $\mathcal{P}_k$ is an operator to single out the coefficients of $e_k(x)$. In practice, this operation is equivalent to identifying the coefficients of $e_k(x)$ by inspection and separating them out (to be more mathematical, we must assume that $\{e_k\}$ makes a function ring with polynomial coefficients, and this is usually the case in pattern formations).

If ψ is a conserved quantity (i.e., ψ integrated over the spatial domain is constant), then it is often the case that the perturbation term $N[\psi]$ takes the form $\Delta M[\psi]$, where Δ is the Laplacian with respect to $\boldsymbol{r}$ and M is some function(al) of ψ. It is impossible to respect this conservation law order by order with the above naive perturbation calculation. Thus, we need a nonperturbative method to impose the conservation law. The most natural approach is to rewrite the original equation as

$$\tilde{L}\psi \equiv \Delta^{-1}L\psi = \epsilon M[\psi] + H, \tag{28}$$

where H is a spatial harmonic function. The solution to the zeroth order equation $\tilde{L}\psi_0 = H$ may be written as

$$\psi_0(x) = \sum_k A_k e_k(x). \tag{29}$$

The subsequent development is quite parallel to the above "nonconserved" case. From an analog of Eq. (17) we obtain

$$\begin{aligned}&\sum_k \tilde{L}(\partial_x+\partial_\chi)A_{Rk}(\chi)e_k(x)\\ &\quad = \sum_k \tilde{L}(\partial_x+\partial_\chi)P^k(\chi,A_R)e_k(x) + H(x).\end{aligned} \tag{30}$$

Applying $(\partial_x+\partial_\chi)^2$ to the above equation, we obtain

$$L(\partial_x+\partial_\chi)A_{Rk}(\chi)e_k(x) = L(\partial_x+\partial_\chi)P^k(\chi,A_R)e_k(x). \tag{31}$$

Then, we find the proto-RG equation of the same form as before, i.e., Eq. (25).

The first order proto-RG equation in the conserved case can be obtained similarly by noting

$$\begin{aligned}&\mathcal{P}_k L(\partial_x+\partial_\chi)Y_1^{[r]}(x,\{\chi^m\},A)\\ &\quad = \mathcal{P}_k(\partial_x+\partial_\chi)^2 M\left[\sum_k A_{Rk}(\chi)e_k(x)\right],\end{aligned} \tag{32}$$

or [see Eq. (20)]

$$\mathcal{L}_k(\partial_\chi)P_1^k(\chi,A_R) = \mathcal{P}_k(\partial_x+\partial_\chi)^2 M\left[\sum_k A_{Rk}(\chi)e_k(x)\right]\Bigg|_{x=\chi}. \tag{33}$$

Thus, we find that

$$\mathcal{L}_k(\partial_\chi)A_{Rk}(\chi)=\mathcal{P}_k(\partial_x+\partial_\chi)^2 M\left[\sum_k A_{Rk}(\chi)e_k(x)\right]\Bigg|_{x=\chi}. \tag{34}$$

IV. ILLUSTRATIONS

We shall illustrate the proto-RG approach with the aid of the phase-field crystal (PFC) equation:

$$\partial_t\psi = \partial_{\boldsymbol{r}}^2\left[-\epsilon\psi + \left(\partial_{\boldsymbol{r}}^2+k_0^2\right)^2\psi + s\psi^2+\psi^3\right], \tag{35}$$

where ψ is a real scalar order parameter field. The $\partial_{\boldsymbol{r}}$ is a gradient operator (∇) with respect to the position vector $\boldsymbol{r}$, and $\partial_{\boldsymbol{r}}^2$ is the Laplacian (Δ). ϵ is a bifurcation parameter. It is well known that, when $s=0$, the competition between the surface energy contribution given by the $\partial_{\boldsymbol{r}}^2\psi$ term in the square brackets on the RHS of Eq. (35) and the curvature energy term $\propto(\partial_{\boldsymbol{r}}^2)^2\psi$ gives rise to spatially modulated structures (called stripes or lamellae) with a period of $\sim 2\pi/k_0$ for $\epsilon>0$. For $s\neq 0$, the hexagonal and the bcc structure exist as a stable phase.

Since we wish to consider the situation where each term of $-\epsilon\psi+s\psi^2+\psi^3$ in Eq. (35) plays an equally relevant role in dynamics, we scale ψ as $\sqrt{\epsilon}\psi$ and $\sqrt{\epsilon}s$ as s and denote the scaled ψ and s with the same symbols. The model (35) then reads

$$\partial_t\psi = \partial_{\boldsymbol{r}}^2\left[\left(\partial_{\boldsymbol{r}}^2+k_0^2\right)^2\psi + \epsilon(-\psi+s\psi^2+\psi^3)\right],$$

i.e.,

$$L(\partial_x)\psi = \epsilon\Delta M(\psi), \tag{36}$$

where

$$L(\partial_x) = \partial_t - \partial_{\boldsymbol{r}}^2\left(\partial_{\boldsymbol{r}}^2+k_0^2\right)^2, \tag{37}$$

$$M(\psi) = -\psi + s\psi^2+\psi^3. \tag{38}$$

Since the actual calculation for Eq. (36) is a little involved, we first illustrate the approach using the simpler and nonconserved version of Eq. (36), which is the Swift-Hohenberg (SH) equation:

$$\frac{\partial}{\partial t}\psi = \epsilon(\psi-\psi^3) - \left(\partial_{\boldsymbol{r}}^2+k_0^2\right)^2\psi. \tag{39}$$

For Eq. (39),

$$L(\partial_x) = \partial_t + \left(\partial_{\boldsymbol{r}}^2+k_0^2\right)^2. \tag{40}$$

From this, we get the bases of the null space as

$$e_k(x) = \exp[\omega(\boldsymbol{k})t + i\boldsymbol{k}\cdot\boldsymbol{r}] \tag{41}$$

with the dispersion relation $\omega(\boldsymbol{k}) = -(k^2-k_0^2)^2$, where $k=|\boldsymbol{k}|$. In the following, however, for simplicity, we consider the case of time-independent unperturbed states (stationary patterns in space as the basic patterns), so we assume $k=k_0$. Other operators in the general discussion above read [see Eq. (19) with $\chi=\{\tau,\boldsymbol{\rho}\}$]

$$\mathcal{L}(\partial_x,\partial_\chi) = \partial_\tau + \left\{\left[(\partial_{\boldsymbol{r}}+\partial_{\boldsymbol{\rho}})^2\right]^2 - \left(\partial_{\boldsymbol{r}}^2\right)^2\right\} + 2k_0^2\left[(\partial_{\boldsymbol{r}}+\partial_{\boldsymbol{\rho}})^2-\partial_{\boldsymbol{r}}^2\right] \tag{42}$$

and [see Eq. (20)]

$$\mathcal{L}_{\boldsymbol{k}}(\partial_\chi) = \partial_\tau + \Box_{\boldsymbol{k}}^2(\boldsymbol{\rho}), \tag{43}$$

where $\Box_{\boldsymbol{k}}(\boldsymbol{\rho}) = \partial_{\boldsymbol{\rho}}^2 + 2i\boldsymbol{k}\cdot\partial_{\boldsymbol{\rho}}$ ($k=k_0$ is assumed). Therefore, the amplitude for $e_{\boldsymbol{k}}(x)$ obeys the following proto-RG equation [see Eqs. (21) and (27)] to order ϵ:

$$\mathcal{L}_{\boldsymbol{k}}(\partial_\chi)A_{R\boldsymbol{k}}(\chi) = \epsilon\mathcal{P}_{\boldsymbol{k}}\left(\psi_0-\psi_0^3\right). \tag{44}$$

If we assume a striped pattern as the unperturbed pattern, then $\psi_0 = Ae^{i\boldsymbol{k}\cdot\boldsymbol{r}}+\text{c.c.}$, where c.c. denotes the complex conjugate. In Eq. (44) A in ψ_0 must be replaced with $A_R(\chi)$. Therefore, after replacing $\chi\to x$ and writing A for A_R accordingly, we obtain

$$\partial_t A + \Box^2 A = \epsilon(A-3|A|^2A), \tag{45}$$

where $\Box\equiv\Box_{\boldsymbol{k}}(\boldsymbol{r})$. This operator is a rotationally covariant operator [16].

Now, let us return to the PFC equation Eq. (36). The crucial difference from the case of SH equation Eq. (39) is that there is a zero-wave-vector mode [17], so even if the unperturbed pattern is stationary, we have the zeroth order solution in the form

$$\psi_0(x) = B + \left(\sum_{j=1}^m A_j e_{\boldsymbol{k}_j}(x) + \text{c.c.}\right). \tag{46}$$

Here, B is the amplitude of the zero-wave-vector mode, and

$$e_{\boldsymbol{k}_j}(x) = \exp[\omega(\boldsymbol{k}_j)t + i\boldsymbol{k}_j\cdot\boldsymbol{r}], \tag{47}$$

with the dispersion relation $\omega(\boldsymbol{k}_j) = -k_j^2(k_j^2-k_0^2)^2$ (but again, we set $k_j\equiv|\boldsymbol{k}_j| = k_0$ here). Each possible planform is characterized by m pairs of wave vectors $\{\boldsymbol{k}_i,-\boldsymbol{k}_i\}$ with critical wave numbers $k_i=k_0$, and the respective wave vectors add up to satisfy the resonant condition $\sum_{i=1}^m\boldsymbol{k}_i = 0$. With the structural form of (46), the equilibrium states of the PFC model (36) are found [18] to be stripes ($m=1$), hexagons ($m=3$), and bcc ($m=6$). In the present discussion we restrict ourselves to these planforms.

Other operators in the general discussion above read

$$\mathcal{L}(\partial_x,\partial_\chi) = \partial_\tau - (\partial_{\boldsymbol{r}}+\partial_{\boldsymbol{\rho}})^2\left[(\partial_{\boldsymbol{r}}+\partial_{\boldsymbol{\rho}})^2+k_0^2\right]^2 + \partial_{\boldsymbol{r}}^2\left(\partial_{\boldsymbol{r}}^2+k_0^2\right)^2 \tag{48}$$

and

$$\mathcal{L}_{\boldsymbol{k}_j}(\partial_\chi) = \partial_\tau - \left[\Box_j(\boldsymbol{\rho})-k_0^2\right]\Box_j^2(\boldsymbol{\rho}), \tag{49}$$

$$\mathcal{L}_{\boldsymbol{0}}(\partial_\chi) = \partial_\tau - \partial_{\boldsymbol{\rho}}^2\left(\partial_{\boldsymbol{\rho}}^2+k_0^2\right)^2, \tag{50}$$

with $\Box_j(\boldsymbol{\rho})\equiv\Box_{\boldsymbol{k}_j}(\boldsymbol{\rho})$, $j=1,\ldots,m$.

Now let us evaluate the first-order proto-RG equation Eq. (34), first for the case $m=3$ corresponding to the hexagonal pattern in two-dimensional systems. The pattern is supported by resonant interaction among a triplet of modes:

$$\begin{aligned} \boldsymbol{k}_1 &= k_0\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right), \quad \boldsymbol{k}_2 = k_0(0,1), \\ \boldsymbol{k}_3 &= k_0\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right). \end{aligned} \tag{51}$$

Then, we find

$$\begin{aligned} M(\psi_0) = {} & 6A_{R1}A_{R2}A_{R3} + 6A_{R1}^*A_{R2}^*A_{R3}^* - B_R \\ & + 6|A_{R1}|^2B_R + 6|A_{R2}|^2B_R + 6|A_{R3}|^2B_R + B_R^3 \\ & + s\,(2|A_{R1}|^2 + 2|A_{R2}|^2 + 2|A_{R3}|^2 + B_R^2) \\ & - \big[e^{i\boldsymbol{k}_1\cdot\boldsymbol{r}}\big\{A_{R1} - 3|A_{R1}|^2A_{R1} - 6|A_{R2}|^2A_{R1} \\ & - 6|A_{R3}|^2A_{R1} - 6A_{R2}^*A_{R3}^*B_R - 3A_{R1}B_R^2 \\ & - 2s(A_{R2}^*A_{R3}^* + A_{R1}B_R)\big\} \\ & + e^{i\boldsymbol{k}_2\cdot\boldsymbol{r}}\big\{A_{R2} - 6|A_{R1}|^2A_{R2} - 3|A_{R2}|^2A_{R2} \\ & - 6|A_{R3}|^2A_{R2} - 6A_{R3}^*A_{R1}^*B_R - 3A_{R2}B_R^2 \\ & - 2s(A_{R3}^*A_{R1}^* + A_{R2}B_R)\big\} \\ & + e^{i\boldsymbol{k}_3\cdot\boldsymbol{r}}\big\{A_{R3} - 6|A_{R1}|^2A_{R3} - 6|A_{R2}|^2A_{R3} \\ & - 3|A_{R3}|^2A_{R3} - 6A_{R1}^*A_{R2}^*B_R - 3A_{R3}B_R^2 \\ & - 2s(A_{R1}^*A_{R2}^* + A_{R3}B_R)\big\} + \cdots + \text{c.c.}\big]. \end{aligned} \tag{52}$$

In the above the ellipsis represents the nonfundamental mode contributions with wave vectors $2\boldsymbol{k}_j, 3\boldsymbol{k}_j, \boldsymbol{k}_m - \boldsymbol{k}_n, 2\boldsymbol{k}_m + \boldsymbol{k}_n$, where $m \neq n$.

Combining Eq. (52) with Eq. (34), we find for A_{R1}

$$\begin{aligned} & \partial_\tau A_{R1} - \left(\mathcal{L}_{k_1} - k_0^2\right)\mathcal{L}_{k_1}^2 A_{R1} \\ & \quad = \epsilon\left(\mathcal{L}_{k_1} - k_0^2\right)\big[\left(-1 + 2sB_R + 3B_R^2\right)A_{R1} + 2(s + 3B_R) \\ & \quad\quad \times A_{R2}^*A_{R3}^* + 3A_{R1}\{|A_{R1}|^2 + 2(|A_{R2}|^2 + |A_{R3}|^2)\}\big]. \end{aligned} \tag{53}$$

The appropriate cyclic permutation of the indices {1,2,3} gives the other equations for A_{R2} and A_{R3}. For B_R, the result is

$$\begin{aligned} & \partial_\tau B_R - \left(\partial_{\boldsymbol{\rho}}^2 + k_0^2\right)^2\partial_{\boldsymbol{\rho}}^2 B_R \\ & \quad = \epsilon\partial_{\boldsymbol{\rho}}^2\big\{-B_R + sB_R^2 + B_R^3 + 2(s + 3B_R)(|A_{R1}|^2 \\ & \quad\quad + |A_{R2}|^2 + |A_{R3}|^2) + 6(A_{R1}A_{R2}A_{R3} + \text{c.c.})\big\}. \end{aligned} \tag{54}$$

Therefore, we have finally obtained the proto-RG equations for the hexagonal patterns. Replacing the now dummy variables as $A_R \to A, B_R \to B, \boldsymbol{\rho} \to \boldsymbol{r}, \tau \to t$, they read

$$\begin{aligned} \partial_t A_j = {} & \left(\Box_j - k_0^2\right)\Bigg\{\left(\Box_j^2 - \epsilon\right)A_j + \epsilon(2sB + 3B^2)A_j \\ & + 2\epsilon(s + 3B)\prod_{\substack{i=1 \\ i\neq j}}^{3} A_i^* + 3\epsilon\left(2\sum_{i=1}^{3}|A_i|^2 - |A_j|^2\right)A_j\Bigg\}, \end{aligned} \tag{55}$$

$$\begin{aligned} \partial_t B = {} & \nabla^2\Bigg\{\left(\nabla^2 + k_0^2\right)^2 B + \epsilon(-B + sB^2 + B^3) \\ & + 2\epsilon(s + 3B)\sum_{i=1}^{3}|A_i|^2 + 6\epsilon\left(\prod_{i=1}^{3}A_i + \text{c.c.}\right)\Bigg\}, \end{aligned} \tag{56}$$

where $\Box_j, j = 1,2,3$ are the rotationally covariant operators, $\Box_j \equiv \nabla^2 + 2i\boldsymbol{k}_j \cdot \nabla$. These equations are identical to the amplitude equations given in Ref. [19], which are obtained with the multiple-scales method truncated at $O(\epsilon^{7/2})$. We emphasize, however, that the latter results are to be regarded as ones which have been obtained by enforcing the conservation by hand. The reason is that the finite order result in the multiple-scales method always breaks the conservation law, as expounded in Ref. [4].

The amplitude equation for the stripe pattern formation can be obtained from our results (55) and (56) by simply setting $A_1 = A, A_2 = A_3 = 0$. In particular, for $s = 0$ the resulting equations agree with those derived earlier in Ref. [4].

The case of $m = 6$, i.e., the bcc crystalline state in three dimensions can be treated in the same way by repeating the steps outlined above for the hexagonal structure. The basic wave vectors in Eq. (46) are

$$\begin{aligned} \boldsymbol{k}_1 &= \frac{k_0}{\sqrt{2}}(1,1,0), \quad \boldsymbol{k}_2 = \frac{k_0}{\sqrt{2}}(1,0,1), \\ \boldsymbol{k}_3 &= \frac{k_0}{\sqrt{2}}(0,1,1), \quad \boldsymbol{k}_4 = \frac{k_0}{\sqrt{2}}(0,1,-1), \\ \boldsymbol{k}_5 &= \frac{k_0}{\sqrt{2}}(1,-1,0), \quad \boldsymbol{k}_6 = \frac{k_0}{\sqrt{2}}(-1,0,1). \end{aligned} \tag{57}$$

Thus for $M(\psi_0)$ we have

$$M(\psi_0) = \tilde{B} + \sum_{p=1}^{6}\left(\tilde{A}_p e^{i\boldsymbol{k}_p\cdot\boldsymbol{r}} + \text{c.c.}\right) + \cdots, \tag{58}$$

where the ellipsis represents the nonfundamental mode contributions and

$$\begin{aligned} \tilde{B} = {} & 6(A_1^*A_2A_4 + A_2^*A_3A_5 + A_1A_3^*A_6 + A_4A_5A_6 + \text{c.c.}) \\ & - B + 6B(|A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2 + |A_5|^2 \\ & + |A_6|^2) + B^3 + 2s(|A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2 \\ & + |A_5|^2 + |A_6|^2) + sB^2, \end{aligned} \tag{59}$$

$$\begin{aligned} \tilde{A}_1 = {} & -A_1 + 3A_1(|A_1|^2 + 2|A_2|^2 + 2|A_3|^2 + 2|A_4|^2 + 2|A_5|^2 \\ & + 2|A_6|^2) + 6A_3A_4A_5 + 6A_2A_5^*A_6^* + 6A_2A_4B \\ & + 6A_3A_6^* + 3A_1B^2 + 2s(A_2A_4 + A_3A_6^* + A_1B), \end{aligned} \tag{60}$$

$$\begin{aligned} \tilde{A}_4 = {} & -A_4 + 3A_4(2|A_1|^2 + 2|A_2|^2 + 2|A_3|^2 + |A_4|^2 \\ & + 2|A_5|^2 + 2|A_6|^2) + 6A_1A_3^*A_5^* + 6A_2^*A_3A_6^* \\ & + 6A_1A_2^*B + 6A_5^*A_6^*B + 3A_4B^2 \\ & + 2s(A_1A_2^* + A_5^*A_6^* + A_4B). \end{aligned} \tag{61}$$

The coefficients $\tilde{A}_2$ and $\tilde{A}_3$ are obtained by separately operating cyclic permutations on {1,2,3} and {4,5,6} in Eq. (60). For example, the term $6A_2A_5^*A_6^*$ in $\tilde{A}_1$ becomes

$6A_3A_6^*A_4^*$ in $\tilde{A}_2$ and $6A_1A_4^*A_5^*$ in $\tilde{A}_3$. Similarly, $\tilde{A}_5$ and $\tilde{A}_6$ are obtained by the same cyclic permutations performed on Eq. (61). Therefore, we find the proto-RG equations for the bcc crystal formation as follows (writing A_1 for A_{R1} and so on):

$$\frac{\partial B}{\partial t} = \nabla^2\left[\left(\nabla^2+k_0^2\right)^2 B + \epsilon(-B+sB^2+B^3) + 2\epsilon(s+3B)\sum_{p=1}^{6}|A_p|^2 + 6\epsilon(A_4A_5A_6 + A_1^*A_2A_4 + A_2^*A_3A_5 + A_3^*A_1A_6 + \text{c.c.})\right], \tag{62}$$

$$\begin{aligned}\frac{\partial A_j}{\partial t} = \left(\Box_j - k_0^2\right)\Bigg[&\left(\Box_j^2-\epsilon\right)A_j + \epsilon(2sB+3B^2)A_j + 2\epsilon(s+3B)(A_kA_m + A_iA_\ell^*)\\ &+6\epsilon(A_iA_mA_n + A_kA_\ell^*A_n^*) + 3\epsilon A_j\left(2\sum_{p=1}^{6}|A_p|^2 - |A_j|^2\right)\Bigg],\end{aligned} \tag{63}$$

$$\begin{aligned}\frac{\partial A_m}{\partial t} = \left(\Box_m - k_0^2\right)\Bigg[&\left(\Box_m^2-\epsilon\right)A_m + \epsilon(2sB+3B^2)A_m + 2\epsilon(s+3B)(A_jA_k^* + A_\ell^*A_n^*)\\ &+6\epsilon(A_iA_k^*A_\ell^* + A_i^*A_jA_n^*) + 3\epsilon A_m\left(2\sum_{p=1}^{6}|A_p|^2 - |A_m|^2\right)\Bigg],\end{aligned} \tag{64}$$

where the indices (i,j,k) and (ℓ,m,n) are cyclic permutations of (1,2,3) and (4,5,6), respectively. To our knowledge, the multiple-scales analysis is not available for the amplitude equations of the bcc crystalline formation. However, Yeon *et al.* [19] obtained the same result as above employing what they called a variational approach. This approach, however, does not fall under the class of singular perturbation methods, and since this approach does not have any parameter to regulate the order of approximation, it cannot be systematic and must remain ad hoc.

V. REMARKS

The derivation of the proto-RG equation is not the final goal of system reduction; the system reduction is not yet completed. Do not forget that these equations are called *proto*-RG equations because usually they require further modifications [2,3]. First of all, we must specify what we wish to observe, or more precisely, we must specify on what space-time scale we wish to study the system. If we wish to observe a system to a given scale of precision, all the scales up to the specified precision must be consistently preserved (e.g., if order ϵ^3 is the specified precision, all the terms not exceeding order ϵ^3 must be collected). Thus, the consistency of the orders of various terms is probably the chief guiding principle to obtain physically meaningful reduced equations. In other words, we should demand that the obtained equation describe a certain intermediate asymptotic processes allowed to the system under study.

Consider, for example, the proto-RG equation Eq. (45) for striped patterns exhibited by the SH equation. If one wishes to describe a diffusive behavior of the order parameter ψ, a natural requirement is $\partial_t \sim \partial_r^2 \sim \epsilon$ [this expression implies that these derivatives are of the order specified (e.g., $\nabla\psi \sim \epsilon^{1/2}$)]. If $k_0 \sim 1$ (this is the usual interpretation; we are interested in the global and slow change of the pattern of the basic scale of order 1), the operator $\Box$ consists of two operators of order ϵ and of order $\epsilon^{1/2}$. Thus, to order ϵ, $\Box^2 = (2i\boldsymbol{k}\cdot\nabla)^2$, and (45) gives

$$\partial_t A = 4(\boldsymbol{k}\cdot\nabla)^2 A + \epsilon(A - 3|A|^2A), \tag{65}$$

which loses the rotational covariance in the original proto-RG equation. Therefore, if we wish to describe the diffusive space-time dynamics with rotational covariance, since keeping the whole $\Box^2$ requires retention of $O(\epsilon^2)$ terms, the $O(\epsilon)$ reduction (45) is not enough; we need the next order reduction [i.e., the proto-RG equation to order $O(\epsilon^2)$ is required] [3]. Incidentally, do not misunderstand the proto-RG approach as a method to preserve appropriate symmetries. As emphasized above, the approach is the simplest reduction method that allows reduction; the preservation of symmetry is a byproduct (and it is a side issue).

Instead, if we are interested in the spatial pattern of order 1 (i.e., $k_0 \sim 1$) with evolution on the space-time scale $\partial_t \sim \epsilon$ and $\partial_r \sim \epsilon^{1/4}$, the proto-RG equation (45) is consistent to order ϵ; without dropping or adding any term, we may interpret it as a properly reduced equation to $O(\epsilon)$.

A similar conclusion applies to the proto-RG equations (55) and (56) for striped patterns in the PFC equation. With $\partial_t \sim \epsilon^{3/2}, \partial_r \sim \epsilon^{1/4}$, and $k_0 \sim 1$, we may consistently interpret these proto-RG equations as properly reduced equations (consistent to order $\epsilon^{3/2}$; notice that no further contribution comes from the RHS).

On the other hand, if pattern evolution of the scale $\partial_t \sim \epsilon$, $\partial_r \sim \epsilon^{1/2}$ (this is the scale the multiple-scales analysis employs) is of interest to us, the reduced equations in which the rotational invariance and conservation law are preserved are obtained only at $O(\epsilon^3)$. As can easily be guessed, higher order reduction then requires us to retain numerous terms on the RHS of the proto-RG equations; the number of terms is expected to increase exponentially. This implies that, for example, the $O(\epsilon^3)$ reduction is almost impractical, defeating the purpose of system reduction.

The reader might think that the reductive renormalization approach is quite similar to the multiple-scales (MS) expansion approach that has been standard in systematic system reductions. Notice, however, that there is a crucial difference. As is clear from the above discussion, the derivation of the RG equation does not aim at the reduction of the system that is consistent in space-time scales; the method concentrates on removing secular space-time dependence that ruins any naive reduction. Only after obtaining the reduced equation that is guaranteed to have no secular discrepancy from the original equation do we start paying attention to the scaling consistency.

One important aspect of the present proto-RG formalism is that the proto-RG equation respects the conservation law irrespective of the order in ϵ to which the perturbation expansion is truncated. On the other hand, it is not the case with the MS formalism. (Although this point has already been emphasized in Ref. [4], we here recapitulate it for the sake of adding clarity to our present formulation.) To see this, let us take up the proto-RG equation (55) for the PFC model. Without losing generality of the following argument, we simply consider the particular case of striped patterns with $s = 0$. Then the proto-RG equation for A takes the form

$$\partial_t A = \left(\Box - k_0^2\right)[(\Box^2 - \epsilon)A + 3\epsilon(|A|^2 + B^2)A + \text{h.o.t.}], \tag{66}$$

where h.o.t. denotes the higher order (in ϵ) terms. First, notice that

$$\frac{d}{dt}\int d\boldsymbol{x}\, A(t,\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0, \tag{67}$$

owing to the presence of the operator $\Box - k_0^2$ in (66). This is in accord with the order parameter (ψ) conservation inherent in the PFC equation. Now consider the pattern evolution of the scales $\partial_t \sim \epsilon$, $\partial_{\boldsymbol{x}} \sim \epsilon^{1/2}$ as in the MS analysis. In the MS analysis the expansion parameter is $\delta \equiv \epsilon^{1/2}$, and the operator $\Box$ acting on the amplitudes (A, B) operates in the combination

$$2i\delta\boldsymbol{k}\cdot\boldsymbol{\nabla}_{\boldsymbol{X}} + \delta^2\nabla_{\boldsymbol{X}}^2,$$

where $\boldsymbol{\nabla}_{\boldsymbol{X}}$ operates on the slow variables $\boldsymbol{X} = \delta\boldsymbol{x}$. Thus the conserving operator $\Box - k_0^2$ is mixed order in ϵ. This fact then forces an inescapable conclusion that in the MS analysis the finite-order result always breaks the conservation law. For example, the correct term proportional to

$$\epsilon^2\left(\Box - k_0^2\right)A|A|^4$$

will appear on the RHS of Eq. (66) if the amplitude equation is truncated at $O(\delta^6)$, whereas at $O(\delta^4)$ we only obtain $-\epsilon^2 k_0^2 A|A|^4$, which violates the conservation. Since at any finite order there always appear such nonconserving terms, we cannot escape the aforementioned conclusion [20].

How reliable is the lowest nontrivial order reduction satisfying a particular planform? For the case of ODEs there is a theorem due to Chiba [6], demonstrating (roughly speaking) the C^1 diffeomorphism between the invariant manifold of the original equation and that of the RG equation. More precisely, the original vector field and that governing the renormalized perturbation result are diffeomorphic. That is, they are qualitatively the same. Thus, for example, if the RG equation has a stable limit cycle as in the case of the van der Pol equation, this proves that the original van der Pol equation has a stable limit cycle. There is no analogous theorem known for PDEs, but if the spatial pattern in a big box is not very different from the infinite space case, as is usually the case, we could reduce the PDE as a set of ODEs. Then, Chiba's theorem applies. Thus, we may conclude that even for the PDE case the order-ϵ reductive renormalization result is qualitatively the same (diffeomorphic) as the original result.

Our general conjecture is as follows: If the original ODE or PDE is (C^1) structurally stable (or at least Ω stable for an invariant set Ω), then the reductive RG result and that of the original equation are qualitatively the same (C^1 diffeomorphic). In the case of maps (i.e., the dynamical systems defined by maps), a necessary and sufficient condition for the original system and the C^1-perturbed system to be diffeomorphic (i.e., the system is C^1 structurally stable) is that the system is normally hyperbolic [21]. Unfortunately, as far as we are aware, there is no differential equation counterpart of this equivalence theorem. What Chiba demonstrated is that if this theorem holds for ODE, the reductive renormalization preserves the qualitative features of the structurally stable system. Thus, our conjecture should be a very natural one.

Since we may generally expect that, for sufficiently small ϵ, the reduced system to order ϵ should be qualitatively the same as the original system, one might be tempted to obtain higher order reduction results (and to order ϵ^2 they may still be reasonably practical) to understand the cases with slightly larger ϵ. However, if there is no new bifurcation for larger ϵ, the ϵ order (i.e., the lowest nontrivial order) result should be enough. If there is a new bifurcation, then it may be practical to devise the reduction around this new bifurcation point so that the lowest nontrivial order result would be used.

APPENDIX: RENORMALIZABILITY TO ALL ORDERS

The purpose of this appendix is to demonstrate that the renormalization constant determined solely from the resonant secular terms can successfully eliminate all the secular terms in the perturbation series to all orders. This appendix is self-contained.

We consider an autonomous equation (no explicit x dependence, where x collectively denotes all the independent variables)

$$L\psi = \epsilon N[\psi], \tag{A1}$$

where L is a linear differential operator with constant coefficients and we assume that its eigenfunctions e_i are plane waves (or we assume that their derivatives are proportional to themselves) [22]. We consider a set Λ of such eigenfunctions (not necessarily the totality of the eigenfunctions, but as in the illustrations given in the main part of this paper, Λ may be chosen to be a set of eigenfunctions generating a finite group). As a notational convention we identify Λ and the set of suffixes of the functions in Λ (i.e., if $\Lambda = \{e_1,e_2,e_3\}$, the set $\{1,2,3\}$ is also denoted as Λ). The general solution to the unperturbed equation $L\psi_0 = 0$ reads

$$\psi_0 = \sum_{i\in\Lambda} A_i e_i(x). \tag{A2}$$

We write the formal expansion of the solution to Eq. (A1) as

$$\psi(x,A) = \sum A_i e_i(x) + \epsilon\psi_1(x,A) + \epsilon^2\psi_2(x,A) + \cdots, \tag{A3}$$

where A collectively denotes $\{A_i\}_{i\in\Lambda}$.

Let $X \subset \Lambda$, and write $e_X = \prod_{i\in X} e_i$. Let us define L_X by $Lfe_X = e_X L_X f$. If we may assume $L_X e_i \neq 0$ for X containing more than one element, then a special solution to $Lf = \psi\, e_i$ satisfies $f = \phi\, e_i$, so $L_i\phi = \psi$. This implies that ϕ is secular [not a bounded function (usually polynomial) of x]. For example, for a constant c, $L_i c = 0$ because $Lce_i = 0$. Hence, $Lfe_i = e_i$ implies $L_i f = 1$, so f must be unbounded for the usual L.

First, let us look at the RG procedure up to the third order in ϵ. This will help to introduce streamlined notations and will serve to initiate a mathematical induction later.

The first order perturbation equation reads (symbolically)

$$L\psi_1(x,A) = N[Ae(x)] = N\left[\sum_i A_i e_i\right]. \tag{A4}$$

$N[f]$ is usually a polynomial of f and its derivatives. The derivatives of e_i can be written in terms of e_i. Hence, N may be decomposed as

$$N\left[\sum_i A_i e_i\right] = \sum_X n_X(Ae). \tag{A5}$$

Here, Ae collectively denotes $\{A_i e_i\}$, and $n_X(Ae)$ is a monomial $\prod_{i\in X} A_i e_i$ (times a constant), where X consists of elements of Λ with specified multiplicities; for example, $X = \{1,1,3,4,7,7,7\}$ denotes (using the Hadamard notation) $(Ae)^X = (A_1e_1)^2 A_3e_3A_4e_4(A_7e_7)^3$. Sometimes $e^X = e^Y$ for $X \neq Y$. We do not reduce and combine such products to a single monomial. Notice that this structure Eq. (A5) is uniquely determined by N itself (up to the just mentioned reduction). Hence, a special solution may always be written as

$$\psi_1 = \sum_X L^{-1} n_X(Ae). \tag{A6}$$

Consequently, we have ψ to order ϵ as

$$\psi^{[1]} = Ae + \epsilon\sum_X L^{-1} n_X(Ae). \tag{A7}$$

Here, the first term Ae denotes ψ_0 (while Ae as a variable of n_X denotes the set $\{A_i e_i\}$ throughout the appendix). A is renormalized as [which is the same as Eq. (7)]

$$A_i = A_{Ri} + \epsilon Z_1^i A_{Ri} + \epsilon^2 Z_2^i A_{Ri} + \cdots + \epsilon^k Z_k^i A_{Ri} + \cdots, \tag{A8}$$

where the suffix $i \in \Lambda$ will henceforth be suppressed unless otherwise noted. The renormalization procedure to order ϵ starts with

$$\psi^{[1]} = (1+\epsilon Z_1)A_R e + \epsilon\sum_X L^{-1} n_X(A_R e). \tag{A9}$$

Therefore, the following renormalization condition for each $i \in \Lambda$ removes the secular term to the first order:

$$Z_1^i A_{Ri} e_i + L^{-1} n_i(A_R e) = 0. \tag{A10}$$

This secular term is, needless to say, a resonant secular term. Thus, we have

$$\psi^{[1]} = A_R e + \epsilon\psi_{R1}, \tag{A11}$$

where we have introduced the following notation to keep renormalized and nonsecular terms without distinguishing them:

$$\psi_{R1} = Z_1 A_R e + \sum_X L^{-1} n_X(A_R e). \tag{A12}$$

We understand that Z_1 kills the secular term in the second term.

The second order perturbation equation reads

$$L\psi_2(x,A) = O[\epsilon] \text{ terms of } N\left[Ae + \epsilon\sum_X L^{-1} n_X(Ae)\right]. \tag{A13}$$

We need the Taylor expansion of n_X:

$$\begin{aligned} n_X(Ae + \epsilon B(x)e) = {} & n_X(Ae) + \epsilon\sum_i \partial_i n_X(Ae) B_i(x) e_i \\ & + \frac{\epsilon^2}{2}\sum_{i,j} \partial_i\partial_j n_X(Ae) B_i(x)e_i B_j(x)e_j \\ & + \cdots, \end{aligned} \tag{A14}$$

where ∂_i is the partial differentiation with respect to $A_i e_i$. We abbreviate this as

$$\begin{aligned} n_X(Ae + \epsilon B(x)e) = {} & n_X(Ae) + \epsilon n'_X(Ae)B(x)e \\ & + \frac{\epsilon^2}{2} n''_X(Ae)[B(x)e]^2 + \cdots. \end{aligned} \tag{A15}$$

Generally, we write

$$\sum_{i_1,\cdots,i_k\in\Lambda} \partial_{i_1}\cdots\partial_{i_k} n_X(Ae) F_{i_1}\cdots F_{i_k} = n_X^{(k)}(Ae)F^k, \tag{A16}$$

as long as the summations of suffixes run over all possible elements in Λ. Eq. (A13) reads, with these notations,

$$L\psi_2(x,A) = \sum_Y n'_Y(Ae)\sum_X L^{-1} n_X(Ae). \tag{A17}$$

Here, the dummy summation variables X and Y both run over the same suffix set (determined by N). Therefore,

$$\psi_2(x,A) = L^{-1}\sum_X n'_X(Ae)\sum_Y L^{-1} n_Y(Ae). \tag{A18}$$

Here, L^{-1} acts on everything to its right (generally, until it hits the unpaired closed parenthesis, if any). Thus, the bare perturbation expression for ψ to order ϵ^2 reads

$$\begin{aligned} \psi^{[2]} = {} & Ae + \epsilon\sum_X L^{-1} n_X(Ae) \\ & + \epsilon^2 L^{-1}\sum_X n'_X(Ae)\sum_Y L^{-1} n_Y(Ae). \end{aligned} \tag{A19}$$

The renormalization procedure is to replace A with $A_R + \epsilon Z_1 A_R + \cdots$ and then to Taylor expand Eq. (A19) with respect to ϵ.

At this juncture, notice that, since L^{-1} is a linear operator, the expansion with respect to ϵ and application of L^{-1} commute. For example, we obviously have

$$L^{-1} f(x+\epsilon g(x)) = L^{-1} f(x) + \epsilon L^{-1} f'(x) g(x) + \tfrac{1}{2}\epsilon^2 L^{-1} f''(x) g^2(x) + o[\epsilon^2]. \quad \text{(A20)}$$

Therefore, as long as we do not change the order of L^{-1} and functions of x and stick to the convention that it acts on everything to its right (until it hits the unpaired closed parenthesis, if any), we can virtually perform any transformation without calculating the explicit result of operating L^{-1}.

Let us return to the perturbative calculation. The needed expansion result reads (to order ϵ^2)

$$\begin{aligned}\psi^{[2]} &= (1+\epsilon Z_1 + \epsilon^2 Z_2) A_R e + \epsilon \sum_X L^{-1} n_X (A_R e + \epsilon Z_1 A_R e) + \epsilon^2 \sum_X L^{-1} n'_X (A_R e) \sum_Y L^{-1} n_Y (A_R e) \\ &= A_R e + \epsilon \left[Z_1 A_R e + \sum_X L^{-1} n_X (A_R e) \right] + \epsilon^2 \left[Z_2 A_R e + \sum_X L^{-1} n'_X (A_R e) \left\{ Z_1 A_R e + \sum_Y L^{-1} n_Y (A_R e) \right\} \right]. \end{aligned} \quad \text{(A21)}$$

We may write this as a renormalized perturbation form:

$$\psi^{[2]} = A_R e + \epsilon \psi_{R1} + \epsilon^2 \psi_{R2}. \quad \text{(A22)}$$

Notice that

$$\psi_{R2} = Z_2 A_R e + \sum_X L^{-1} n'_X (A_R e) \left\{ Z_1 A_R e + \sum_Y L^{-1} n_Y (A_R e) \right\} \quad \text{(A23)}$$

$$= Z_2 A_R e + \sum_X L^{-1} n'_X (A_R e) \psi_{R1}. \quad \text{(A24)}$$

The term in { } has no singularity; this implies the singularities that have not been renormalized by the lower order calculations (i.e., the nonresonant secular terms to the second order $Y_2^{[nr]}$) do not show up. The singularity produced anew from the last term is the new second order singularity, which is killed by Z_2.

The third order perturbative equation reads, in terms of self-explanatory abbreviations,

$$L\psi_3(x,A) = O[\epsilon^2] \text{ terms of } N[Ae + \epsilon\psi_1 + \epsilon^2 \psi_2 \cdots] = N'[Ae]\psi_2 + (1/2) N''[Ae]\psi_1^2. \quad \text{(A25)}$$

Therefore,

$$\psi_3 = L^{-1} \sum_Z n'_Z(Ae) \sum_Y L^{-1} n'_Y(Ae) \sum_X L^{-1} n_X(Ae) + \frac{1}{2} L^{-1} \sum_Z n''_Z(Ae) \left[\sum_X L^{-1} n_X(Ae) \right]^2. \quad \text{(A26)}$$

The renormalization procedure to order ϵ^3 is as follows:

$$\begin{aligned}\psi^{[3]} &= (1+\epsilon Z_1 + \epsilon^2 Z_2 + \epsilon^3 Z_3) A_R e + \epsilon \sum_X L^{-1} n_X ((1+\epsilon Z_1 + \epsilon^2 Z_2) A_R e) \\ &\quad + \epsilon^2 \sum_Y L^{-1} n'_Y ((1+\epsilon Z_1) A_R e) \sum_X L^{-1} n_X ((1+\epsilon Z_1) A_R e) \\ &\quad + \epsilon^3 \left[\sum_Z L^{-1} n'_Z(Ae) \sum_Y L^{-1} n'_Y(Ae) \sum_X L^{-1} n_X(Ae) + \frac{1}{2} L^{-1} \sum_Z n''_Z(Ae) \left(\sum_X L^{-1} n_X(Ae) \right)^2 \right] \\ &= A_R e + \epsilon \left[Z_1 A_R e + \sum_X L^{-1} n_X(A_R e) \right] + \epsilon^2 \left[Z_2 A_R e + \sum_X L^{-1} n'_X(A_R e) \left\{ Z_1 A_R e + \sum_Y L^{-1} n_Y(A_R e) \right\} \right] \\ &\quad + \epsilon^3 \left[Z_3 A_R e + \sum_X L^{-1} n'_X(A_R e) \left\{ Z_2 A_R e + \sum_Y L^{-1} n'_Y(A_R e) \left(Z_1 A_R e + \sum_Z L^{-1} n_Z(A_R e) \right) \right\} \right. \\ &\quad \left. + \frac{1}{2} \sum_X L^{-1} n''_X(A_R e) \left(Z_1 A_R e + \sum_Y L^{-1} n_Y(A_R e) \right)^2 \right]. \end{aligned} \quad \text{(A27)}$$

Recalling the results (A12) and (A23), we can write this in a renormalized perturbation form:

$$\psi^{[3]} = A_R e + \epsilon\psi_{R1} + \epsilon^2\psi_{R2} + \epsilon^3\psi_{R3}, \tag{A28}$$

where ψ_{R3}, i.e., the coefficient of ϵ^3 in (A27), is rewritten as

$$\psi_{R3} = Z_3 A_R e + \sum_X L^{-1} n'_X(A_R e)\psi_{R2} + \frac{1}{2}\sum_X L^{-1} n''_X(A_R e)\psi_{R1}^2. \tag{A29}$$

An important observation is that the renormalized perturbation series has a branched and nested structure. When ψ_1 and ψ_2 appear in the calculation of $\psi^{[3]}$, their secular terms are exactly killed by Z_1 and Z_2, and the newly produced secular term from the resonant term of order ϵ^3 is killed by Z_3.

Introducing the abbreviations $B = Ae$ and $M = L^{-1}N$, we can make the structure exhibited above more transparent. Our problem reads

$$\psi = B + \epsilon M[\psi]. \tag{A30}$$

Iterative substitution solves this equation as

$$\psi = B + \epsilon M[B + \epsilon M[B + \epsilon M[B + \cdots]]]. \tag{A31}$$

If we renormalize this, we have, setting $A_R e = R$, i.e., $B = R + \epsilon Z R$, where $\epsilon Z = \epsilon Z_1 + \epsilon^2 Z_2 + \cdots$,

$$\psi = R + \epsilon Z R + \epsilon M[R + \epsilon Z R + \epsilon M[R + \epsilon Z R + \epsilon M[R + \cdots]]]. \tag{A32}$$

Let us write $\psi_{Rk} = Y_k$. Then, generally, we can write ψ to order ϵ^k, $\psi^{[k]}$, as

$$\psi^{[k]} = R + \epsilon Y_1 + \cdots + \epsilon^k Y_k = R + \epsilon Z R + \epsilon M[\psi^{[k-1]}], \tag{A33}$$

where now $\epsilon Z = \epsilon Z_1 + \cdots + \epsilon^{k-1} Z_{k-1} + \epsilon^k Z_k$. Thus, the calculation above can be formally streamlined as

$$\begin{aligned}
\psi^{[1]} &= R + \epsilon(Z_1 R + M), \quad M \equiv M[R],\\
\psi^{[2]} &= R + \epsilon Y_1 + \epsilon^2(Z_2 R + M' Y_1),\\
\psi^{[3]} &= R + \epsilon Y_1 + \epsilon^2 Y_2 + \epsilon^3\big(Z_3 R + M' Y_2 + M'' Y_1^2/2\big),\\
\psi^{[4]} &= R + \epsilon Y_1 + \epsilon^2 Y_2 + \epsilon^3 Y_3 + \epsilon^4\big(Z_4 R + M' Y_3\\
&\quad + M'' Y_1 Y_2 + M''' Y_1^3/6\big),
\end{aligned} \tag{A34}$$

and so on. Notice that

$$\epsilon^k Y_k = R + \epsilon Z R + \epsilon M[\psi^{[k-1]}] - \psi^{[k-1]}. \tag{A35}$$

Therefore, if Z_{k-1} is determined by the resonant secular terms in $\psi^{[l]}$ ($l = 1,\ldots,k-1$), which have been successfully renormalized, then the remaining secular term in $\psi^{[k]}$ is the resonant secular term of order ϵ^k, which is removed by Z_k. We have already demonstrated such relations up to $k = 3$. Therefore, we may conclude that the renormalization constants are determined solely by the resonant secular terms.

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[1] H. Mori and Y. Kuramoto, *Dissipative Structure and Chaos* (Springer, New York, 1998).

[2] K. Nozaki and Y. Oono, Phys. Rev. E **63**, 046101 (2001).

[3] K. Nozaki, Y. Oono, and Y. Shiwa, Phys. Rev. E **62**, R4501 (2000); Y. Shiwa, *ibid.* **63**, 016119 (2000).

[4] Y. Shiwa, Prog. Theor. Phys. **125**, 871 (2011).

[5] B. P. Athreya, N. Goldenfeld, and J. A. Dantzig, Phys. Rev. E **74**, 011601 (2006); Y. Shiwa, *ibid.* **79**, 013601 (2009); N. Goldenfeld, B. P. Athreya, and J. A. Dantzig, *ibid.* **79**, 013602 (2009).

[6] H. Chiba, SIAM J. Appl. Dyn. Syst. **7**, 895 (2008); **8**, 1066 (2009); J. Differ. Equations **246**, 1991 (2009).

[7] R. E. Lee De Ville, A. Harkin, M. Holzer, K. Josić, and T. Kaper, Physica D **237**, 1029 (2008).

[8] Y. Oono, *The Nonlinear World: Conceptual Analysis and Phenomenology* (Springer, New York, 2012), Sec. 3.9.

[9] K. R. Elder, M. Katakowski, M. Haataja, and M. Grant, Phys. Rev. Lett. **88**, 245701 (2002); K. R. Elder and M. Grant, Phys. Rev. E **70**, 051605 (2004).

[10] J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).

[11] Y. Oono, Int. J. Mod. Phys. B **14**, 1327 (2000).

[12] Since L is a partial differential operator, if it is defined on a set of functions whose domain is not compact, its spectrum cannot generally be simple (eigenvalues may not exhaust its spectrum). In this paper, however, the reduction is always around an unperturbed solution that is in a finite dimensional subspace of the null space of L.

[13] We proceed quite formally here, so we need not worry about the spectrum of L, but if we put the system in a big but compact box, L is just an ordinary differential operator. Thus, there is no use in expounding the true meaning of the summation in this formula. Formally, it could be an integration or an infinite summation, but for the usual practical examples (as illustrated in Sec. IV) the summation is a finite summation.

[14] If an honest detailed calculation is done order by order, χ does not remain. However, the bare order by order calculation and the renormalized order by order calculation are different, so this procedure is the simplest.

[15] If an honest detailed calculation is performed, no χ will remain; however, if a simplified calculation is done, χ might remain because all the relations are not utilized. Thus, the recipe described here gives the same result as the exact one.

[16] G. H. Gunaratne, Qi Ouyang, and H. L. Swinney, Phys. Rev. E **50**, 2802 (1994).

[17] See the second paper cited in Ref. [5]; we particularly point out that qualitatively new instabilities with no analog in the absence of the zero mode have been found in the (two-dimensional) PFC model in H. Ohnogi and Y. Shiwa, Physica D **237**, 3046 (2008).

[18] K. R. Elder, N. Provatas, J. Berry, P. Stefanovic, and M. Grant, Phys. Rev. B **75**, 064107 (2007).

[19] D.-H. Yeon, Z.-F. Huang, K. R. Elder, and K. Thornton, Philos. Mag. **90**, 237 (2010).

[20] Here it is worth reemphasizing the point illustrated by Eq. (65). Unlike in PDEs with the conservation law, the MS analysis in PDEs without the conservation law can preserve the rotational covariance symmetry at some finite orders, as first explained in Ref. [16].

[21] M. Hirsch, C. Pugh, and M. Shub, Bull. Am. Math. Soc. **76**, 1015 (1970); R. Mañé, Trans. Am. Math. Soc. **246**, 261 (1978).

[22] The auxiliary conditions (e.g., the boundary condition) are included in the definition of the operator.