Spectral relations between products and powers of isotropic random matrices

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We show that the limiting eigenvalue density of the product of n identically distributed random matrices from an isotropic unitary ensemble is equal to the eigenvalue density of nth power of a single matrix from this ensemble, in the limit when the size of the matrix tends to infinity. Using this observation, one can derive the limiting density of the product of n independent identically distributed non-Hermitian matrices with unitary invariant measures. In this paper we discuss two examples: the product of n Girko-Ginibre matrices and the product of n truncated unitary matrices. We also provide evidence that the result holds also for isotropic orthogonal ensembles.

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I. INTRODUCTION

Free probability theory arose from merging the concepts of noncommutative probability and free independence. Since the link between free probability theory and random matrix theory was established [1], several new results have been proven in an easy way in the limit of infinitely large random matrices [2–4]. In this paper we demonstrate a simple albeit quite counterintuitive result that the spectral density of the product of n free, identically distributed random matrices from an isotropic unitary ensemble (IUE) is equal to the spectral density of the nth power of a single matrix from this ensemble in the limit of infinite matrix size. The proof is based on the multiplicative properties of the S transform and the Haagerup-Larsen theorem [5].

The motivation for the present work comes from the observation made in Refs. [6–9] that the eigenvalue density of n independent Girko-Ginibre matrices [10,11] is the same as the eigenvalue density of the nth power of a single Girko-Ginibre matrix in the limit of infinite size. This observation leads to the question whether this is a feature of only this particular class of matrices or there exists a larger class of matrices that have this property. In the present paper we show that there indeed exists a larger class of matrices sharing this property, a class of random isotropic matrices. We begin with defining isotropic matrices. Then we present the main result and its derivation. Finally we outline a few simple applications related to the recent interest in the literature. In particular we apply our result to the product of Girko-Ginibre matrices and rederive the limiting density obtained in Ref. [6]. We also consider classes of truncated unitary [12] and orthogonal matrices. We compare our predictions to Monte Carlo simulations and identify finite size corrections to the limiting distribution. We conclude the paper with a summary.

II. ISOTROPIC RANDOM MATRICES

It is convenient to introduce the concept of isotropic random matrices in analogy to isotropic complex random variables z that have a circularly symmetric probability distribution depending only on the module $|z|$. Using the polar decomposition, one can write $z = re^{i\phi}$, where r is a real non-negative random variable and ϕ is a random variable (phase) with a uniform distribution on $[0,2\pi)$. Isotropic random matrices are defined by a straightforward generalization of isotropic complex random variables. A square $N \times N$ matrix x is said to be an isotropic random matrix if it has a polar decomposition $x = hu$ in which h is a positive semidefinite Hermitian random matrix and u is a unitary random matrix independent of h that is distributed on the unitary group $U(N)$ with the Haar measure. In short, u is a Haar unitary matrix. The random matrix h plays the role of the radial part of x. Such random matrices form an ensemble of isotropic unitary matrices. More precisely, this means that we consider ensembles of random matrices with a probability measure that is right invariant under multiplication by a unitary matrix or, in other words, that depends only on $xx^\dagger$. An example is an ensemble generated by the partition function [13,14]

$$Z = \int Dx\, e^{-N\mathrm{Tr}V(x^\dagger x)}, \tag{1}$$

where $Dx = \prod_{ij} d(\mathrm{Re}x_{ij})d(\mathrm{Im}x_{ij})$ is a flat measure and $V(a)$ is a polynomial in a. Another natural class of IUE matrices consists of random matrices of the form $x = vdu$, where d is an $N \times N$ diagonal random matrix having real positive random eigenvalues drawn independently from the given probability distribution and v and u are two independent random unitary matrices with the Haar measure on the unitary group $U(N)$. Isotropic random matrices have been considered, for example, in the context of isotropic random inputs for multiple antenna [15]. By analogy one can also consider the isotropic orthogonal ensemble (IOE) given by the decomposition $x = so$, with s being a positive semidefinite real symmetric matrix and o being a Haar orthogonal matrix. In this case x is a real random matrix with a probability measure that is right invariant under multiplication by an orthogonal matrix or, in other words, that depends only on xx^T.

Isotropic matrices for $N \to \infty$ are called R diagonal in mathematical literature [16]. In this paper we prefer to stick to the name isotropic (or IUE or IOE) even in the large-N limit. Isotropic unitary ensemble matrices have an eigenvalue distribution independent of the polar angle on the complex

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plane. In the limit $N \to \infty$ one can find an explicit relation between the eigenvalue densities of the matrix h^2 and of the matrix x [5,13,14,17]. We briefly recall this relation below. We note that the angular independence of the eigenvalue density does not imply that the random matrix is isotropic. For example, a block diagonal matrix of the form

$$x = \begin{pmatrix} h_1 u_1 & 0 \\ 0 & h_2 u_2 \end{pmatrix}, \tag{2}$$

where h_1 and h_2 are independent Hermitian matrices of dimensions N_1 and N_2, $N_1 + N_2 = N$, and u_1 and u_2 are Haar unitary matrices on $U(N_1)$ and $U(N_2)$, respectively, has a circularly symmetric eigenvalue density in the complex plane, but is not isotropic since the block structure of the matrix is not preserved under right multiplication by every unitary matrix from the $U(N)$ group.

III. MAIN RESULT

The main result of this paper is as follows. Consider n identically distributed isotropic matrices $x_1, x_2, \ldots, x_n$ generated independently from a given IUE. In the limit $N \to \infty$ the eigenvalue density of the product $X_n = x_1 x_2 \cdots x_n$ is identical to the eigenvalue density of the nth power x^n of a single matrix x from this ensemble (e.g., $x = x_1$). In other words, the probability that a randomly chosen eigenvalue of X_n lies within a circle of radius r: $\mathrm{Prob}(\lambda_{X_n} < r)$ approaches for $N \to \infty$ the probability that a randomly chosen eigenvalue of x^n lies within the same circle: $\mathrm{Prob}(\lambda_x^n < r)$. One can use this observation to derive the eigenvalue density of the product $X_n = x_1 x_2 \cdots x_n$ if the eigenvalue density of x is known. In particular one can immediately show that the eigenvalue distribution of the product of n independent Girko-Ginibre matrices has a simple form

$$\rho(z,\bar{z}) = \frac{1}{\pi n} |z|^{-2+2/n} \quad \text{for } |z| \leqslant 1 \tag{3}$$

and zero for $|z| > 1$, in agreement with Refs. [6–9]. It is interesting to note that the matrices $X_n X_n^\dagger$ obtained from the products X_n of Girko-Ginibre matrices generate a Fuss-Catalan family of distributions [18] that have, however, a much more complicated limiting eigenvalue density [19]. Another interesting case is the product of n independent truncated unitary matrices [12] that is

$$\rho(z,\bar{z}) = \frac{\kappa}{n\pi} |z|^{-2+2/n} (1 - |z|^{2/n})^{-2} \quad \text{for} \quad |z| \leqslant \left(\frac{1}{1+\kappa}\right)^{n/2} \tag{4}$$

and zero otherwise. The truncated matrices have dimensions $N \times N$. They are obtained by removing L columns and L rows from an $(N + L) \times (N + L)$ Haar unitary random matrix. The result holds for $N \to \infty$ and $\kappa = L/N$ fixed.

This is a counterintuitive result, so let us stress that it holds only in the limit $N \to \infty$. For finite N the eigenvalue distributions of the product of $x_1 \cdots x_n$ and of the power x^n differ. The difference, however, disappears when N tends to infinity, as we illustrate below. The finite-N distribution of eigenvalues for the product n independent Girko-Ginibre matrices has been analytically derived in Ref. [20].

IV. DERIVATION

Consider an IUE of random matrices $x = hu$ of dimensions $N \times N$. In the large-N limit the random matrices can be represented as free random variables. In this case one can apply the Haagerup-Larsen theorem [5], which relates the eigenvalue density of x to the eigenvalue density of h^2 by the following formula:

$$S_{h^2}(F_x(r) - 1) = \frac{1}{r^2}, \tag{5}$$

where $F_x(r)$ is the cumulative density function for the density of eigenvalues of x on the complex plane and $S_{h^2}(x)$ is the S transform for the matrix h^2. The cumulative density function

$$F_x(r) = \int_{|z| \leqslant r} d^2 z \rho_x(z,\bar{z}) = 2\pi \int_0^r ds s \varrho_x(s) = \int_0^r ds p_x(s) \tag{6}$$

can be interpreted as the fraction of eigenvalues of x in the circle of radius r centered at the origin of the complex plane. It is related to the eigenvalue density $\rho_x(z,\bar{z}) = \varrho_x(|z|)$, which depends on the distance from the origin $|z|$. The integrand $dp_x(s) = 2\pi ds s \varrho_x(s)$ is interpreted as the probability of finding eigenvalues of x in a ring of radii $|z|$ and $|z| + d|z|$:

$$F_x'(r) = p_x(r) = 2\pi r \varrho_x(r). \tag{7}$$

The prime denotes the derivative with respect to the radial variable. The cumulative density function $F_x(r)$ enters Eq. (5) through the argument of the S transform $S_{h^2}(z)$, which is related to the eigenvalue density $\rho_{h^2}(\lambda)$ of the matrix h^2 (see Appendix A). The Haagerup-Larsen theorem states also [5,13,14,17] that the support of the eigenvalue density of x is a ring of radii $R_{\min}$ and $R_{\max}$ or a disk (if $R_{\min} = 0$):

$$R_{\max}^2 = \int_0^\infty d\lambda \lambda \rho_{h^2}(\lambda), \quad R_{\min}^{-2} = \int_0^\infty d\lambda \lambda^{-1} \rho_{h^2}(\lambda). \tag{8}$$

For an R-diagonal (isotropic) matrix x given by the radial decomposition $x = hu$, where h is Hermitian and u is a Haar unitary matrix, the two matrices $xx^\dagger = h^2$ and $x^\dagger x = u^\dagger h^2 u$ have identical eigenvalues and therefore the S transforms for $xx^\dagger$ and $x^\dagger x$ are identical: $S_{xx^\dagger}(z) = S_{x^\dagger x}(z) = S_{h^2}(z)$. This means that Eq. (5) can be written as

$$S_{x^\dagger x}(F_x(r) - 1) = \frac{1}{r^2}. \tag{9}$$

Let us now apply this equation to the product of n identically distributed R-diagonal (isotropic) matrices $X_n = x_1 \cdots x_n$. The resulting matrix has eigenvalues identical to $H_n u_n$, where $H_n^2 = X_n^\dagger X_n$, so we can apply Eq. (9) by replacing in this equation x by X_n:

$$S_{X_n^\dagger X_n}\left(F_{X_n}(r) - 1\right) = \frac{1}{r^2}. \tag{10}$$

The S transform for the matrix $X_n^\dagger X_n$, which appears in Eq. (10), can be replaced by the S transforms for individual terms in the product. Indeed, writing

$$X_n^\dagger X_n = x_n^\dagger X_{n-1}^\dagger X_{n-1} x_n, \tag{11}$$

where $X_{n-1} = x_1 \cdots x_{n-1}$, we see that

$$S_{X_n^\dagger X_n} = S_{X_{n-1}^\dagger X_{n-1}} S_{x_n^\dagger x_n} \tag{12}$$

since, due to the cyclic properties of the trace, the moments of $x_n^\dagger X_{n-1}^\dagger X_{n-1} x_n$ are identical to those of $x_n x_n^\dagger X_{n-1}^\dagger X_{n-1}$. Thus the moments of $x_n x_n^\dagger$ are identical to those of $x_n^\dagger x_n$. By applying Eq. (12) recursively we eventually obtain

$$S_{X_n^\dagger X_n} = \prod_{i=1}^{n} S_{x_i^\dagger x_i}. \tag{13}$$

By taking into account that all x_i are identically distributed and having the same S transform (which we denote by $S_{x^\dagger x}$) we can write Eq. (13) as

$$S_{X_n^\dagger X_n} = S_{x^\dagger x}^n. \tag{14}$$

Inserting this into Eq. (10) we get

$$S_{x^\dagger x}\big(F_{X_n}(r) - 1\big) = \frac{1}{r^{2/n}}. \tag{15}$$

This equation is identical in form to Eq. (9) except that on the left-hand side $F_x(r)$ is replaced by $F_{X_n}(r)$ and on the right-hand side r is replaced by $r^{1/n}$. From this observation it immediately follows that

$$F_{X_n}(r) = F_x(r^{1/n}) = F_{x^n}(r). \tag{16}$$

The equality (16) follows from the fact that eigenvalues of the matrix x^n are equal to the nth power of the corresponding eigenvalues of x: $F_{x^n}(r) \equiv \mathrm{Prob}(|\lambda|^n \leqslant r) = \mathrm{Prob}(|\lambda| \leqslant r^{1/n}) \equiv F_x(r^{1/n})$. So we see that indeed the product of n identically distributed isotropic matrices $X_n = x_1 x_2 \cdots x_n$ has the same eigenvalue distribution as the nth power x^n of a single matrix in the product. In practice, the eigenvalue distribution of X_n can be calculated directly from the eigenvalue distribution of a single matrix x by substituting $r \to r^{1/n}$ in the cumulative distribution function $F_x(r)$ [Eq. (6)]. The corresponding eigenvalue densities may be found using Eq. (7). They read

$$p_{X_n}(r) = \frac{1}{n} r^{1/n-1} p_x(r^{1/n}) \tag{17}$$

and

$$\varrho_{X_n}(r) = \frac{1}{n} r^{2/n-2} \varrho_x(r^{1/n}). \tag{18}$$

V. APPLICATIONS

Let us apply these formulas to a couple of examples. First consider Girko-Ginibre matrices [10,11] that have a uniform distribution $\varrho_x(r) = 1/\pi$ inside the unit circle $|z| \leqslant 1$. We have

$$F_x(r) = 2\int_0^r r' dr' = r^2 \quad \text{for } r \leqslant 1 \tag{19}$$

and 1 otherwise. For the product of n independent Girko-Ginibre matrices we have [Eq. (16)]

$$F_{X_n}(r) = r^{2/n} \quad \text{for } r \leqslant 1 \tag{20}$$

and one otherwise. Taking the derivative with respect to r [Eq. (7)], we find the corresponding densities

$$p_{X_n}(r) = \frac{2}{n} r^{2/n-1}\theta(1-r) \tag{21}$$

and

$$\varrho_{X_n}(r) = \frac{1}{\pi n} r^{2/n-2}\theta(1-r), \tag{22}$$

where θ denotes the Heaviside step function. This result agrees with that obtained using different methods in Refs. [6,21–24], as mentioned in the Introduction of the paper.

As the second example we consider the product of n truncated unitary matrices [12]. The cumulative eigenvalue distribution of a single matrix from this ensemble is

$$F_x(r) = \frac{\kappa r^2}{1-r^2} \quad \text{for} \quad r \leqslant \left(\frac{1}{1+\kappa}\right)^{1/2} \tag{23}$$

and 1 otherwise. The coefficient $\kappa = L/N$ is the ratio of the number of rows and columns L removed from a Haar unitary matrix of dimensions $(N+L)\times(N+L)$. This truncation leaves a matrix of dimensions $N \times N$. In Appendix B we show how to derive this result using free random variables. The corresponding density reads

$$\varrho_x(r) = \frac{\kappa}{\pi}(1-r^2)^{-2}\theta\left(\left(\frac{1}{1+\kappa}\right)^{1/2} - r\right). \tag{24}$$

Using Eq. (16) we find the distribution of eigenvalues for the product of n such matrices:

$$F_{X_n}(r) = \kappa \frac{r^{2/n}}{1-r^{2/n}} \quad \text{for} \quad r \leqslant \left(\frac{1}{1+\kappa}\right)^{n/2} \tag{25}$$

and 1 otherwise. The corresponding eigenvalue density is

$$\varrho_{X_n}(r) = \frac{\kappa}{n\pi} r^{2/n-2}(1-r^{2/n})^{-2}\theta\left(\left(\frac{1}{1+\kappa}\right)^{n/2} - r\right). \tag{26}$$

VI. NUMERICAL COMPARISON AND FINITE SIZE EFFECTS

In order to cross-check our results, we use Monte Carlo simulations to generate (sample) finite size random matrices from the ensembles in question. Agreement between the analytical formula (3) or (4) and numerical results is observed, taking into account finite size corrections. The shape of the obtained distributions $p(r) = F'(r)$ [Eq. (7)] is shown in Fig. 1. In the limit $N \to \infty$, the distributions have compact support and a sharp drop at the edge. For finite N the spectra do not have a sharp threshold, but instead tend to zero continuously in an extended crossover region. The difference between the product of independent matrices and the corresponding power of a single matrix is visible in this region (Fig. 2). The eigenvalue density of the product of independent matrices approaches the theoretical curve faster than that of the corresponding power of a single matrix. Only radial distributions $p(r) = F'(r)$ [Eq. (7)] are shown since eigenvalue densities are circularly symmetric on the complex plane. The shape of the finite size corrections for the Girko-Ginibre distribution was discussed in Refs. [21,22]. The exact shape of this crossover behavior has been analytically derived recently in Ref. [20].

We also performed numerical simulations for the products of truncated orthogonal matrices as an example of multiplication of IOE matrices. In the large-N limit both the IUE

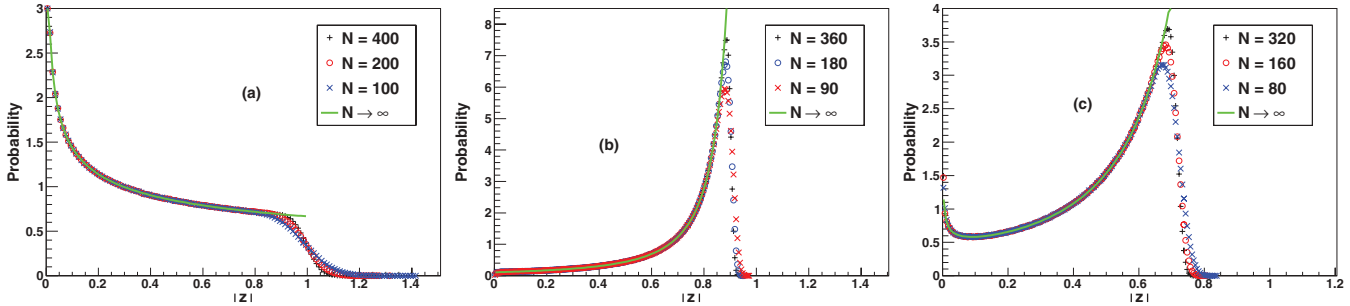

FIG. 1. (Color online) Numerical verification of theoretical formulas (a) (3) and (b) and (c) (4) for the radial part $p(r) = F'(r)$ [Eq. (7)] of the mean spectral density $\rho(z,\bar{z})$ of the product of independent matrices. (a) Numerical histograms for the product of three independent Gaussian random matrices $N = 400$ (black pluses), $N = 200$ (red circles), and $N = 100$ (blue crosses) compared to the theoretical prediction for $N \to \infty$ (solid green line). Each histogram is made for 10^7 eigenvalues. The numerical histograms approach the theoretical curve as the size of matrices increases. (b) Plot analogous to (a) for the product of two independent truncated unitary matrices with ratio $\kappa = \frac{1}{9}$ and $N = 360$ (black pluses), $N = 180$ (red circles), and $N = 90$ (blue crosses). Each histogram is made for 9×10^6 eigenvalues. (c) Plot analogous to (a) and (b) for the product of three independent truncated unitary matrices with ratio $\kappa = \frac{1}{4}$ and $N = 320$ (black pluses), $N = 160$ (red circles), and $N = 80$ (blue crosses). Each histogram represents 8×10^6 eigenvalues.

and IOE densities are expected to have the same limiting distribution, while for finite N the distribution in the IOE case is expected to display a characteristic pattern that weakly breaks the circular symmetry of the eigenvalue distribution on the complex plane. More precisely, one expects that a fraction of eigenvalues accumulates on the real axis and disappears from a narrow depletion region close to the axis. The effect was first discussed for real Girko-Ginibre matrices [25,26] and later also for orthogonal truncated matrices [27]. It is known to be a finite size effect in the sense that the fraction of eigenvalues forming the pattern tends to zero for $N \to \infty$, so the full circular symmetry of the eigenvalue density is restored in the limit. In fact, this is exactly what we see in our numerical simulations of the product of truncated matrices. First we observe that the radial distribution of eigenvalues of the product of two truncated unitary matrices is identical to the case of truncated orthogonal matrices except in a small region close to $r = 0$ [see Fig. 3(a)]. In Fig. 3(b) we compare finite size distributions for the product of the IUE (bottom part) and the IOE (top part). We see that the IUE distribution is circularly symmetric up to the statistical noise while the IOE distribution has an elongated shape close to the real axis, as expected. Finally, in Fig. 3(c) we show the full spectrum on which one can clearly see an accumulation of eigenvalues on the real axis.

VII. DISCUSSION

In this paper we have shown a simple relation between the spectral properties of the product of n identically distributed isotropic random matrices from the given IUE and spectral properties of nth power of a single matrix from this ensemble. We stress a nonintuitive aspect of this result that the product of independent matrices has the same eigenvalue density as the product of fully correlated (identical) matrices. In a sense it is a self-averaging effect: A single random matrix from an isotropic ensemble is a good representative to describe products of matrices from this ensemble in the limit $N \to \infty$.

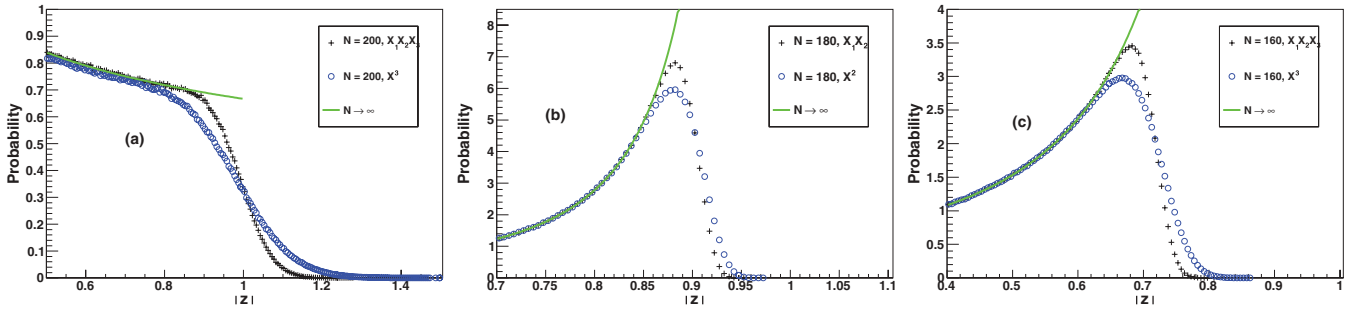

FIG. 2. (Color online) Numerical analysis of the finite size effects for the radial part $p(r) = F'(r)$ [Eq. (7)] of the mean spectral density $\rho(z,\bar{z})$ of the product of independent matrices in comparison to the power of a single matrix. (a) Numerical histograms for the product of three independent Gaussian random matrices $N = 200$ (black pluses) and one matrix raised to the third power for $N = 200$ (blue circles) compared to theoretical prediction for $N \to \infty$ (solid green line). Each histogram is made for 10^7 eigenvalues. Plots are zoomed in the region, where the difference in the shape is visible. (b) Plot analogous to (a) for the product of two independent truncated unitary matrices (black pluses) and the second power of a single truncated unitary matrix (blue circles) with ratio $\kappa = \frac{1}{9}$ and $N = 180$. Each histogram is made for 9×10^6 eigenvalues. (c) Plot analogous to (a) and (b), but the product of three independent truncated unitary matrices (black pluses) and the third power of a single truncated unitary matrix (blue circles) with ratio $\kappa = \frac{1}{4}$ and $N = 160$. Each histogram is made for 8×10^6 eigenvalues.

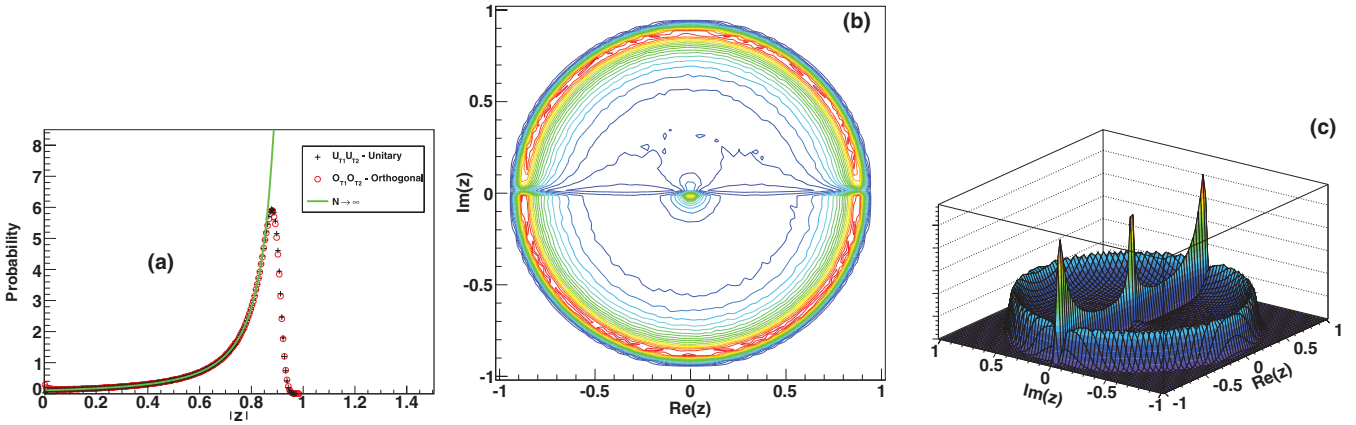

FIG. 3. (Color online) Numerical comparison of the eigenvalues of the product of two truncated orthogonal and unitary matrices. (a) Radial part $p(r) = F'(r)$ [Eq. (7)] of the mean spectral density $\rho(z,\bar{z})$ for unitary (black pluses) and orthogonal (red circles) matrices for $N = 100$ and $\kappa = \frac{1}{9}$. Each histogram is made for 9×10^6 eigenvalues. The theoretical prediction for $N \to \infty$ is shown for comparison (solid green line). (b) Full eigenvalue distribution of orthogonal (top half of the complex plane) and unitary (bottom half of the complex plane) truncated matrices for $N = 100$ and $\kappa = \frac{1}{9}$. (c) Full eigenvalue distribution of orthogonal truncated matrices for the same N and κ parameters. The real eigenvalue band is clearly visible.

We have supplemented our analytic proof with numerical simulations, allowing us to see how the finite size effects vanish in the thermodynamical limit. The obtained eigenvalue density for finite size Girko-Ginibre matrices agrees with the analytic form conjectured in Refs. [21,22] and derived in Ref. [20]. Our result elucidates the transparent analytic structure noted in several papers on the products of random matrices [6–9,12,21–24,28–33] and provides a powerful tool for the derivation of similar results for products of some application-designed isotropic random matrices of large (infinite) size.

ACKNOWLEDGMENTS

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APPENDIX A

In this Appendix we briefly recall basic facts about the S transform, introduced by Voiculescu in free random probability [34]. Consider a Hermitian random matrix a. One usually defines the Green's function

$$G_a(z) = \lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}(z-a)^{-1} \rangle = \int d\lambda \frac{\rho_a(\lambda)}{z-\lambda}, \tag{A1}$$

which is directly related to the eigenvalue density $\rho_a(\lambda)$. The complex variable z in the Stieltjes transform (A1) lies outside the cut on the real axis corresponding to the support of $\rho_a(\lambda)$. Note that the density is a function of a real variable while the Green's function is a function of a complex variable. The Green's function generates moments μ_{ak} (if they exist)

$$G_a(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{\mu_{ak}}{z^{k+1}} \tag{A2}$$

of the eigenvalue density

$$\mu_{ak} = \lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}\, a^k \rangle = \int d\lambda \rho_a(\lambda) \lambda^k. \tag{A3}$$

Sometimes it is more convenient to use another generating function, given by a power series in z rather than in $1/z$:

$$\psi_a(z) = \frac{1}{z} G_a\left(\frac{1}{z}\right) - 1 = \sum_{k=1}^{\infty} \mu_{ak} z^k \tag{A4}$$

and to introduce its functional inverse χ_a:

$$\chi_a[\psi_a(z)] = \psi_a[\chi_a(z)] = z, \tag{A5}$$

which can also be expressed as a power series in z if the first moment is nonzero: $\mu_{a1} \neq 0$. The S transform for the matrix a is related to the χ transform as

$$S_a(z) = \frac{1+z}{z} \chi_a(z). \tag{A6}$$

The relevance of the S transform in free probability stems from the fact that it allows one to concisely formulate the law of free multiplication. The S transform of the product of two free (independent) matrices from invariant ensembles is a product of the S transforms of individual matrices:

$$S_{ab}(z) = S_a(z) S_b(z). \tag{A7}$$

The multiplication law was formulated in free random probability [34], but it also can be rederived in a random matrix setup using field theoretical techniques for the summation of planar Feynman diagrams and it can be generalized to non-Hermitian matrices [23].

APPENDIX B

In this Appendix we rederive the distribution of a single unitary truncated matrix (23) using free probability and the Haagerup-Larsen theorem. We first construct the density of an

$(N+L)\times(N+L)$ matrix $y = pu$, where

$$p = \mathrm{diag}(\underbrace{1,\ldots,1}_{N},\underbrace{0,\ldots,0}_{L}) \tag{B1}$$

is a projection matrix and u is an Haar unitary matrix of dimensions $(N+L)\times(N+L)$. In order to calculate the S transform for the projector $p = p^2$ we first observe that all moments of p are equal $\mu_k = N/(N+L)$. Hence $\psi_p(z) = \frac{N}{N+L}\frac{z}{1-z}$ [Eq. (A4)], $\chi_p = \frac{z}{N/(N+L)+z}$ [Eq. (A5)], and eventually [Eq. (A6)]

$$S_p = \frac{1+z}{N/(N+L)+z}. \tag{B2}$$

Inserting this into Eq. (5) we find

$$F_y(r) = \frac{L}{N+L}\frac{1}{1-r^2} \quad \text{for} \quad r \leqslant \sqrt{N/(N+L)} \tag{B3}$$

and 1 otherwise. We see that $F_y(0) = L/(L+N)$. This means that there are L eigenvalues equal to zero. They are inherited from the zero eigenvalues of the projector. The matrix y is of dimensions $(N+L)\times(N+L)$. When we remove L last columns and L last rows we are left with an upper-left block of dimensions $N\times N$ that has no trivial zero eigenvalues. We denote the matrix corresponding to this block by x, whose cumulative eigenvalue distribution is given in Eq. (23),

$$F_x(r) = \frac{N+L}{N}\left(F_x(r) - \frac{L}{N+L}\right) = \frac{L}{N}\frac{r^2}{1-r^2} \quad \text{for} \quad r \leqslant \sqrt{N/(N+L)} \tag{B4}$$

and one otherwise. The term $-L/(N+L)$ removes L zero eigenvalues out of all $(N+L)$ eigenvalues and the factor $(N+L)/N$ restores the total normalization.

[1] See D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free Random Variables*, CRM Monograph Series Vol. 1 (American Mathematical Society, Providence, RI, 1992), for a noncommutative probability approach to free products with applications to random matrices, operator algebras, and harmonic analysis of free groups.

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