Non-Markovian stationary probability density for a harmonic oscillator in an electromagnetic field

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We calculate the exact solution of the Fokker-Planck equation for the stationary-state probability density of a harmonic oscillator embedded in an electromagnetic field. The magnetic field is assumed to be a constant and the electric field an external stochastic force with the properties of a Gaussian and exponentially correlated noise (Ornstein-Uhlenbeck process). In this work, we first study the problem in the absence of the magnetic field, then we obtain the complete solution and corroborate that the latter reduces to the former when the magnetic field is suppressed.

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I. INTRODUCTION

To describe a variety of physical, chemical, and biological phenomena wherein noise plays a fundamental role, two theoretical approaches, Markovian and non-Markovian, have been developed [1–34]. The Markovian processes are easier to handle than the non-Markovian ones because a lot of the former admit exact analytical solutions, whereas for the latter ones it is not an easy task to extract the exact statistical information. This is indeed the case in trying to solve explicitly the Fokker-Planck (FP) equation for the time-dependent probability density function for linear and nonlinear potentials [6–10], to give just an example. However, substantial advances have been achieved when the non-Markovian Langevin equation is driven by a Gaussian exponentially correlated noise [Ornstein-Uhlenbeck (OU) noise] [5–24]. Within this context, the ordinary Brownian harmonic oscillator driven by external, exponentially correlated noise has been a topic of great interest, widely studied in the literature [8–11]. It is important to notice the small number of contributions on Brownian motion in a magnetic field driven by Gaussian colored noise proposed in the literature [27,28], compared to those given for the same problem driven by a Gaussian white noise (GWN) [25–34].

The diffusion process in a plasma, studied as a Brownian motion problem across a magnetic field, was solved a long time ago [25,26] and assumed to arise from local fluctuations of the electric field which induce collisions between particles in a Brownian motion–like manner. In [25] it was considered a situation in which the density gradient occurs in a direction orthogonal to the magnetic field where the magnetic pressure is dominant. The ion velocity is much less than the electron velocities so that the mean friction dynamics on an ion will be proportional to the ion velocity. After these pioneer works, the studies given in this context have considered such fluctuations as a thermal noise where the fluctuation-dissipation relation holds [25–33]. In this case, a balance between both the fluctuating and friction forces originating from the same environment allows the system to reach the equilibrium state. However, when the fluctuating and dissipative forces originate from different environments, the system reaches a nonequilibrium steady state. This kind of fluctuating force is known as an external noise and it leads to breakdown of the fluctuation-dissipation relation. The problem of Brownian motion of a charged harmonic oscillator in an electromagnetic field driven by an external colored noise for the electric field is addressed here. In a more realistic situation, a randomly fluctuating electric field should produce a randomly fluctuating magnetic field. In this case, the solution of the problem would require the combined effect of Maxwell equations considered as a set of stochastic equations strongly coupled to a generalized Langevin equation. However, this task is beyond our purposes in this work. It is also true that a radiation reaction occurs for a single charged particle moving in an external field leading to the appearance of a damping force also called Lorentz friction force [35]. It will be shown in Sec. III that in the study of Brownian motion the damping radiation force is much less than the Stokes friction dynamics, and thus can be safely ignored.

It is our purpose in this contribution to study the plasma diffusion process across a constant magnetic field when the electric field satisfies the properties of an OU noise. Our study is mainly related to the calculation of the exact solution of the FP equation for the stationary-state probability density (SSPD) of a charged Brownian harmonic oscillator. Due to the external nature of the electromagnetic field, one can keep a constant magnetic field greater than the amplitude of the external fluctuating electric field, which we can consider also to be dominant over the internal noise (local fluctuations of the electric field). The SSPD of the FP equation in an electromagnetic field supports the recent results [34] obtained via the Langevin equation, for the statistics of the initial conditions required to characterize the decay of nonlinear unstable states driven by an OU process for the electric field. The robustness of our analytical results is also corroborated when they are compared with the numerical simulation showing excellent agreement.

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Our work is then structured as follows: in order to compare with our theoretical results, in Sec. II we calculate the SSPD for the ordinary harmonic oscillator driven by an external OU process. In Sec. III we extend the problem to the case in the presence of an electromagnetic field. We give our concluding remarks in Sec. IV and, at the end of our work, two Appendixes with the explicit calculations are included.

II. HARMONIC OSCILLATOR DRIVEN BY EXTERNAL COLORED NOISE

The Langevin equation for a Brownian harmonic oscillator of mass m in the presence of an external noise $\boldsymbol{\mu}(t)$ which satisfies the OU process is given by

$$m\frac{d\mathbf{v}}{dt} = -\alpha\mathbf{v} - k\mathbf{r} + \boldsymbol{\mu}(t), \tag{1}$$

where $\alpha > 0$ is the friction coefficient, k is the harmonic force constant, and $\boldsymbol{\mu}(t)$ is the external fluctuating force satisfying the property of a Gaussian exponentially correlated noise with zero mean value and correlation function

$$\langle \mu_i(t)\mu_j(t')\rangle = \frac{\lambda}{\tau}\,\delta_{ij}e^{-|t-t'|/\tau}, \quad i,j = x,y,z, \tag{2}$$

with λ and τ being the intensity and correlation time of the noise, respectively. We are interested in the calculation of the SSPD in the overdamped approximation of the above Langevin equation, leading in this case to

$$\dot{\mathbf{r}} = -a\mathbf{r} + \alpha^{-1}\boldsymbol{\mu}(t), \tag{3}$$

where $a = k/\alpha$. Due to the properties of the Gaussian colored noise, the stochastic processes (3) can be seen as two Markovian processes described by [5,8–10,14]

$$\dot{\mathbf{r}} = -a\mathbf{r} + \alpha^{-1}\boldsymbol{\mu}, \tag{4}$$

$$\dot{\boldsymbol{\mu}} = -\frac{1}{\tau}\boldsymbol{\mu} + \frac{1}{\tau}\boldsymbol{\xi}(t), \tag{5}$$

where $\xi(t)$ is a Gaussian white noise with zero mean value and correlation function

$$\langle \xi_i(t)\xi_j(t')\rangle = 2\lambda\,\delta_{ij}\delta(t-t'). \tag{6}$$

Equation (5) describes an OU process, equivalent to Eq. (2). To check that this is indeed the case, we first integrate Eq. (5) to obtain

$$\boldsymbol{\mu}(t) = e^{-t/\tau}\boldsymbol{\mu}_0 + \frac{1}{\tau}\int_0^t e^{-(t-t')/\tau}\boldsymbol{\xi}(t')dt', \tag{7}$$

with $\boldsymbol{\mu}(t=0) = \boldsymbol{\mu}_0$. For this process, we assume an initially Gaussian distribution function such that $\langle \mu_{0i}\rangle = 0$ and $\langle \mu_{0i}\mu_{0j}\rangle = (\lambda/\tau)\,\delta_{ij}$. In this case, we can see that $\langle \mu(t)\rangle = 0$ and, if we assume that $\langle \mu_{0i}\xi_j\rangle = 0$, then

$$\langle \mu_i(t_1)\mu_j(t_2)\rangle = \frac{\lambda}{\tau}\delta_{ij}e^{-(t_1+t_2)/\tau} + \frac{1}{\tau^2}\int_0^{t_1}\int_0^{t_2} e^{-(t_1+t_2-t_1'-t_2')/\tau} \times \langle \xi_i(t_1')\xi_j(t_2')\rangle dt_1' dt_2', \tag{8}$$

which, after integration, becomes

$$\langle \mu_i(t_1)\mu_j(t_2)\rangle = \frac{\lambda}{\tau}\delta_{ij}e^{-(t_1+t_2)/\tau} + \frac{\lambda}{\tau}\delta_{ij}[e^{-|t_1-t_2|/\tau} - e^{-(t_1+t_2)/\tau}] = \frac{\lambda}{\tau}\delta_{ij}e^{-|t_1-t_2|/\tau} \tag{9}$$

consistent with Eq. (2); therefore, $\boldsymbol{\mu}$ is a stationary OU process. Now we can establish the FP equation for the joint probability density $P(\mathbf{r},\boldsymbol{\mu};t)$ associated with Eqs. (4) and (5). This equation reads

$$\frac{\partial P}{\partial t} = a\nabla_{\mathbf{r}}\cdot(\mathbf{r}P) - \frac{\boldsymbol{\mu}}{\alpha}\cdot\nabla_{\mathbf{r}}P + \frac{1}{\tau}\nabla_{\mu}\cdot(\boldsymbol{\mu}P) + \frac{\lambda}{\tau^2}\nabla_{\mu}^2 P. \tag{10}$$

As far as we know, the explicit analytical solution for the time-dependent $P(\mathbf{r},\boldsymbol{\mu};t)$ has not yet been reported in the literature. However, due to the Gaussian noise character and the fact that the Langevin equation is linear, the stationary solution denoted by $P_{\rm st}(\mathbf{x}) \equiv P_{\rm st}(\mathbf{r},\boldsymbol{\mu})$ is proposed as a Gaussian distribution function given by [1,2]

$$P_{\rm st}(\mathbf{x}) = N\exp\left[-\frac{1}{2}\sum_{i,j=1}^{6}\sigma_{ij}^{-1}(x_i - \langle x_i\rangle)(x_j - \langle x_j\rangle)\right], \tag{11}$$

where $N = 1/(2\pi)^3\sqrt{\det\sigma_{ij}}$ is the normalization constant with the vector $\mathbf{x} = (\mathbf{r},\boldsymbol{\mu}) = (x,y,z,\mu_x,\mu_y,\mu_z)$. The matrix σ_{ij} represents the steady-state correlation matrix of the random variables x_i and σ_{ij}^{-1} its inverse. Due to the structure of Eqs. (4) and (5) the total SSPD will be given by $P_{\rm st}(\mathbf{r},\boldsymbol{\mu}) = P_{1\rm st}(x,\mu_x)P_{2\rm st}(y,\mu_y)P_{3\rm st}(z,\mu_z)$; then we proceed to calculate the SSPD for two variables, say $P_{1\rm st}(x,\mu_x)$. For such a purpose, we identify the matrix elements $\sigma_{11} = \langle x^2\rangle_{\rm st}$, $\sigma_{22} = \langle \mu^2\rangle_{\rm st}$, and $\sigma_{12} = \langle x\mu\rangle_{\rm st} = \langle \mu x\rangle_{\rm st} = \sigma_{21}$. The elements of the 2×2 matrix σ_{ij} can be easily calculated from Eqs. (4) and (5), yielding

$$\sigma_{11} = \frac{\lambda}{a\alpha^2(1+a\tau)}, \quad \sigma_{12} = \frac{\lambda}{\alpha(1+a\tau)}, \quad \sigma_{22} = \frac{\lambda}{\tau}, \tag{12}$$

and $\langle x\rangle_{\rm st} = 0$, $\langle \mu\rangle_{\rm st} = 0$. Since $\det\sigma_{ij} = [\lambda/\tau(1+a\tau)]\sigma_{11}$, the matrix elements of σ_{ij}^{-1} are

$$\sigma_{11}^{-1} = \frac{a\alpha^2(1+a\tau)^2}{\lambda}, \quad \sigma_{22}^{-1} = \frac{\tau(1+a\tau)}{\lambda}, \quad \sigma_{12}^{-1} = \sigma_{21}^{-1} = -\frac{a\alpha\tau(1+a\tau)}{\lambda}. \tag{13}$$

Hence, for just two random variables, the SSPD given by Eq. (11) reads as

$$P_{1\rm st}(x,\mu_x) = \widehat{N}\exp\left[-Ax^2 + Bx\mu_x - C\mu_x^2\right], \tag{14}$$

with $\widehat{N} = [\sqrt{a\tau}(1+a\tau)/2\pi\alpha D]$ the normalization constant and

$$A = \frac{a(1+a\tau)^2}{2D}, \quad B = \frac{a\tau(1+a\tau)}{\alpha D}, \quad C = \frac{\tau(1+a\tau)}{2\alpha^2 D}, \tag{15}$$

with D being a redefinition of the noise intensity given by $D = \lambda/\alpha^2$. We get analogous expressions for $P_{2\rm st}(y,\mu_y)$ and $P_{3\rm st}(z,\mu_z)$ and therefore the solution for the stationary state of the FP Eq. (10) can be written as

$$P_{\rm st}(\mathbf{r},\boldsymbol{\mu}) = \widehat{N}^3\exp[-A|\mathbf{r}|^2 + B\mathbf{r}\cdot\boldsymbol{\mu} - C|\boldsymbol{\mu}|^2]. \tag{16}$$

After some straightforward algebra it can be shown that the marginal probability densities are

$$P_{\rm st}(\mathbf{r}) = \left(\frac{1}{2\pi D_e/a}\right)^{3/2}\exp\left(-\frac{|\mathbf{r}|^2}{2D_e/a}\right), \tag{17}$$

$$P_{\rm st}(\boldsymbol{\mu}) = \left(\frac{1}{2\pi\lambda/\tau}\right)^{3/2}\exp\left(-\frac{|\boldsymbol{\mu}|^2}{2\lambda/\tau}\right), \tag{18}$$

where the quantity $D_e = D/(1 + a\tau)$ is a rescaling of the noise intensity D. The result given in Eq. (17) shows that the contribution of the external colored noise amounts to a renormalization of the noise intensity D by the factor $1/(1 + a\tau)$, where the correlation time τ plays a relevant role. On the other hand, due to the Gaussian character of the noise $\boldsymbol{\mu}$ with zero mean value and correlation function as given by Eq. (2), it is also stationary as shown in Eq. (5). Its stationary distribution is that given by Eq. (18), which has been obtained as additional information from Eq. (16) as expected, because we already know that $\langle \mu_i \rangle_{\rm st} = 0$ and $\langle \mu_i^2 \rangle_{\rm st} = \lambda/\tau$.

III. HARMONIC OSCILLATOR IN AN ELECTROMAGNETIC FIELD DRIVEN BY EXTERNAL COLORED NOISE

Now consider that the above Brownian harmonic oscillator has a charge q and is under the action of an electromagnetic field. We take into account a constant magnetic field pointing along the z axis, that is $\mathbf{B} = (0,0,B)$, and an external fluctuating electric field $\bar{\mathbf{E}}(t) = (E_x, E_y, E_z)$ satisfying the OU process. Due to the orientation of the magnetic field, the Langevin equation can be split into two independent processes: one is given along the magnetic field and the other on the x-y plane orthogonal to this field. The SSPD describing the stationary process parallel to the magnetic field (along the z axis) is quite similar to the z component given in Sec. II and therefore we just pay attention to the process in the x-y plane. Before we establish the Langevin equation for the charged particle, it must be mentioned that for a single particle moving in an external field a damping radiation occurs. This damping radiation is also called the Lorentz friction force and it is given by $\mathbf{F}_L = 2e^2\ddot{\mathbf{r}}/3(4\pi\epsilon_0)c^3$, with $\epsilon_0 = 8.854 \times 10^{-12}$ C^2/Nm^2 being the vacuum electric permittivity [35]. By comparing this radiation damping force with the Brownian friction force (Stokes force) given by $\mathbf{F}_S = -\alpha\dot{\mathbf{r}}$, we can estimate the order of magnitude of both forces. For a charged Brownian particle we can consider an ion with a charge $q = ne$, where $n = 10$ and $e \sim 10^{-19}$ C, $|\mathbf{r}|$ can be considered approximately as the size l of the particle, and $l \sim R$, with R being its radius. Hence $|\ddot{\mathbf{r}}| \sim l/\tau^3$, where τ is a characteristic time which can be estimated as $\tau \sim l^2/D = R^2/D$ and D is the diffusion coefficient. In this case, the amplitude of the Lorentz force $F_L \sim (e^2/\epsilon_0 c^3)|\ddot{\mathbf{r}}| \sim (10^{-36}/10^{12}) l/\tau^3 = 10^{-48} D^3/R^5$. For a Brownian particle the amplitude of the friction dynamics is $F_s = \alpha|\dot{\mathbf{r}}|$, where $\alpha \sim \eta R$ is the friction coefficient, η being the viscosity and thus $F_s \sim \eta R l/\tau \sim \eta D$. So that a comparison between the amplitude of both frictional forces leads to $F_L/F_S \sim 10^{-48} D^2/\eta R^5$. For water the viscosity is of the order of $\eta \sim 10^{-3}$ kg/ms and the diffusion coefficient $D \sim 10^{-5}$ m^2/s. For a Brownian particle of the order of microns (10^{-6} m) or nanometers (10^{-9}), the ratio $F_L/F_S \sim 10^{-25}$ or $F_L/F_S \sim 10^{-10}$, respectively, which are much less than the unity. Due to this fact, the damping radiation in the study of plasma diffusion considered as a classical Brownian motion can be neglected. Hence the Langevin equation describing the process in the x-y plane can be written as

$$m\frac{d\mathbf{u}}{dt} = -\alpha\mathbf{u} + \frac{q}{c}\mathbf{u} \times \mathbf{B} - k\mathbf{r} + \boldsymbol{\mu}(t), \quad (19)$$

where $\mathbf{r} = (x,y)$, $\mathbf{u} = d\mathbf{r}/dt = (u_x,u_y)$, with $\boldsymbol{\mu} = q\mathbf{E}(t) = (\mu_x,\mu_y)$ satisfying the property of a Gaussian colored noise with zero mean value and correlation function

$$\langle \mu_i(t)\mu_j(t')\rangle = \frac{\lambda}{\tau}\delta_{ij}e^{-|t-t'|/\tau}, \quad i,j = x,y. \quad (20)$$

The overdamped Langevin equation is now given as

$$\frac{d\mathbf{r}}{dt} = -\widetilde{\Gamma}\mathbf{r} + \alpha_e^{-1}\Lambda\boldsymbol{\mu}(t), \quad (21)$$

where the matrices $\widetilde{\Gamma}$ and Λ are defined as

$$\widetilde{\Gamma} = \begin{pmatrix} \widetilde{a} & \widetilde{\Omega} \\ -\widetilde{\Omega} & \widetilde{a} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & C_0 \\ -C_0 & 1 \end{pmatrix}, \quad (22)$$

with $\widetilde{a} = k/\alpha_e$, $\widetilde{\Omega} = \widetilde{a}C_0$, and $\alpha_e = \alpha(1 + C_0^2)$; $C_0 = qB/c\alpha$ is a dimensionless constant. In a way similar to Sec. II, we split Eq. (21) into a set of four Markovian differential equations, written in vectorial form as

$$\dot{\mathbf{r}} = -\widetilde{\Gamma}\mathbf{r} + \alpha_e^{-1}\Lambda\boldsymbol{\mu}, \quad (23)$$

$$\dot{\boldsymbol{\mu}} = -\frac{1}{\tau}\boldsymbol{\mu} + \frac{1}{\tau}\boldsymbol{\xi}(t), \quad (24)$$

where, due to the structure of matrix $\widetilde{\Gamma}$, Eq. (23) represents a coupled pair of differential equations, although the processes $\boldsymbol{\mu} = (\mu_x,\mu_y)$ are independent. The term $\boldsymbol{\xi}(t) = (\xi_x,\xi_y)$ also satisfies the property of a GWN as before. The FP equation for the joint probability density $P^m(\mathbf{r},\boldsymbol{\mu};t)$ is given by

$$\frac{\partial P^m}{\partial t} = \nabla_{\mathbf{r}} \cdot (\widetilde{\Gamma}\mathbf{r}P^m) - \frac{1}{\alpha_e}\Lambda\boldsymbol{\mu} \cdot \nabla_{\mathbf{r}}P^m + \frac{1}{\tau}\nabla_{\mu} \cdot (\boldsymbol{\mu}P^m) + \frac{\lambda}{\tau^2}\nabla_{\mu}^2 P^m. \quad (25)$$

The explicit solution of Eq. (25), as well as its stationary-state solution, is reported here. Again the SSPD is proposed to satisfy the Gaussian distribution function (11), now for the vector $\mathbf{x} = (x,y,\mu_x,\mu_y)$ and the normalization constant $N = 1/(2\pi)^2\sqrt{\det\sigma_{ij}}$. The 4×4 matrix σ_{ij}, its inverse, as well as the SSPD, are all explicitly calculated in Appendix A, yielding the exact result

$$P_{\rm st}^m(\mathbf{r},\boldsymbol{\mu}) = N\exp[-\mathcal{A}|\mathbf{r}|^2 + \mathcal{B}(\mathbf{r}\cdot\boldsymbol{\mu}) - \mathcal{C}|\boldsymbol{\mu}|^2 + \mathcal{D}(\mathbf{r}\times\boldsymbol{\mu})_z], \quad (26)$$

where N as well as the coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are given in Appendix A. We also show explicitly in Appendix B that the solution (26) is indeed the exact solution of Eq. (25). The marginal SSPDs $P_{\rm st}^m(\mathbf{r})$ and $P_{\rm st}^m(\boldsymbol{\mu})$ can be calculated after integration over the noise space $\boldsymbol{\mu}$ and the configuration space $\mathbf{r}$, respectively. After some algebra we arrive at the expressions

$$P_{\rm st}^m(\mathbf{r}) = \frac{1}{2\pi\sigma_{11}}\exp\left(-\frac{|\mathbf{r}|^2}{2\sigma_{11}}\right), \quad (27)$$

$$P_{\rm st}^m(\boldsymbol{\mu}) = \frac{1}{2\pi(\lambda/\tau)}\exp\left(-\frac{|\boldsymbol{\mu}|^2}{2\lambda/\tau}\right), \quad (28)$$

where the exact expression of σ_{11} is given by Eq. (A4) of Appendix A, which can also be written as $\sigma_{11} = D_m(1 + \widetilde{a}\tau)/\widetilde{a}[(1 + \widetilde{a}\tau)^2 + (\widetilde{\Omega}\tau)^2]$ and $D_m = (\lambda/\alpha^2)/(1 + C_0^2) = D/(1 + C_0^2)$. The result (27) is obtained for all τ values, being strongly coupled to the magnetic field through the term $\widetilde{\Omega}\tau$. Again, due to the fact that the noise term $\boldsymbol{\mu} = q\mathbf{E}(t)$ satisfies the properties of a Gaussian colored noise (OU noise), its stationary probability function is given by Eq. (28), as expected. Its derivation from Eq. (26) justifies the consistency of this equation itself. Equation (27) tells us that the position of the charged particle reaches the stationary state and the width of its distribution is measured by an effective noise intensity $D_m(1+\widetilde{a}\tau)/[(1+\widetilde{a}\tau)^2+(\widetilde{\Omega}\tau)^2]$. This effective noise intensity is quantified by the cooperative effect of the friction coefficient, the magnetic field, and the noise correlation time.

To verify our theoretical results we compare them with those obtained in the previous section in the absence of the magnetic field ($C_0 = 0$). In this case, it is clear that the parameters are $\alpha_e = \alpha$ and $\widetilde{a} = (k/\alpha) = a$. It can be readily shown that the coefficients $\mathcal{A} = A, \mathcal{B} = B, \mathcal{C} = C, \mathcal{D} = 0$, and the normalization constant $N = 1/(2\pi)^2 H$ given in Eq. (26) reduces to $N = a\tau(1+a\tau)^2/(2\pi)^2\lambda D$, which is quite similar to $\widehat{N}^2$, with $\widehat{N}$ defined in Sec. II. So, for zero magnetic field, the SSPD (26) reduces to the SSPD given by Eq. (16) when we write $P_{\mathrm{st}}(\mathbf{r},\boldsymbol{\mu}) = \widehat{N}^2 P_{1\mathrm{st}}(x,\mu_x)P_{2\mathrm{st}}(y,\mu_y)$. In a similar way, for the marginal SSPD $P^m_{\mathrm{st}}(\mathbf{r})$ given by Eq. (27) we can check that σ_{11} reduces to $\sigma_{11} = \lambda/\alpha^2 a(1+a\tau) = D_e/a$ and therefore $P^m_{\mathrm{st}}(\mathbf{r})$ reduces to Eq. (17) when $P_{\mathrm{st}}(\mathbf{r}) = P_{1\mathrm{st}}(x)P_{\mathrm{st}}(y) = (1/2\pi D_e/a)\exp(-|\mathbf{r}|^2/2D_e/a)$.

An interesting approximation comes out when we consider that the magnetic field is weakly coupled to the noise correlation time such that $\widetilde{\Omega}\tau \ll 1$. In this case σ_{11} reduces to $\sigma_{11} = D_m/\widetilde{a}(1+\widetilde{a}\tau)$, and thus

$$P^m_{\mathrm{st}}(\mathbf{r}) = \frac{1}{2\pi D_m/\widetilde{a}(1+\widetilde{a}\tau)}\exp\left(-\frac{|\mathbf{r}|^2}{2D_m/\widetilde{a}(1+\widetilde{a}\tau)}\right), \tag{29}$$

where it is seen that the presence of the magnetic field in Eq. (29) with respect to the colored noise $D/a(1+a\tau)$ induces a rescaling of the parameters D and a by a factor $1/(1+C_0^2)$ leading to D_m and $\widetilde{a}$, respectively.

To see the robustness of our exact analytical results, they have been compared with the numerical simulation of Eqs. (23) and (24). In Fig. 1(a) we compare the singled-valued marginal SSPDs (27) and (28) for the fieldless case ($C_0 = 0$), showing clearly a remarkable match between both results. The single-valued marginal SSPDs are possible due to the independence of the Langevin equations in the absence of the magnetic field. In fact, the SSPD given by Eq. (28) is the same as Eq. (18). The marginal SSPD (27) takes into account the presence of the magnetic field and is given in two dimensions. For practical purposes in the plot, we transform it in terms of the variable $R^2 = x^2 + y^2$, given as a result the normalized SSPD $P^m_{\mathrm{st}}(R) = (R/\sigma_{11})\exp(-R^2/2\sigma_{11})$. This probability function is compared with the numerical simulation results as shown in Fig. 1(b), where again the agreement between both results is excellent.

IV. CONCLUDING REMARKS

In this work we have obtained the exact analytical expression of the SSPD $P^m_{\mathrm{st}}(\mathbf{r},\boldsymbol{\mu})$ for a Brownian harmonic oscillator under the influence of a constant magnetic field and

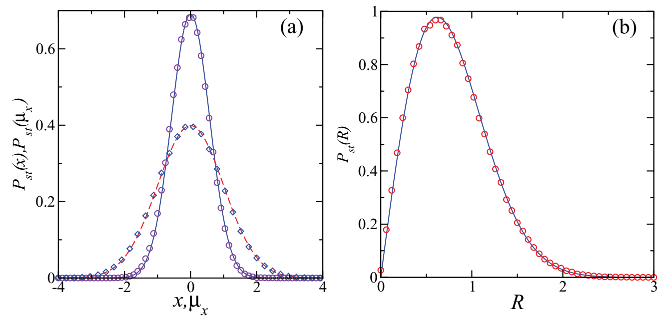

FIG. 1. (Color online) (a) Marginal SSPDs $P_{\mathrm{st}}(x)$ (continuous line) and $P_{\mathrm{st}}(\mu_x)$ (dashed line) obtained from Eqs. (27) and (28) for the set of parameters $C_0 = 0$, $k = \alpha = 1$, and $\lambda = \tau = 0.5$. Circles and diamonds correspond to numerical results obtained from direct simulation of Eqs. (23) and (24). (b) Marginal SSPD $P^m_{\mathrm{st}}(R)$ obtained for the same set of parameters as in (a), except for $C_0 = 1$; again, symbols correspond to numerical simulation results.

a randomly fluctuating electric field modeled as a Gaussian exponentially correlated noise. Due to the non-Markovian character introduced by the electric field the SSPD $P^m_{\mathrm{st}}(\mathbf{r},\boldsymbol{\mu})$ given in Eq. (26) has been derived through the usual procedure which extends the space of variables, whereby a Markovian problem results. The SSPD has been obtained for all values of noise correlation which is shown to be strongly coupled to the magnetic field through the term $\widetilde{\Omega}\tau$, as shown in Eqs. (A4)–(A6). The marginal probabilities $P^m_{\mathrm{st}}(\mathbf{r})$ and $P^m_{\mathrm{st}}(\boldsymbol{\mu})$ are calculated after marginal integration of Eq. (26). In the particular case of weak coupling between the magnetic field and the noise correlation time, for which $\widetilde{\Omega}\tau \ll 1$, the exact SSPD (27) reduces to Eq. (29). In this case the presence of the magnetic field induces, with respect to the colored noise, a renormalization in the noise intensity D and the parameter a. As shown in Fig. 1, our theoretical results agree in an excellent way with the numerical simulation results.

Finally, we mention two problems in which our present contribution has been useful: the first one is related to the recent study on the decay of unstable states driven by a Gaussian colored noise under the action of an electromagnetic field [34]. The decay process is characterized through the statistics of the first passage time distribution [16,22], which strongly depends on the initial distribution function. The variance of this initial distribution was calculated only via the Langevin equation in [34] and it is exactly the same as Eq. (A4) of the present work if the parameter $\widetilde{a}$ is replaced by $\widetilde{a}_0$. Therefore, the solution of the stationary-state FP equation provides another alternative way of how to calculate the statistics of such an initial condition. The other problem is related with the work per unit time fluctuation theorem for a charged Brownian harmonic oscillator in an electromagnetic field for a Gaussian exponentially correlated noise. This problem has been successfully achieved and recently submitted elsewhere.

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APPENDIX A: $P^m_{st}(\mathbf{r},\boldsymbol{\mu})$ CALCULATION IN AN ELECTROMAGNETIC FIELD

Here we calculate the SSPD solution of the FP Eq. (25). We write the SPD as

$$P^m_{st}(\mathbf{x}) = N\exp[-(\mathbf{x}^T\cdot\boldsymbol{\sigma}^{-1}\cdot\mathbf{x})/2], \tag{A1}$$

where $N = 1/(2\pi)^2\sqrt{\det\sigma_{ij}}$, $\mathbf{x}^T$ is the transpose of the vector $\mathbf{x}$, and $\boldsymbol{\sigma}^{-1}$ is the inverse of the 4×4 matrix $\boldsymbol{\sigma}\equiv\sigma_{ij}$ such that

$$\boldsymbol{\sigma} = \begin{pmatrix} \langle x^2\rangle_{\rm st} & \langle xy\rangle_{\rm st} & \langle x\mu_x\rangle_{\rm st} & \langle x\mu_y\rangle_{\rm st} \\ \langle yx\rangle_{\rm st} & \langle y^2\rangle_{\rm st} & \langle y\mu_x\rangle_{\rm st} & \langle y\mu_y\rangle_{\rm st} \\ \langle \mu_x x\rangle_{\rm st} & \langle \mu_x y\rangle_{\rm st} & \langle \mu_x^2\rangle_{\rm st} & \langle \mu_x\mu_y\rangle_{\rm st} \\ \langle \mu_y x\rangle_{\rm st} & \langle \mu_y y\rangle_{\rm st} & \langle \mu_y\mu_x\rangle_{\rm st} & \langle \mu_y^2\rangle_{\rm st} \end{pmatrix}. \tag{A2}$$

We already know that $\sigma_{33} = \langle\mu_x^2\rangle = \langle\mu_y^2\rangle_{\rm st} = \sigma_{44} = \lambda/\tau$ and that $\sigma_{34} = \langle\mu_x\mu_y\rangle_{\rm st} = \langle\mu_y\mu_x\rangle_{\rm st} = \sigma_{43} = 0$. It is also clear that $\sigma_{12} = \langle xy\rangle = \langle yx\rangle_{\rm st} = \sigma_{21}$, $\sigma_{13} = \langle x\mu_x\rangle_{\rm st} = \langle\mu_x x\rangle_{\rm st} = \sigma_{31}$, $\sigma_{14} = \langle x\mu_y\rangle_{\rm st} = \langle\mu_y x\rangle_{\rm st} = \sigma_{41}$, $\sigma_{23} = \langle y\mu_x\rangle_{\rm st} = \langle\mu_x y\rangle_{\rm st} = \sigma_{32}$, and $\sigma_{24} = \langle y\mu_y\rangle_{\rm st} = \langle\mu_y y\rangle_{\rm st} = \sigma_{42}$. Therefore, we have only seven coefficients to be determined which can be calculated from Eqs. (23) and (24). These equations lead to a set of seven coupled algebraic equations given by

$$\begin{aligned}
\widetilde{a}\sigma_{11} + \widetilde{a}\sigma_{22} - \frac{1}{\alpha_e}[\sigma_{13}+\sigma_{24}] - \frac{C_0}{\alpha_e}[\sigma_{14}-\sigma_{23}] &= 0,\\
\widetilde{a}\sigma_{11} - \widetilde{a}\sigma_{22} + 2\widetilde{\Omega}\sigma_{12} + \frac{1}{\alpha_e}[\sigma_{24}-\sigma_{13}] - \frac{C_0}{\alpha_e}[\sigma_{14}+\sigma_{23}] &= 0,\\
\widetilde{\Omega}\sigma_{11} - \widetilde{\Omega}\sigma_{22} - 2\widetilde{a}\sigma_{12} + \frac{1}{\alpha_e}[\sigma_{14}+\sigma_{23}] - \frac{C_0}{\alpha_e}[\sigma_{24}-\sigma_{13}] &= 0,\\
(1+\widetilde{a}\tau)\sigma_{13} + \widetilde{\Omega}\tau\sigma_{23} - \frac{\lambda}{\alpha_e} &= 0,\\
(1+\widetilde{a}\tau)\sigma_{23} - \widetilde{\Omega}\tau\sigma_{13} + \frac{C_0\lambda}{\alpha_e} &= 0,\\
(1+\widetilde{a}\tau)\sigma_{24} - \widetilde{\Omega}\tau\sigma_{14} - \frac{\lambda}{\alpha_e} &= 0,\\
(1+\widetilde{a}\tau)\sigma_{14} + \widetilde{\Omega}\tau\sigma_{24} - \frac{C_0\lambda}{\alpha_e} &= 0.
\end{aligned} \tag{A3}$$

After some intricate algebra we get the following results: $\sigma_{11} = \langle x^2\rangle_{\rm st} = \langle y^2\rangle_{\rm st} = \sigma_{22}$, $\sigma_{12} = \langle xy\rangle_{\rm st} = \sigma_{21} = 0$, $\sigma_{13} = \langle x\mu_x\rangle_{\rm st} = \langle y\mu_y\rangle_{\rm st} = \sigma_{24}$, and $\sigma_{14} = \langle x\mu_y\rangle_{\rm st} = -\langle y\mu_x\rangle_{\rm st} = -\sigma_{23}$, where

$$\sigma_{11} = \frac{\lambda(1+\widetilde{a}\tau)}{\widetilde{a}\alpha^2(1+C_0^2)[(1+\widetilde{a}\tau)^2+(\widetilde{\Omega}\tau)^2]}, \tag{A4}$$

$$\sigma_{13} = \frac{\lambda[(1+\widetilde{a}\tau)+C_0\widetilde{\Omega}\tau]}{\alpha_e[(1+\widetilde{a}\tau)^2+(\widetilde{\Omega}\tau)^2]}, \tag{A5}$$

$$\sigma_{14} = \frac{\lambda C_0}{\alpha_e[(1+\widetilde{a}\tau)^2+(\widetilde{\Omega}\tau)^2]}. \tag{A6}$$

The correlation matrix $\boldsymbol{\sigma}\equiv\sigma_{ij}$ is then

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} & \sigma_{14}\\ 0 & \sigma_{11} & -\sigma_{14} & \sigma_{13}\\ \sigma_{13} & -\sigma_{14} & \sigma_{33} & 0\\ \sigma_{14} & \sigma_{13} & 0 & \sigma_{33}\end{pmatrix}, \tag{A7}$$

with its inverse given by

$$\boldsymbol{\sigma}^{-1} = \frac{1}{H}\begin{pmatrix} \sigma_{33} & 0 & -\sigma_{13} & -\sigma_{14}\\ 0 & \sigma_{33} & \sigma_{14} & -\sigma_{13}\\ -\sigma_{13} & \sigma_{14} & \sigma_{11} & 0\\ -\sigma_{14} & -\sigma_{13} & 0 & \sigma_{11}\end{pmatrix}, \tag{A8}$$

and $H = \sigma_{11}\sigma_{33} - \sigma_{13}^2 - \sigma_{14}^2$, where H is related to $\det\sigma_{ij} = H^2$. Taking into account Eq. (A8) and the vector $\mathbf{x} = (x,y,\mu_x,\mu_y)$ we conclude that the exact SSPD reads as

$$P^m_{\rm st}(\mathbf{r},\boldsymbol{\mu}) = N\exp[-\mathcal{A}|\mathbf{r}|^2 + \mathcal{B}\mathbf{r}\cdot\boldsymbol{\mu} - \mathcal{C}|\boldsymbol{\mu}|^2 + \mathcal{D}(\mathbf{r}\times\boldsymbol{\mu})_z], \tag{A9}$$

where $N = 1/(2\pi)^2\sqrt{\det\sigma_{ij}} = 1/4\pi^2 H$ and

$$\mathcal{A} = \frac{\sigma_{33}}{2H},\quad \mathcal{B} = \frac{\sigma_{13}}{H},\quad \mathcal{C} = \frac{\sigma_{11}}{2H},\quad \mathcal{D} = \frac{\sigma_{14}}{H}. \tag{A10}$$

APPENDIX B: PROOF OF THE STATIONARY SOLUTION OF THE FOKKER-PLANCK EQ. (25)

In this Appendix we will show explicitly that our solution given by Eq. (26) is effectively the stationary solution of Eq. (25). For this purpose we use the following representations for the derivatives, matrices, and vectors:

$$\nabla_{\boldsymbol{\xi}} \equiv \frac{\partial}{\partial\xi_i}\hat{\mathbf{e}}_i,\quad L\equiv L_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j,\quad \boldsymbol{\xi}\equiv\xi_i\hat{\mathbf{e}}_i, \tag{B1}$$

where $\boldsymbol{\xi}$ represents the vectors $\mathbf{r}$ and $\boldsymbol{\mu}$, and the matrix L the matrices $\widetilde{\Gamma}$ and Λ; the scalar product $\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j = \delta_{ij}$ will be used. We first identify the product $\widetilde{\Gamma}\mathbf{r}\equiv\widetilde{\Gamma}\cdot\mathbf{r} = \widetilde{\Gamma}_{jk}x_l\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k\cdot\hat{\mathbf{e}}_l$, and then $\widetilde{\Gamma}\mathbf{r}P^m_{\rm st} = \widetilde{\Gamma}_{jk}x_lP^m_{\rm st}\hat{\mathbf{e}}_j(\hat{\mathbf{e}}_k\cdot\hat{\mathbf{e}}_l) = \widetilde{\Gamma}_{jk}x_kP^m_{\rm st}\hat{\mathbf{e}}_j$. Hence each term of Eq. (25) is given by

$$\begin{aligned}
\nabla_{\mathbf{r}}\cdot\left(\widetilde{\Gamma}\mathbf{r}P^m_{\rm st}\right) &= \frac{\partial}{\partial x_i}\widetilde{\Gamma}_{jk}x_kP^m_{\rm st}\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j = \frac{\partial}{\partial x_i}\widetilde{\Gamma}_{ik}x_kP^m_{\rm st}\\
&= \frac{\partial}{\partial x}(\widetilde{\Gamma}_{11}x+\widetilde{\Gamma}_{12}y)P^m_{\rm st}\\
&\quad+\frac{\partial}{\partial y}(\widetilde{\Gamma}_{21}x+\widetilde{\Gamma}_{22}y)P^m_{\rm st},
\end{aligned} \tag{B2}$$

$$\begin{aligned}
\frac{1}{\tau}\nabla_{\boldsymbol{\mu}}\cdot\left(\boldsymbol{\mu}P^m_{\rm st}\right) &= \frac{1}{\tau}\frac{\partial}{\partial\mu_i}\mu_jP^m_{\rm st}\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j = \frac{1}{\tau}\frac{\partial}{\partial\mu_i}\mu_iP^m_{\rm st}\\
&= \frac{1}{\tau}\frac{\partial}{\partial\mu_x}\mu_xP^m_{\rm st} + \frac{1}{\tau}\frac{\partial}{\partial\mu_y}\mu_yP^m_{\rm st},
\end{aligned} \tag{B3}$$

$$\begin{aligned}
-\frac{1}{\alpha_e}\Lambda\boldsymbol{\mu}\cdot\nabla_{\mathbf{r}}P^m_{\rm st} &= -\frac{1}{\alpha_e}\Lambda_{ij}\mu_j\hat{\mathbf{e}}_i\cdot\frac{\partial}{\partial x_k}P^m_{\rm st}\hat{\mathbf{e}}_k\\
&= -\frac{1}{\alpha_e}\Lambda_{ij}\mu_j\frac{\partial}{\partial x_i}P^m_{\rm st}\\
&= -\frac{1}{\alpha_e}\left(\Lambda_{11}\mu_x\frac{\partial}{\partial x}P^m_{\rm st} + \Lambda_{12}\mu_y\frac{\partial}{\partial x}P^m_{\rm st}\right)\\
&\quad-\frac{1}{\alpha_e}\left(\Lambda_{21}\mu_x\frac{\partial}{\partial y}P^m_{\rm st} + \Lambda_{22}\mu_y\frac{\partial}{\partial x}P^m_{\rm st}\right),
\end{aligned} \tag{B4}$$

$$\frac{\lambda}{\tau^2}\nabla^2_{\boldsymbol{\mu}}P^m_{\rm st} = \frac{\lambda}{\tau^2}\left(\frac{\partial^2}{\partial\mu_x^2}P^m_{\rm st} + \frac{\partial^2}{\partial\mu_y^2}P^m_{\rm st}\right). \tag{B5}$$

By substituting the matrix elements of $\widetilde{\Gamma}$ and Λ, and calculating the corresponding derivatives of $P^m_{\rm st}$, we get, after a long but straightforward algebra, that the right-hand side of Eq. (25)

can be written in the following way:

$$[\mathbb{C}_0 - \mathbb{C}_1|\mathbf{r}|^2 + \mathbb{C}_2\mathbf{r}\cdot\boldsymbol{\mu} + \mathbb{C}_3(\mathbf{r}\times\boldsymbol{\mu})_z - \mathbb{C}_4|\boldsymbol{\mu}|^2]P_{\rm st}^m, \quad \text{(B6)}$$

where the $\mathbb{C}_i$ coefficients are given by

$$\mathbb{C}_0 = \frac{2}{\tau}(1+\widetilde{a}\tau) - \frac{4\lambda}{\tau^2}\mathcal{C}, \quad \text{(B7)}$$

$$\mathbb{C}_1 = 2\widetilde{a}\mathcal{A} - \frac{\lambda}{\tau^2}\mathcal{B}^2 - \frac{\lambda}{\tau^2}\mathcal{D}^2, \quad \text{(B8)}$$

$$\mathbb{C}_2 = \frac{1}{\tau}(1+\widetilde{a}\tau)\mathcal{B} + \widetilde{a}C_0\mathcal{D} + \frac{2}{\alpha_e}\mathcal{A} - \frac{4\lambda}{\tau^2}\mathcal{BC}, \quad \text{(B9)}$$

$$\mathbb{C}_3 = \frac{1}{\tau}(1+\widetilde{a}\tau)\mathcal{D} - \widetilde{a}C_0\mathcal{B} + \frac{2C_0}{\alpha_e}\mathcal{A} - \frac{4\lambda}{\tau^2}\mathcal{DC}, \quad \text{(B10)}$$

$$\mathbb{C}_4 = \frac{1}{\alpha_e}\mathcal{B} + \frac{C_0}{\alpha_e}\mathcal{D} + \frac{2}{\tau}\mathcal{C} - \frac{4\lambda}{\tau^2}\mathcal{C}^2. \quad \text{(B11)}$$

To prove that $P_{\rm st}^m$ is the solution of Eq. (25) we must show that Eq. (B6) is identically zero. This must be true if each of the constants $\mathbb{C}_i = 0$. For such a purpose we must first notice from Eqs. (A4)–(A6) that the relation $\sigma_{11} = (1/\widetilde{a}\alpha_e)(\sigma_{13} + C_0\sigma_{14})$ is satisfied. In this case, it can be shown that the H parameter defined in Appendix A is equal to

$$H = \sigma_{11}\sigma_{33} - \sigma_{13}^2 - \sigma_{14}^2 = \left(\frac{\lambda}{\alpha_e\widetilde{a}\tau} - \sigma_{13}\right)\sigma_{13} + \left(\frac{\lambda C_0}{\alpha_e\widetilde{a}\tau} - \sigma_{14}\right)\sigma_{14}. \quad \text{(B12)}$$

Using the explicit expression of σ_{13} and σ_{14} it is also shown that

$$\frac{\lambda}{\alpha_e\widetilde{a}\tau} - \sigma_{13} = \frac{\lambda(1+\widetilde{a}\tau)}{\alpha_e\widetilde{a}\tau\Delta}, \qquad \frac{\lambda}{\alpha_e\widetilde{a}\tau} - \sigma_{14} = \frac{\lambda C_0}{\alpha_e\widetilde{a}\tau\Delta}(\Delta - \widetilde{a}\tau), \quad \text{(B13)}$$

where $\Delta \equiv (1+\widetilde{a}\tau)^2 + (\widetilde{\Omega}\tau)^2$. It is also possible to show that

$$\left(\frac{\lambda}{\alpha_e\widetilde{a}\tau} - \sigma_{13}\right)\sigma_{13} = \frac{\lambda^2}{\alpha_e^2\widetilde{a}\tau\Delta^2}(\Delta + \widetilde{a}C_0^2\tau), \quad \text{(B14)}$$

$$\left(\frac{\lambda}{\alpha_e\widetilde{a}\tau} - \sigma_{14}\right)\sigma_{14} = \frac{\lambda^2C_0^2}{\alpha_e^2\widetilde{a}\tau\Delta^2}(\Delta - \widetilde{a}\tau). \quad \text{(B15)}$$

Finally, substituting Eqs. (B14) and (B15) into Eq. (B12) we obtain

$$H = \frac{\lambda^2(1+C_0^2)}{\alpha_e^2\widetilde{a}\tau\Delta} = \frac{\lambda}{\tau}\frac{\lambda}{\alpha^2\widetilde{a}(1+C_0^2)\Delta} = \frac{\lambda}{\tau}\frac{\sigma_{11}}{(1+\widetilde{a}\tau)}. \quad \text{(B16)}$$

We can now verify, according to the coefficient $\mathcal{C} = \sigma_{11}/2H$ and Eq. (B16), that the $\mathbb{C}_0$ is given by

$$\mathbb{C}_0 = \frac{2}{\tau}(1+\widetilde{a}\tau) - \frac{4\lambda}{\tau^2}\frac{\sigma_{11}}{2H} = \frac{2}{\tau}(1+\widetilde{a}\tau) - \frac{2}{\tau}(1+\widetilde{a}\tau) = 0. \quad \text{(B17)}$$

For coefficient $\mathbb{C}_1$ we have

$$\mathbb{C}_1 = 2\widetilde{a}\mathcal{A} - \frac{\lambda}{\tau^2}\mathcal{B}^2 - \frac{\lambda}{\tau^2}\mathcal{D}^2 = \frac{\widetilde{a}\lambda}{\tau H} - \frac{\lambda}{\tau^2}\frac{\sigma_{13}^2}{H^2} - \frac{\lambda}{\tau^2}\frac{\sigma_{14}^2}{H^2} = \frac{\lambda}{\tau^2H^2}[\widetilde{a}\tau H - \sigma_{13}^2 - \sigma_{14}^2] = \frac{\lambda}{\tau^2H^2}\left[\frac{\lambda}{\tau}\sigma_{11} - \sigma_{13}^2 - \sigma_{14}^2 - H\right] = 0. \quad \text{(B18)}$$

The $\mathbb{C}_2$ coefficient is

$$\mathbb{C}_2 = \frac{1}{\tau}(1+\widetilde{a}\tau)\mathcal{B} + \widetilde{a}C_0\mathcal{D} + \frac{2}{\alpha_e}\mathcal{A} - \frac{4\lambda}{\tau^2}\mathcal{BC} = \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{13} - \frac{2\lambda}{\tau^2}\frac{\sigma_{11}}{H}\frac{\sigma_{13}}{H} + \widetilde{a}C_0\frac{\sigma_{14}}{H} + \frac{\lambda}{\alpha_e\tau H}. \quad \text{(B19)}$$

Again, according to Eq. (B16), the first two terms of Eq. (B19) reduce to

$$\frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{13} - \frac{2\lambda}{\tau^2}\frac{\sigma_{11}}{H}\frac{\sigma_{13}}{H} = \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{13} - \frac{2(1+\widetilde{a}\tau)}{\tau H}\sigma_{13} = -\frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{13}, \quad \text{(B20)}$$

and the last two terms of Eq. (B19) reduce to

$$\widetilde{a}C_0\frac{\sigma_{14}}{H} + \frac{\lambda}{\alpha_e\tau H} = \frac{1}{\tau H}\left[\frac{\lambda}{\alpha_e} + \frac{\lambda\widetilde{a}\tau C_0^2}{\alpha_e\Delta}\right] = \frac{\lambda}{\alpha_e\tau H}\left[1 + \frac{\widetilde{a}\tau C_0^2}{\Delta}\right] = \frac{(1+\widetilde{a}\tau)}{\tau H}\frac{\lambda[(1+\widetilde{a}\tau) + C_0\widetilde{\Omega}\tau]}{\alpha_e\Delta} = \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{13}, \quad \text{(B21)}$$

and therefore, due to Eqs. (B20) and (B21),

$$\mathbb{C}_2 = \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{13} - \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{13} = 0. \quad \text{(B22)}$$

In a similar way

$$\mathbb{C}_3 = \frac{1}{\tau}(1+\widetilde{a}\tau)\mathcal{D} - \widetilde{a}C_0\mathcal{B} + \frac{2C_0}{\alpha_e}\mathcal{A} - \frac{4\lambda}{\tau^2}\mathcal{DC} = \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{14} - \frac{2\lambda}{\tau^2}\frac{\sigma_{11}}{H}\frac{\sigma_{14}}{H} - \widetilde{a}C_0\frac{\sigma_{13}}{H} + \frac{\lambda C_0}{\alpha_e\tau H}. \quad \text{(B23)}$$

The first two terms of Eq. (B23) read as

$$\frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{14} - \frac{2\lambda}{\tau^2}\frac{\sigma_{11}}{H}\frac{\sigma_{14}}{H} = \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{14} - \frac{2(1+\widetilde{a}\tau)}{\tau H}\sigma_{14} = -\frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{14}, \quad \text{(B24)}$$

and the last two terms of Eq. (B23) reduce to

$$\frac{\lambda C_0}{\alpha_e\tau H} - \frac{\widetilde{a}C_0}{H}\sigma_{13} = \frac{C_0}{\tau H}\left[\frac{\lambda}{\alpha_e} - \widetilde{a}\tau\sigma_{13}\right] = \frac{\lambda C_0}{\alpha_e\tau H}\left[1 - \frac{\widetilde{a}\tau[1+\widetilde{a}\tau + C_0\widetilde{\Omega}\tau]}{\Delta}\right] = \frac{(1+\widetilde{a}\tau)}{\tau H}\frac{\lambda C_0}{\alpha_e\Delta} = \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{14}. \quad \text{(B25)}$$

Again, according to Eqs. (B24) and (B25), we get

$$\mathbb{C}_3 = \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{14} - \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{14} = 0. \quad \text{(B26)}$$

Lastly, for the coefficient $\mathbb{C}_4$ we obtain

$$\mathbb{C}_4 = \frac{1}{\alpha_e}\mathcal{B} + \frac{C_0}{\alpha_e}\mathcal{D} + \frac{2}{\tau}\mathcal{C} - \frac{4\lambda}{\tau^2}\mathcal{C}^2 = \frac{1}{\alpha_e}\frac{\sigma_{13}}{H} + \frac{C_0}{\alpha_e}\frac{\sigma_{14}}{H} + \frac{1}{\tau}\frac{\sigma_{11}}{H} - \frac{\lambda}{\tau^2}\frac{\sigma_{11}^2}{H^2}$$

$$= \frac{1}{\alpha_e H}(\sigma_{13} + C_0\sigma_{14}) + \frac{1}{\tau H}\sigma_{11} - \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{11}$$
$$= \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{11} - \frac{(1+\widetilde{a}\tau)}{\tau H}\sigma_{11} = 0, \qquad \text{(B27)}$$

where we have used the expression of σ_{11} in terms of σ_{13} and σ_{14} given above. In conclusion, because all the coefficients $\mathbb{C}_i = 0$ the stationary probability density $P^m_{st}(\mathbf{r},\boldsymbol{\mu})$ given by Eq. (26) is the exact solution of the FP equation established in Eq. (25).

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