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Effects of random and deterministic discrete scale invariance on the critical behavior of the Potts model

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The effects of disorder on the critical behavior of the q-state Potts model in noninteger dimensions are studied by comparison of deterministic and random fractals sharing the same dimensions in the framework of a discrete scale invariance. We carried out intensive Monte Carlo simulations. In the case of a fractal dimension slightly smaller than two $d_f \simeq 1.974\,636$, we give evidence that the disorder structured by discrete scale invariance does not change the first order transition associated with the deterministic case when $q = 7$. Furthermore the study of the high value $q = 14$ shows that the transition is a second order one both for deterministic and random scale invariance, but that their behavior belongs to different university classes.

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I. INTRODUCTION

Dimensional perturbations suggest a generalization of the concepts of critical phenomena (renormalization, scaling laws, universality) to noninteger dimensions: ϵ expansions are able to provide sets of critical exponents even if the space dimension is not an integer [1]. The idea that fractals could be physical objects able to support such a generalization led Gefen *et al.* [2] to study the critical behavior of the Ising model on deterministic fractal structures with the help of a renormalization method. As a main result, it turns out that besides the space dimension, topological details of the fractal structure (lacunarity, connectivity, ramification order) have an influence on the values of the critical exponents. It is now well established that the critical behavior on deterministic fractal structures can be understood in the framework of a weak universality [3–7]: a universality class does not only depend on the order parameter dimension, the space dimension, and the interaction range, but also on topological details of the fractal structure. Since no analytical general theory is hitherto available, most results have been obtained with the help of numerical simulations; in the case of deterministic fractals, these methods come up against peculiar difficulties which have been discussed in Ref. [5]. It is only recently that the critical behavior of the Ising model in a noninteger dimension disordered system has been studied [8], since on top of the difficulties mentioned above, thermodynamical quantities must be averaged over the disorder. It should be pointed out that the Harris criterion [9], which states that the disorder is relevant if the specific heat of the "pure" model diverges, cannot predict what should happen in noninteger dimensions, since the translation invariance is broken. In other words, the critical behavior of deterministic fractals cannot provide clues concerning random fractals. Two main results have been found [8]: (i) The existence of a critical point exhibiting features close to a random fixed point of a renormalization process; although the disorder is structured by fractality, some thermodynamical quantities lack self-averaging. (ii) The set of critical exponents is clearly different from the one predicted by dimensional perturbation: averaging over the disorder does not restore any hypothetical translational invariance underlying the ϵ expansions. This paper is devoted to the questions brought up at the end of the conclusion in Ref. [8]: Let us recall that, in translationally invariant systems with a space dimension d, the order of the phase transition of the q-state Potts model changes from the second to the first one as q is increased; thus, two regions can be distinguished in the (d,q) plane; the question of the extrapolation of this phase diagram to noninteger dimensions where the symmetry changes from translation invariance to discrete scale invariance without disorder has been studied for deterministic fractals [10–12]; moreover an approximate analytical result has been provided by Andelman and Berker [13]. Furthermore, the introduction of disorder in translationally invariant systems can, under some conditions linked to the Harris criterion, induce a second order transition from a pure system exhibiting a first order one. In the light of these results, the purpose of this work is to achieve a comparison between the effects of deterministic and random discrete scale invariance on the critical behavior of the Potts model. Such a task will provide a more general picture of phase transitions in noninteger dimensions with hierarchically constrained quenched disorder. From an experimental point of view, let us recall that random fractals, whose critical behavior can nowadays be thoroughly investigated by means of numerical methods [8], are more able to model physical systems (silica aerogels, polymers, magnetic domains, low dimensional self-similar structures, networks, etc.) than the deterministic ones.

II. THEORETICAL BACKGROUND AND SIMULATION METHODS

A. Deterministic and random discrete scale invariance

The fractal structures we deal with are constructed according to an iteration process involving dilations from a square generating cell of side l where N_{oc} subsquares among the l^2 available ones are occupied: at the next step, a dilation by a factor l enlarges this generating cell, each occupied subsquare is once more divided into l^2 ones, and N_{oc} among them are occupied. This process is then iterated as many times as wished. In the deterministic case the occupied subsquares are at each step chosen according to an invariant rule (for instance always the upper left corner) indexed by g; in the random case, this rule is left out since the subsquares are chosen randomly. The lateral size of a lattice at the iteration step k is $L = l^k$, and the number $N = (N_{oc})^k$ of occupied subsquares grows as a power law of L, which enables us to define the Hausdorff fractal dimension as the associated exponent $d_f = \ln(N_{oc})/\ln(l)$. We shall denote $\epsilon(L)$ the set $\{\epsilon_i\}$ of occupation numbers associated with the construction process of a lattice $-\epsilon_i = 0$ if a subsquare is deleted, $\epsilon_i = 1$ otherwise, and $\mathcal{G}_\epsilon^k$ is the fractal graph constructed when connecting the centers of first neighboring occupied subsquares. We shall denote $SR_\epsilon(l^2,N_{oc},k)$ a realization of a random Sierpinski carpet built up from a finite number k of iteration steps and $SP_g(l^2,N_{oc},k)$ the deterministic Sierpinski carpet built up according to the unique set $\epsilon(L)$ defined by the rule g. It should be pointed out that the scale invariance of Sierpinski carpets is discrete [14]: this invariance—exact in the deterministic case, statistic in the random one—is constrained by a geometrical series with a reason l. When probing the lattice at scales which do not satisfy such a constraint, log periodic modulations around the leading power law behavior appear; log periodic oscillations bear the mark of discrete scaling invariance, and can be taken in account in an imaginary part of the associated exponent [14]. Such oscillations have recently been brought out in the framework of nonequilibrium critical dynamics [15,16] in the deterministic case. From a statistical point of view, the discrete character of the random scale invariance arises in the behavior of the configuration space with the system size, and can be more conveniently expressed by calculating the configuration entropy $S = k_B \ln W$ where k_B is the Boltzmann constant and W is the number of possible configurations. For a given set $\{l,N_{oc},k\}$, and provided that l and N_{oc} are chosen in such a way that $SR_\epsilon(l^2,N_{oc},k)$ always overlaps the percolation cluster, W is equal to

$$\binom{l^2}{N_{oc}}^{1+N_{oc}+N_{oc}^2+\cdots+N_{oc}^{k-1}};$$

hence, the configuration entropy per site is equal to

$$s_{SR} = k_B \frac{1-1/(N_{oc})^k}{N_{oc}-1} \ln \binom{l^2}{N_{oc}},$$

where

$$\binom{l^2}{N_{oc}}$$

designates the binomial coefficients. Furthermore, fractal structures can be obtained in the case of the usual diluted site quenched disorder, where sites of a translationally invariant lattice of size L are randomly occupied with a probability p equal to the associated percolation threshold: $p_c \simeq 0.592\,746$ on a two-dimensional square lattice [17,18] where the fractal dimension of the percolation cluster is $D_f = 91/48 \simeq 1.895\,83$. So, the scale invariance is no more discrete in the sense that it can be probed for any value of the length between 1 and L, and the configuration entropy s_P scales as L^{β_p/ν_p} where β_p and ν_p are exponents of the percolation problem; $\beta_p/\nu_p = 5/48$ in the two-dimensional case [19]. Hence, the distinction between random fractality with and without discrete scale invariance is striking, since s_{SR} tends towards a finite limit as L tends to infinity while s_P grows with L; the disorder is strongly constrained by discrete scale invariance. In spite of such a constraint, we showed [8] that the finite size scaling of the Ising model on $SR_\epsilon(3^2,8,k)$ lacks self-averaging, [20] thus that the fixed point of the renormalization associated with the second order transition exhibits features of a random one.

B. Potts model on fractal structures

The Hamiltonian of the q-state Potts [21] model on Sierpinski fractals is defined by placing spins at the center of the occupied squares, in such a way that it reads

$$H = -J \sum_{\langle i,j \rangle} \epsilon_i \epsilon_j \delta(\sigma_i,\sigma_j), \tag{1}$$

where σ_i and σ_j designate the spin states at the occupied sites i and j, and can take the integer values $1,2,\ldots,q$. The sum runs over the nearest-neighbor spins; $\delta(\sigma_i,\sigma_j)$ is equal to 1 if $\sigma_i = \sigma_j$, and is equal to 0 if $\sigma_i \neq \sigma_j$. $J > 0$ is the exchange coupling constant between two nearest-neighbor spins. For a given size L and a given spin configuration, the order parameter of the phase transition reads

$$m_L = \frac{q\rho_L - 1}{q-1}, \tag{2}$$

$\rho_L = \max\{N_1/N,\ldots,N_q/N\}$ where N_{q_0} is the number of spins whose state is q_0.

In order to focus on the effects of q and the nature of fractality on the phase transitions, we shall deal with fractal structures able to exhibit a long range ferromagnetic order for the Ising model ($q = 2$). Percolation is a necessary condition for the occurrence of such a critical behavior on Sierpinski fractals; thus, we shall study two different fractal dimensions, in such a way that this condition is always fulfilled independently of the value of k both for deterministic and random fractals: first $l = 3$ and $N_{oc} = 8$ [$d_f = \ln(8)/\ln(3) \simeq 1.892\,789$], and second $l = 5$ and $N_{oc} = 24$ [$d_f = \ln(24)/\ln(5) \simeq 1.974\,636$]. Moreover, the associated fractal graphs $\mathcal{G}_\epsilon^k$ are always fully connected: the percolation clusters systematically overlap entirely $SR_\epsilon(l^2,N_{oc},k)$ and $SP_g(l^2,N_{oc},k)$ so that the occupation probability p decreases with the lattice size as L^{d_f-2}. A sketch of a phase diagram of the Potts model on deterministic Sierpinski fractals with a square symmetry denoted $SP_a(l^2,N_{oc})$ is presented in Fig. 1; the calculations of noninteger dimension points have been obtained by means of numerical simulation presented in Ref. [10–12]. Such a diagram deserves two comments: (i) A border separating a first order from a second order region in qualitative agreement with the analytical approximation of Ref. [13] can be drawn in the (d_f,q) plane. (ii) Second order phase transitions can occur for high values of q when the fractal dimension is smaller than 2. (iii) Although the translation symmetry is strongly broken, the transitions do not change to second order ones as q is increased when the space dimension lies between 1.9746 and 3.

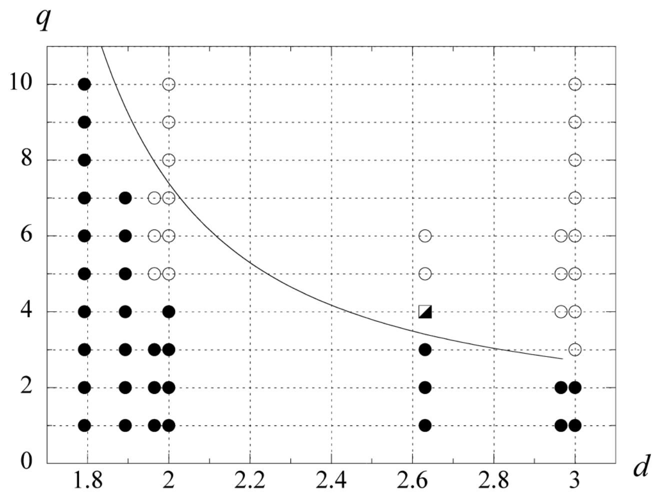

FIG. 1. Phase diagram of the Potts model for deterministic square symmetric fractals $SP_a(l^2,N_{oc})$ as a function of the space dimension d_f and the number of spin states q; the open circles indicate a first order transition whereas the black ones indicate a second order one and the square a weakly first order transition. The full line indicates the approximate Andelman and Berker analytical result.

C. First and second order phase transitions

The difficulties encountered in dealing with simulations of first and second order phase transitions in the case of fractals, together with a finite size analysis of the results, have been set out in Ref. [11]. Let us recall that a bimodal density probability of the energy is not a sufficient condition for a phase transition to be a first order one: The depth of the well separating the peaks associated with the coexistence of the two phases must increase with the system size L at the shifted transition temperature (where the two peaks have the same height). If the correlation length of the ordered domains is high, such an observation should require the simulation of very large systems remaining out of reach of a first order transition suited algorithm like the Wang-Landau [22] one. Moreover, if the correlation length of the ordered domains is much larger that L, but remains finite in the thermodynamical limit, canonical simulations at a fixed temperature T may lead to a pseudocritical behavior whereas the transition is a first order one. A criterion able to discriminate unambiguously from the finite size results between a first and a second order phase transition has been demonstrated by Meyer-Ortmanns and Reisz [23]: Although the order parameter susceptibility $\chi_L(T)$ diverges at the transition temperature as $L \to \infty$ for both first and second order transitions, its behavior as the system size is quite different in each case. In the case of second order transitions, the associated finite size rounded singularity $\chi_L(T)$ rescaled as a function of the deviation from the temperature of the peak $T_c^{\chi}(L)$ do not intersect in their wings. In other words, for a sufficiently large size L_1, there is always a size $L_2 > L_1$ such that $\chi_L[\delta + T_c^{\chi}(L)] > \chi_{L_1}[\delta + T_c^{\chi}(L_1)]$ where $L \geqslant L_2$, and δ denotes a deviation from the critical temperature of the peak. In the case of second order transitions, we call α, β, γ, and ν the critical exponents associated respectively with the singularities in the zero external field thermodynamical limit of the specific heat $c \sim |t|^{-\alpha}$,the order parameter $m \sim (-t)^{\beta}$ with $t < 0$, the susceptibility $\chi \sim |t|^{-\gamma}$, and the correlation length $\xi \sim |t|^{-\nu}$. The exponents γ/ν and $1/\nu$ can be extracted from the finite size behavior [8] of the maximum of the susceptibility $\chi_L^{\max} \propto L^{\gamma/\nu}$ and the first logarithmic derivative of the order parameter $\phi_L^{\max} \propto L^{1/\nu}$.

D. Canonical simulations and disorder averaged thermodynamical quantities

Calling $\mathcal{O}$ an observable (energy, order parameter, etc.) depending both on the distribution $\epsilon(L)$ and a set of spin states $\sigma = \{\sigma_i\}$, the canonical thermodynamical average of $\mathcal{O}$ at temperature T, and at fixed distribution $\epsilon(L)$ reads

$$\langle \mathcal{O}\rangle_T^{\epsilon(L)} = \frac{1}{Z[\epsilon(L)]}\sum_{\{\sigma\}}\mathcal{O}^{\epsilon(L)}(\sigma)e^{-\mathcal{H}^{\epsilon(L)}(\sigma)/k_BT}, \qquad (3)$$

where $\mathcal{H}^{\epsilon(L)}(\sigma)$ designates a realization of the Hamiltonian for a particular spin configuration among the $q^{(N_{oc})^k}$ available ones, $Z[\epsilon(L)]$ is the partition function associated to the sample $SR_\epsilon(l^2,N_{oc},k)$ or $SP_g(l^2,N_{oc},k)$, and the sum runs over the whole set $\{\sigma\}$ of spin configurations.

In the case of random fractals, thermodynamical quantities must be averaged over the disorder, so that the average value of $\mathcal{O}$ can be written

$$\overline{\langle \mathcal{O}_L\rangle_T} = \sum_{\{\epsilon(L)\}}\langle \mathcal{O}\rangle_T^{\epsilon(L)}\mathcal{P}[\epsilon(L)], \qquad (4)$$

where $\mathcal{P}[\epsilon(L)]$ is the weight of the random fractal $SR_\epsilon(l^2,N_{oc},k)$ and the sum runs over the set $\{\epsilon(L)\}$ of possible graphs $\mathcal{G}_\epsilon^k$; since each graph is constructed according to an independent random fractal trial, each one has the same weight in (4). The errors associated to such double averages have different origins which have to be clearly identified before carrying out the simulations; such an analysis has been thoroughly discussed in Sec. C of Ref. [8], and provides the decisive factors in setting up the simulation process, optimizing the computational cost and calculating the error bars.

Let us summarize the main results that have to be kept in mind:

(i) The statistical errors $\delta\langle \mathcal{O}\rangle_{T_0}^{\epsilon(L)}$ associated with the thermal averages of $\mathcal{O}$ for a given realization of the disorder (or a given deterministic structure) are equal to the standard ones enhanced by a factor called statistical inefficiency [24]:

$$\left(\delta\langle \mathcal{O}\rangle_{T_0}^{\epsilon(L)}\right)^2 \simeq \frac{1}{N_s}\left[\langle \mathcal{O}^2\rangle_{T_0}^{\epsilon(L)} - \left(\langle \mathcal{O}\rangle_{T_0}^{\epsilon(L)}\right)^2\right]\left[1 + 2\frac{\tau_{\mathcal{O}}}{\delta\theta}\right], \qquad (5)$$

where N_s is the number of Monte Carlo steps, $\tau_{\mathcal{O}}$ is the integrated autocorrelation time associated with $\mathcal{O}$, and $\delta\theta$ is the unit time associated with one step. $\tau_{\mathcal{O}}$ gives a measure of the critical slowing down, namely the mean value of Monte Carlo steps needed to cancel correlations in the random variable $\mathcal{O}$. In order to minimize the statistical inefficiency, we decided to carry out our canonical simulations with the help of the Wolff cluster algorithm [25]; a spin configuration generated by this algorithm is shown in Fig. 2.

(ii) In order to extract the maximum information from a simulation at a given temperature T_0 we processed the data of the simulations with the help of the histogram method [26], which provides thermal averages in the vicinity of T_0; let

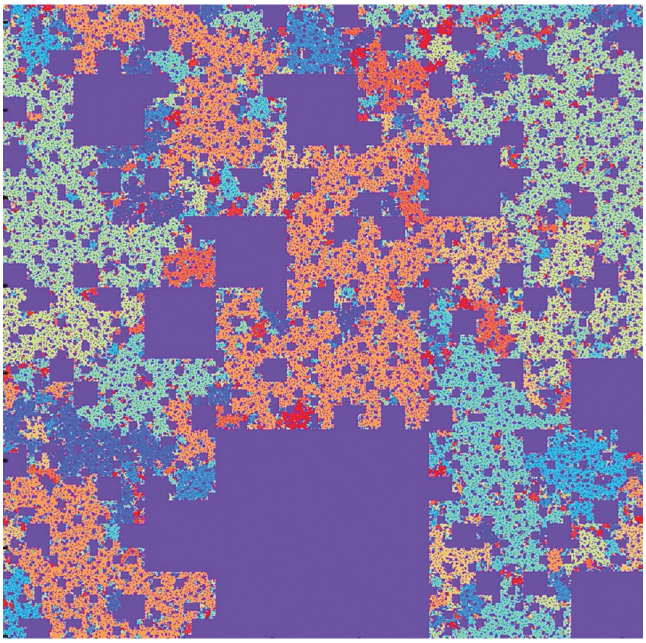

FIG. 2. (Color online) Example of a spin configuration $\sigma = \{\sigma_i\}$ on a random fractal $SR_\epsilon(3^2,8,6)$ (262 144 sites) generated by the Wolff algorithm in the critical region of the 14-state Potts model; each color (one of the 14 grey levels) is associated with a given spin state, except the blue-violet (black) showing the unoccupied squares at different scales. The effect of the discrete scale invariance on the clusters geometry can easily be seen.

us recall that, unfortunately, the reliability range δT of the temperature around the simulation one decreases with the system size [24].

(iii) The double averaging over thermal Boltzmann sampling and over the disorder can introduce bias in some of the estimators, leading to systematical errors. Such a drawback has to be treated with the help of improved estimators [27] if statistical errors on $\overline{\langle \mathcal{O}_L \rangle_T}$ are smaller than this bias. The computational cost at fixed values of T and L is proportional to $\mathcal{N}.N_s$ where $\mathcal{N}$ is the number of disordered configurations. The optimization of this cost rests on the behaviors of the configuration entropy per site s_{SR} mentioned above: The average over the disorder constrained by discrete scaling invariance needs less computational effort than in the case of the site diluted model. Moreover, since increasing the length of the runs enables us to get rid of the use of improved estimators, we decided to keep N_s large enough to ensure that the statistical errors $\delta \langle \mathcal{O} \rangle_{T_0}^{\epsilon(L)}$ over the thermal averages are much smaller than over the disorder. The results of the simulations we carried out in the case of the Ising model support such a choice [8]. This point has anyhow been checked out all along the data processing of the simulation results. Error bars have been calculated according to a jackknife resampling analysis.

III. NUMERICAL RESULTS

A. Fractals $SP_a(5^2,24,k)$ and $SR_\epsilon(5^2,24,k)$: $d_f \simeq 1.974636$

In one of our previous studies of the q-state Potts model on deterministic fractals $SP_a(5^2,24,k)$ with the help of the Wang-Landau algorithm [12], it has been shown that

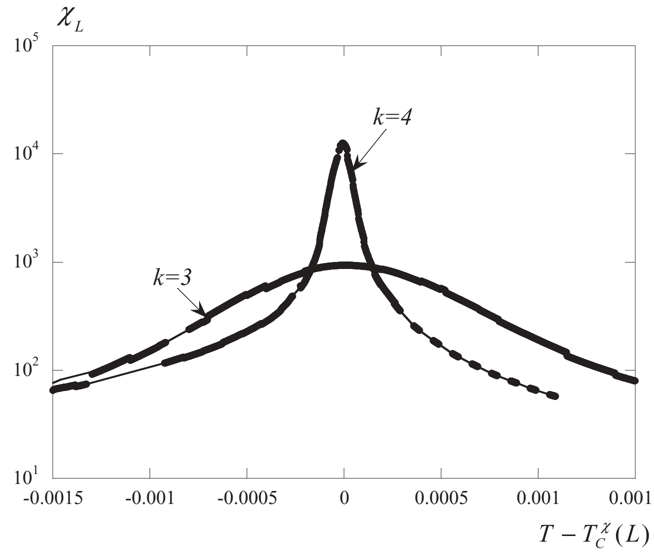

FIG. 3. Susceptibility χ_L of the order parameter for the seven-state Potts model as a function of the shifted temperature $[T - T_c^\chi(L)]$ on the deterministic fractals $SP_a(5^2,24,k)$. Each bold segment is the result of sets of Monte Carlo runs carried out at temperatures close to its middle (or at different points inside the segment), and processed with the help of the histogram method; the interpolating thin lines are guides to the eyes.

the transition is a first order one for $q \geqslant 5$. As already recalled above, the Wang-Landau algorithm, which calculates the density of states over the whole energy range, is very time consuming and does not enable us to simulate very large sizes; we decided to carry out simulations for the value $q = 7$ with the help of the canonical Wolff algorithm in order

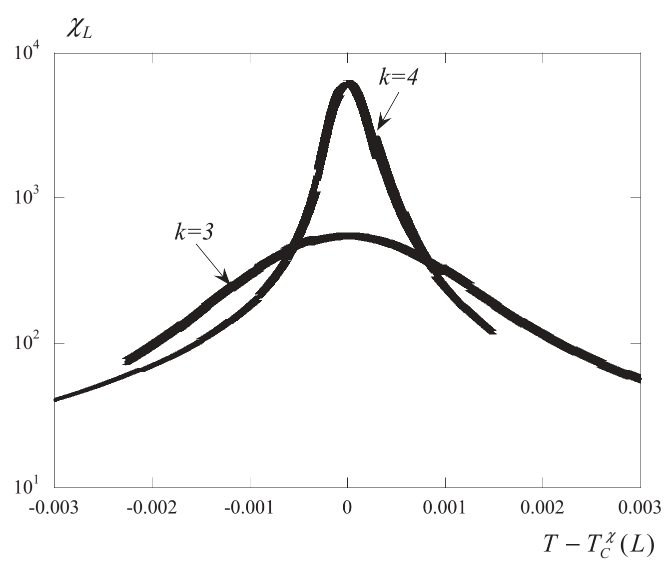

FIG. 4. Susceptibility χ_L of the order parameter for the seven-state Potts model as a function of the shifted temperature $[T - T_c^\chi(L)]$ on the random fractals $SR_\epsilon(5^2,24,k)$. Each bold segment is the result of an average over disorder of a set of Monte Carlo runs carried out at a temperature close to its middle (or at different points inside the segment), and processed with the help of the histogram method; the interpolating thin lines are guides to the eyes.

TABLE I. Values of the maxima of the susceptibility $\chi_L^{\max}$, maxima of the first logarithmic derivative $\phi_L^{\max}$ and temperatures of the associated peaks $T_c^{\chi}(L)$ and $T_c^{\phi}(L)$ for the deterministic fractals $SP_a(3^2,8,k)$, $k \in \{4,5,6,7\}$, with $q = 14$.

L	$T_c^{\chi}(L)$	$\chi_L^{\max}$	$T_c^{\phi}(L)$	$\phi_L^{\max}$
81	0.4891(1)	223(1)	0.490	74.4(5)
243	0.48515	1161(20)	0.48565(40)	147(2)
729	0.483229	6640(60)	0.48353	278(8)
2187	0.48216	39810(100)	0.48232(6)	487(15)

TABLE II. Values of the maxima of the susceptibility $\chi_L^{\max}$, maxima of the first logarithmic derivative $\phi_L^{\max}$, and temperatures of the associated peaks $T_c^{\chi}(L)$ and $T_c^{\phi}(L)$ for the random fractals $SR_\epsilon(3^2,8,k)$, $k \in \{4,5,6,7\}$, with $q = 14$. $\mathcal{N}$ indicates the number of disorder averaged configurations.

L	$T_c^{\chi}(L)$	$\chi_L^{\max}$	$T_c^{\phi}(L)$	$\phi_L^{\max}$	$\mathcal{N}$
81	0.50279(10)	141.3(16)	0.5055(2)	31.3(8)	
243	0.49760(20)	829(15)	0.4991(2)	59.5(20)	
729	0.49538(30)	5100(100)	0.4960(3)	125.9(40)	
2187	0.49428(40)	34185(400)	0.49455	257.5(60)	

to check that the two approaches agree. Many runs were carried out at different temperatures, after a relaxation time long enough to ensure that the thermodynamical equilibrium had been reached; different initial states were chosen and the consistency of the set of canonical simulations carefully checked. The susceptibility peaks at the third ($L = 125$) and fourth ($L = 625$) iteration steps were found to occur at $T_c^{\chi}(L) = 0.723\,40(6)$ and $T_c^{\chi}(L) = 0.722\,82(2)$ respectively. According to the Meyer-Ortmanns and Reisz criterion, the behavior of the susceptibility as a function of the deviation $T - T_c^{\chi}(L)$ from the position of the peaks shown in Fig. 3 clearly confirms that the transition is a first order one.

The investigation of the random fractals $SR_\epsilon(5^2,24,k)$ was carried out as follows: (i) A random set $\epsilon(L)$ of occupation numbers fulfilling the discrete scale invariance was drawn. (ii) Canonical simulations were done on the associated constructed fractal at different temperatures as in the deterministic case. (iii) The disorder averaged thermodynamical quantities were calculated over a set of different random fractals. 200 configurations were investigated at the third iteration step and 80 at the fourth iteration step; the disorder averaged susceptibility exhibits peaks occurring at $T_c^{\chi}(L) = 0.725\,62(9)$ and $T_c^{\chi}(L) = 0.724\,76(3)$ respectively. The finite size behavior of χ_L shown in Fig. 4 enables us to conclude that the phase transition occurring in the random case is not a second order one.

B. Fractals $SP_a(3^2,8,k)$ and $SR_\epsilon(3^2,8,k)$: $d_f \simeq 1.892\,789$

In the results reported in Ref. [11], we gave evidence for the apparition of a double peaked structure in the probability density of the energy for $q = 12$ at the third and fourth iteration step of the fractal $SP_a(3^2,8,k)$. The Wang-Landau algorithm and the computational resources we had at our disposal did not enable us to check if this double peaked structure increased with L; hence, we were not able to determine the order of the transition. Thus, we chose here a higher value of q, namely $q = 14$ (above the Andelman-Berker expected critical value q_C) and carried out simulations with the help of the canonical Wolff algorithm according to the procedure described above. The results associated with the maxima in the susceptibility $\chi_L^{\max}$ and in the first logarithmic derivative of the order parameter $\phi_L^{\max}$ are reported in Table I together with the temperatures of the peaks. The behavior of χ_L as a function of $T - T_c^{\chi}(L)$ investigated until the seventh iteration step ($L = 2187$) reported in Fig. 5 shows that the phase transition of the 14-state Potts model on deterministic

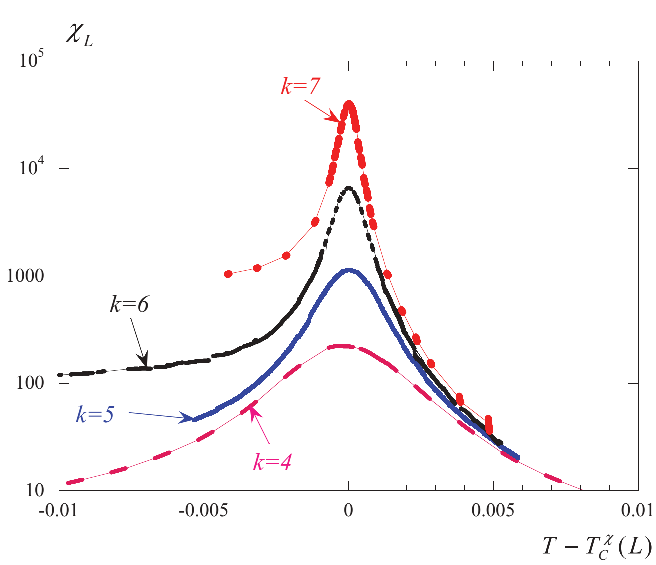

FIG. 5. (Color online) Susceptibility χ_L of the order parameter for the 14-state Potts model as a function of the shifted temperature $[T - T_c^{\chi}(L)]$ on the deterministic fractals $SP_a(3^2,8,k)$ (same procedure as for Fig. 3).

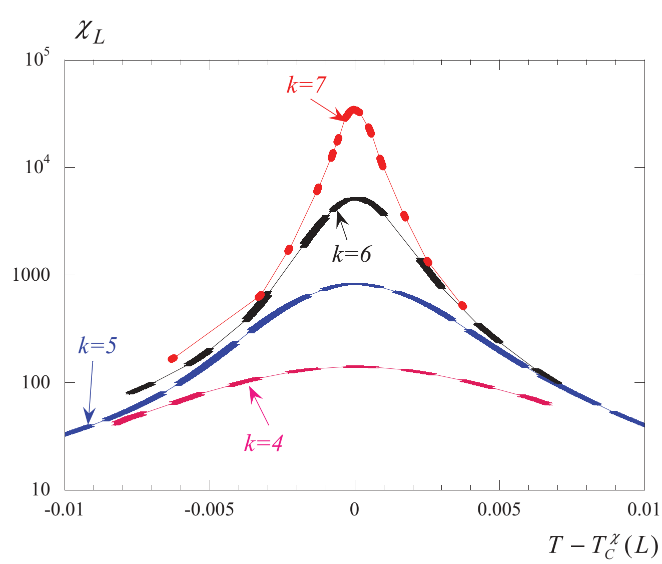

FIG. 6. (Color online) Susceptibility χ_L of the order parameter for the 14-state Potts model as a function of the shifted temperature $[T - T_c^{\chi}(L)]$ on the random fractals $SR_\epsilon(3^2,8,k)$ (same procedure as for Fig. 4).

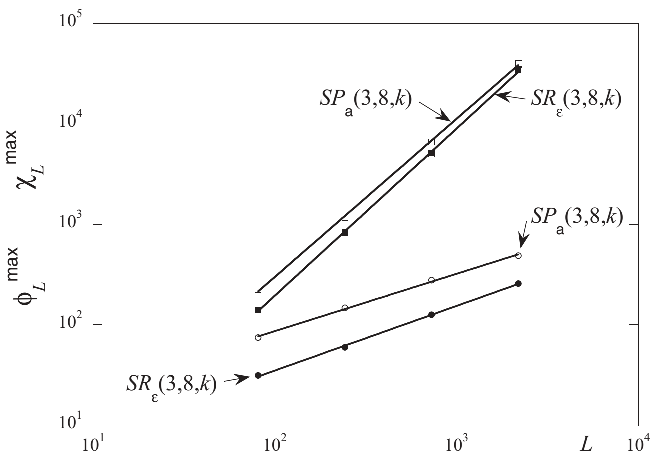

FIG. 7. Finite size behavior of the maxima $\chi_L^{\max}$ (squares) and $\phi_L^{\max}$ (circles) on deterministic fractals $SP_a(3^2,8,k)$ and random fractals $SR_\epsilon(3^2,8,k)$; a slight systematic curvature can be seen in the behavior of $\phi_L^{\max}$ in the deterministic case. The jackknife calculated error bars are smaller than the dots.

fractals $SP_a(3^2,8,k)$ is still a second order one. The finite size behavior of $\chi_L^{\max}$ shown in Fig. 7 leads to the value of the ratio of exponents $\gamma/\nu = 1.60(3)$, significantly different from the fractal dimension, as expected from a second order transition. A comparison of the previously calculated values [5,10,11] of γ/ν for different values of q [1.750(5) when $q = 2$, 1.7013(28) when $q = 3$, and 1.653(10) when $q = 7$] confirms that γ/ν decreases as q is increased. On the other hand, the finite size behavior of $\phi_L^{\max}$ suffers from strong scaling corrections, as already pointed out for the values of q we studied on $SP_a(3^2,8,k)$, and does not enable us to calculate $1/\nu$ in a reliable way. These scaling corrections had been attributed to a topological character of the deterministic fractal [5] and related to the variations of the mean number of first neighbors from an iteration step to the next.

The maxima in $\chi_L^{\max}$ and $\phi_L^{\max}$ and the temperature of the associated peaks obtained from disorder averaged canonical simulations on random fractals $SR_\epsilon(3^2,8,k)$ are reported in Table II. The behavior of, χ_L as a function of $T - T_c^\chi(L)$ reported in Fig. 6 clearly shows that the phase transition of the 14-state Potts model on the random fractals $SR_\epsilon(3^2,8,k)$ is also a second order one. The value of γ/ν obtained from the finite size behavior of $\chi_L^{\max}$ shown in Fig. 7 is $\gamma/\nu = 1{,}69(4)$, significantly different from the deterministic case $SP_a(3^2,8,k)$, and from the two-state Potts model [8] on $SR_\epsilon(3^2,8,k)$ where we found $\gamma/\nu = 1{,}768(3)$. Contrary to the deterministic case, the finite size behavior of $\phi_L^{\max}$ does not suffer from systematic curvature and leads to $1/\nu = 0.650(7)$, close to the value 0.633(3) calculated in the case $q = 2$ on $SR_\epsilon(3^2,8,k)$.

IV. CONCLUSIONS

We focused on the differences between the effects of deterministic and random discrete scale invariance on the phase transition of the Potts model.

(1) In the case of a fractal dimension slightly smaller than 2, $d_f \simeq 1.974\,636$, we chose a value of q large enough ($q = 7$) to ensure the transition in the deterministic case to be a first order one as already shown by means of the calculation of the density of states involving the Wang-Landau algorithm. This result has been confirmed, according to the Meyer-Ortmanns and Reisz criterion, by canonical simulations we carried out. The study of the random case shows that the disorder constrained by fractality does not induce a second order transition. Let us recall that an infinitesimal amount of disorder can induce a second order transition of the Potts model from a transition exhibiting a first order one ($q > 4$) in the translationally invariant two-dimensional case.

(2) In the case of the fractal $d_f \simeq 1.892\,789$ we brought evidence for a second order transition of the Potts model in the deterministic case for a high value of q ($q = 14$ above the critical value of Andelman and Berker), although the Wang-Landau algorithm showed a slightly double peaked structure in the probability density of the energy for small sizes. In the case of disorder constrained by fractality, we were able to show that the transition is a second order one, and that the universality class is different from that of the deterministic case. This result suggests that random discrete scale invariance on low-dimensional systems (above a critical dimension still remaining to be calculated) can exhibit long range order even for high values of the number of states of the Potts model.

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