Mean density of level crossings whose duration exceeds a certain value for a low-friction nonlinear oscillator

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A system that exhibits nonlinear oscillations in the presence of low friction and weak fluctuations is considered. An expression for the mean density of level crossings whose duration exceeds a certain value is derived under the assumption of the high Q factor of the oscillations.

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I. INTRODUCTION

Studies of the level crossings of random processes have been attracting the attention of physicists, mathematicians, and experts in applied sciences for a long time. This is due to the complexity of the analytical solutions of such problems as well as their practical significance.

The importance of studying level crossings was recognized relatively long ago. The first papers in this area were devoted to the theoretical study of the behavior of physical (in particular, oscillatory) systems described by stochastic differential equations. First of all, Pontryagin's fundamental results [1] on the first passage time of Markovian random processes for systems with at least one absorbing boundary should be pointed out.

Intensive statistical analysis of the level crossings of random processes was initiated by Rice's pioneering works [2,3], where the formulas for the mean number of level crossings and distribution of maxima were given for some types of stochastic processes. The works of Tikhonov [4,5] and Stratonovich [6] are also worth mentioning. Interest in this subject has not diminished since then. In subsequent years, the level crossings of stochastic processes have been considered in a number of theoretical and experimental works [7–12]. In several works, the statistics of level crossings for Gaussian processes was studied [3,5,13]. For non-Gaussian processes, there are only a few analytical works regarding the duration distributions of level crossings [3,5,6].

Progress in statistical optics, holography, laser location and laser communication, and the study of random processes has stimulated analysis of the level crossings of light fields [14].

The persistence of a signal level over a time period may also be interpreted as the level crossing of the corresponding stochastic process [15,16]. Persistent properties in different systems, such as twisted nematic liquid crystals [17], laser-polarized ^{129}Xe gas [18], fluctuating steps of Al atoms on Si surface [19], soap froth [20], or droplets on a substrate [21] were experimentally investigated.

Under certain conditions, the mathematical formalism of level crossings can be applied in the study of radio-signal fading resulting from multipath and diffuse propagation in a turbulent medium or reflections from rough surfaces [22]. The duration distribution of level crossings of Nakagami random processes was studied in Ref. [23].

Generally, consideration of the one-dimensional Brownian motion of particles is carried out in the so-called overdamped mode; in this case it is described by the first-order differential equation. Brownian motion of a particle in periodic potentials has been investigated in the low-friction limit [24]. When the friction coefficient is far from these limits, the quasiequilibrium Boltzmann distribution can be applied [25,26]. In general, an escape of a Brownian particle from a potential well is a well-known Kramers problem [27–29].

In this work, we consider the behavior of a nonlinear oscillator in the presence of weak noise in the low-friction limit. For the trajectories, we obtain an expression for the mean density of level crossings (i.e., the average number of them per unit time) whose duration exceeds a certain value. Several specific scenarios are discussed.

II. NONLINEAR OSCILLATOR IN THE PRESENCE OF WEAK NOISE

We consider the one-dimensional motion of a particle of mass m in the potential $U(x) = kx^{2n}$ ($k > 0, n \in \mathbb{Z}^+$). Motion is periodic for the conservative system. The equation of motion of a nonlinear oscillator is written as

$$m\ddot{x} + 2nkx^{2n-1} = 0,$$

where $x(t)$ is the displacement from equilibrium. The solutions to this equation can be represented in terms of elliptic and trigonometric functions depending on n. The oscillation period depends on the energy $\mathscr{E}$ of the particle except for the harmonic case ($n = 1$):

$$T = \frac{\sqrt{2m}}{n}\mathscr{E}^{\frac{1}{2n}-\frac{1}{2}}k^{-\frac{1}{2n}}\mathrm{B}\left(\frac{1}{2},\frac{1}{2n}\right). \tag{1}$$

For an anharmonic oscillator in the presence of noise, the equation is written as

$$m\ddot{x} + \beta\dot{x} + 2nkx^{2n-1} = \zeta(t). \tag{2}$$

Using the notation $\epsilon = \beta/m, \alpha = k/m, \xi(t) = \zeta(t)/\sqrt{m\beta}$, we obtain

$$\ddot{x} + \epsilon\dot{x} + 2n\alpha x^{2n-1} = \sqrt{\epsilon}\xi(t), \tag{3}$$

where $\epsilon = 2\delta$ (δ is decrement) is much less than $1/T$ for high Q-factor oscillations. Under the above assumptions, Eq. (3) describes a system that exhibits nonlinear oscillations in the presence of low friction and weak fluctuations.

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For convenience, we introduce the energy of the oscillator per unit mass:

$$E = \frac{\dot{x}^2}{2} + \alpha x^{2n}. \tag{4}$$

Multiplying Eq. (3) by $\dot{x}$, we get

$$\dot{E} = -\epsilon \dot{x}^2 + \sqrt{\epsilon}\dot{x}\xi(t). \tag{5}$$

Thus, we have two fluctuation equations with the notation $u(x) = \alpha x^{2n}$ introduced:

$$\begin{aligned} \dot{x} &= \sqrt{2[E-u(x)]}, \\ \dot{E} &= -2\epsilon[E-u(x)] + \sqrt{2\epsilon[E-u(x)]}\xi(t). \end{aligned} \tag{6}$$

We assume that the correlation time $\tau_{\rm cor}$ of the stationary random function $\xi(t)$ is much less than the characteristic time of $x(t) \sim T$. In this case, we proceed from the fluctuation equations (6) to the Fokker-Planck equation for the joint probability density $w(x,E,t)$ [6]:

$$\begin{aligned} \frac{\partial w}{\partial t} &= -\frac{\partial}{\partial x}[\sqrt{2[E-u(x)]}w] \\ &\quad + 2\epsilon \frac{\partial}{\partial E}\left[\left(E-u(x)-\frac{K}{4}\right)w\right] \\ &\quad + \epsilon K \frac{\partial^2}{\partial E^2}\{[E-u(x)]w\}, \end{aligned} \tag{7}$$

where $K = \int_{-\infty}^{\infty} \langle \xi(t)\xi(t+\tau)\rangle d\tau$ and $\langle \xi(t)\rangle = 0$.

For random function $x(t)$, the level crossing takes place at instant t_0 if $x(t)$ reaches and then immediately exceeds a certain fixed level b (Fig. 1). Stratonovich showed [6] that the mean density of level crossings of duration greater than τ is given by

$$h(b,\tau) = w_0(b)\int_b^{\infty} \chi(x,\tau)dx, \tag{8}$$

where $w_0(x)$ is the stationary distribution of $x(t)$ and $\chi(x,t)$ is a solution to the Fokker-Planck equation (7) with the following initial and boundary conditions:

$$\chi(x,t) = 0 \quad (t < 0), \quad \chi(b,t) = \delta(t). \tag{9}$$

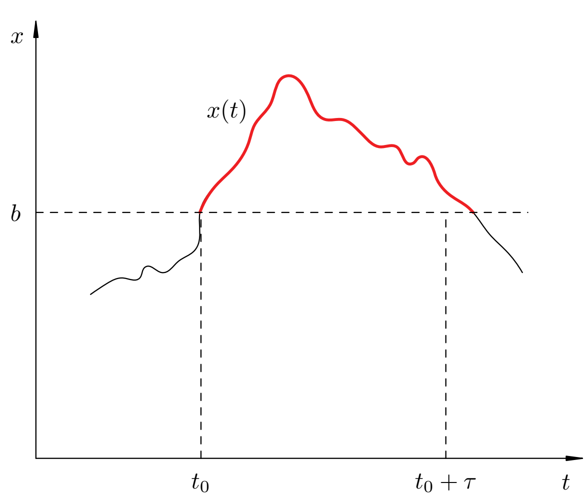

FIG. 1. (Color online) Level crossing of random process.

A. Small and medium durations of level crossing

We assume that $\partial w/\partial t$ and the first term on the right-hand side of Eq. (7) have the same order of magnitude. Then we represent $w(x,E,t)$ in following form:

$$w(x,E,t) = w^{(0)}(x,E,t) + \epsilon w^{(1)}(x,E,t) + \cdots. \tag{10}$$

In zero order of ϵ we get

$$\frac{\partial w^{(0)}}{\partial t} = -\frac{\partial}{\partial x}[\sqrt{2[E-u(x)]}w^{(0)}].$$

Using the method of characteristics, we obtain a general solution of this equation:

$$w^{(0)}(x,E,t) = \frac{\Xi\left(t - \frac{x}{\sqrt{2E}}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2n},1+\frac{1}{2n},\frac{u(x)}{E}\right)\right)}{\sqrt{E-u(x)}}, \tag{11}$$

where $\Xi(z)$ is an arbitrary function. We consider level crossing in the plane $\{x,E\}$ (see Fig. 2). Then the boundary condition must be modified to

$$w^{(0)}(b,E,t) = w_0^+(b,E)\delta(t),$$

where $w_0^+(x,E)$ is the stationary distribution of $\{x,E\}$ with positive velocities. Positivity of velocity near the bound $x = b$ is a necessary condition of level crossing. Taking into account the boundary condition and using the notation

$$\theta(x,E) = \frac{x}{\sqrt{2E}}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2n},1+\frac{1}{2n},\frac{u(x)}{E}\right),$$

we obtain for positive velocities

$$w_+^{(0)}(x,E,t) = w_0^+(b,E)\sqrt{\frac{E-u(b)}{E-u(x)}}\delta[t-\theta(x,E)+\theta(b,E)]. \tag{12}$$

Note that this solution obviously satisfies the necessary initial condition $w_+^{(0)}(x,E,0) = 0, \quad x > b$. It goes over

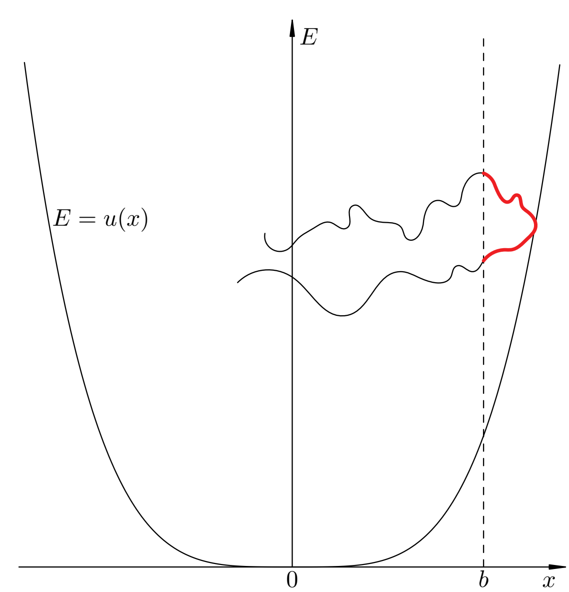

FIG. 2. (Color online) Level crossing of 2D random process in the plane $\{x,E\}$.

continuously into the density for negative velocities

$$w_-^{(0)}(x,E,t) = w_0^+(b,E)\sqrt{\frac{E-u(b)}{E-u(x)}}\delta(t+\theta(x,E)+\theta(b,E)-2\theta[(E/\alpha)^{\frac{1}{2n}},E]). \quad (13)$$

Then we get

$$h(b,\tau,E) = \int_\Omega w_-^{(0)}(x,E,\tau)dx = \sqrt{2[E-u(b)]}\begin{cases} w_0^+(b,E), & \tau \leqslant \frac{\left(\frac{E}{\alpha}\right)^{\frac{1}{2n}}}{\sqrt{E}}\frac{\sqrt{2\pi}\Gamma\left(1+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)}, \quad E \geqslant \Lambda(b,\tau) \\ 0, & \text{otherwise.}\end{cases} \quad (14)$$

These conditions follow from the existence of zero in x of the δ function's argument for fixed b and τ when $b < x \leqslant (\frac{E}{\alpha})^{1/2n}$. Note there is at most one such zero. The integration domain Ω is $u(b) \leqslant u(x) \leqslant E$. $\Lambda(b,\tau)$ is the minimal energy at which the δ function's argument may vanish. It can be found from the following transcendental equation:

$$\tau + \frac{2b}{\sqrt{2\Lambda}}{}_2F_1\left(\frac{1}{2},\frac{1}{2n},1+\frac{1}{2n},\frac{u(b)}{\Lambda}\right) = \left(\frac{\Lambda}{\alpha}\right)^{\frac{1}{2n}}\sqrt{\frac{2\pi}{\Lambda}}\frac{\Gamma\left(1+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)}. \quad (15)$$

The stationary distribution $w_0^+(b,E)$ is easily obtained from Eq. (7):

$$w_0^+(b,E) = \left(\frac{2\alpha}{K}\right)^{\frac{1}{2n}}\frac{1}{2\sqrt{\pi K}\,\Gamma\left(1+\frac{1}{2n}\right)}\frac{e^{-\frac{2E}{K}}}{\sqrt{2[E-u(b)]}}.$$

Integrating Eq. (14) with respect to E, we finally obtain

$$h(b,\tau) = \left(\frac{2\alpha}{K}\right)^{\frac{1}{2n}}\frac{K/4}{\sqrt{\pi K}\,\Gamma\left(1+\frac{1}{2n}\right)}e^{-\frac{2\Lambda(b,\tau)}{K}}. \quad (16)$$

B. Large duration of level crossing

Let us consider level crossings for which the duration is greater than the oscillation period. Following Ref. [6], we find the two-dimensional probability density. Since the friction is low, the energy E is conserved during a significant number of oscillation periods. We assume that a quasiequilibrium distribution characterized by $\partial w/\partial t \sim \epsilon$ is established after some time. Using expression (7), we get

$$\frac{\partial}{\partial x}[\sqrt{2[E-u(x)]}w] = O(\epsilon).$$

Integrating with respect to x, we have

$$w(x,E,t) = \frac{C(E,t)}{\sqrt{E-u(x)}} + O(\epsilon).$$

So in zero order in ϵ the joint probability density $w(x,E,t)$ can be factored to

$$w(x,E,t) = w(x|E)w(E,t).$$

For a fixed value of energy, the normalized conditional probability density is

$$w(x|E) = \begin{cases} \frac{n\alpha^{\frac{1}{2n}}}{\mathrm{B}\left(\frac{1}{2},\frac{1}{2n}\right)}\frac{E^{\frac{1}{2}-\frac{1}{2n}}}{\sqrt{E-u(x)}}, & u(x) < E, \\ 0, & u(x) \geqslant E.\end{cases} \quad (17)$$

This expression can also be obtained using the fact that the time that process $x(t)$ spends in a small neighborhood of x is inversely proportional to the velocity $\dot{x} = \sqrt{2[E-u(x)]}$.

Then the two-dimensional probability density is given by

$$w(x,E,t) = \frac{n\alpha^{\frac{1}{2n}}}{\mathrm{B}\left(\frac{1}{2},\frac{1}{2n}\right)}\frac{E^{\frac{1}{2}-\frac{1}{2n}}w(E,t)}{\sqrt{E-u(x)}}, \quad u(x) < E. \quad (18)$$

Substituting expression (18) into formula (7) and integrating along x, we obtain the Fokker-Planck equation with respect to $w(E,t)$:

$$\frac{\partial w}{\partial t} = \frac{\epsilon K}{2}\frac{\partial^2}{\partial E^2}[\gamma E w] + \epsilon\frac{\partial}{\partial E}\left[\left(\gamma E - \frac{K}{2}\right)w\right]. \quad (19)$$

Here we use the notation

$$\frac{1}{\gamma} = \frac{1}{2} + \frac{1}{2n}.$$

First, we solve Eq. (19) with respect to $w(E,t)$ using the Laplace transform:

$$\frac{\epsilon K}{2}\frac{\partial^2}{\partial E^2}[\gamma E\bar{w}] + \epsilon\frac{\partial}{\partial E}\left[\left(\gamma E - \frac{K}{2}\right)\bar{w}\right] - p\bar{w} = 0, \quad (20)$$

where the bar denotes the transform

$$\bar{w}(E,p) = \int_{0^-}^{\infty} e^{-pt}w(E,t)dt. \quad (21)$$

Expanding the brackets in Eq. (20), we obtain the following equation:

$$\frac{\epsilon\gamma E K}{2}\frac{\partial^2\bar{w}}{\partial E^2} + \epsilon\left[\gamma E + \left(\gamma - \frac{1}{2}\right)K\right]\frac{\partial\bar{w}}{\partial E} + (\epsilon\gamma - p)\bar{w} = 0.$$

The substitution $\eta = -2E/K$ reduces it to the standard form of the confluent hypergeometric equation:

$$\eta\frac{\partial^2\bar{w}}{\partial\eta^2} + \left(2 - \frac{1}{\gamma} - \eta\right)\frac{\partial\bar{w}}{\partial\eta} - \left(1 - \frac{p}{\epsilon\gamma}\right)\bar{w}(\eta,p) = 0.$$

Linearly independent solutions of this equation for arbitrary values of parameters are confluent hypergeometric functions

$\eta^{1/\gamma-1}\Psi(\frac{1}{\gamma}-\frac{p}{\epsilon\gamma},\frac{1}{\gamma},\eta)$ and $\eta^{1/\gamma-1}e^{\eta}\Psi(\frac{p}{\epsilon\gamma},\frac{1}{\gamma},-\eta)$ [30]. Hence the general solution of Eq. (20) can be represented as

$$\bar{w}(E,p) = \left(\frac{2E}{K}\right)^{1/\gamma-1}\left[c_1(p)\Psi\left(\frac{1}{\gamma}-\frac{p}{\epsilon\gamma},\frac{1}{\gamma},-\frac{2E}{K}\right) + c_2(p)e^{-\frac{2E}{K}}\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma},\frac{2E}{K}\right)\right], \tag{22}$$

where the functions $c_{1,2}(p)$ are found below.

With the aid of Eq. (18), we derived the expression for the transform of function $\chi(x,t)$:

$$\begin{aligned}\bar{\chi}(x,p) &= \int_0^\infty \bar{w}(x,E,p)dE \\ &= \int_0^\infty \bar{w}(E,p)w(x|E)dE \\ &= \frac{\alpha^{1/\gamma-1/2}}{\left(\frac{2}{\gamma}-1\right)\mathrm{B}\left(\frac{1}{2},\frac{1}{\gamma}-\frac{1}{2}\right)}\int_{u(x)}^\infty \frac{\bar{w}(E,p)dE}{E^{1/\gamma-1}\sqrt{E-u(x)}}.\end{aligned} \tag{23}$$

The calculation of the necessary integrals is given in Appendix A. Substituting expressions (A3) and (A6) into integral (23), we obtain

$$\bar{\chi}(x,p) = \tilde{c}_1(p)\Psi\left(\frac{1}{\gamma}-\frac{p}{\epsilon\gamma}-\frac{1}{2},\frac{1}{\gamma}-\frac{1}{2},-\frac{2u(x)}{K}\right) + \tilde{c}_2(p)e^{-\frac{2u(x)}{K}}\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(x)}{K}\right).$$

We must use $\tilde{c}_1(p)=0$ because the first term exists only for $\mathrm{Re}\, p<\epsilon-\epsilon\gamma/2$ (see Appendix A), whereas $\lim_{x\to\infty}\exp(-\frac{2u(x)}{K})\Psi(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(x)}{K})=0$ for $\mathrm{Re}\, p>0$.

We choose $\tilde{c}_2(p)$ so the function $\bar{\chi}(x,p)$ satisfies the boundary condition $\bar{\chi}(b,p)=1$ from Eq. (9):

$$\bar{\chi}(x,p) = e^{-2\frac{u(x)-u(b)}{K}}\frac{\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(x)}{K}\right)}{\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)}. \tag{24}$$

The transform of the density of level crossings is given by

$$\bar{h}(b,p) = w_0(b)\int_b^\infty \bar{\chi}(x,p)dx. \tag{25}$$

Substituting integral (B1) into Eq. (25), we have

$$\bar{h}(b,p) = w_0(b)\left(\frac{K}{2\alpha}\right)^{\frac{1}{\gamma}-\frac{1}{2}}\left(\frac{1}{\gamma}-\frac{1}{2}\right) \times\frac{\Psi\left(\frac{p}{\epsilon\gamma}+\frac{3}{2}-\frac{1}{\gamma},\frac{3}{2}-\frac{1}{\gamma},\frac{2u(b)}{K}\right)}{\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)}. \tag{26}$$

The stationary distribution $w_0(b)$ is easily obtained from Eq. (7):

$$w_0(b) = \left[2\Gamma\left(\frac{1}{\gamma}-\frac{1}{2}\right)\left(\frac{K}{2\alpha}\right)^{\frac{1}{\gamma}-\frac{1}{2}}\left(\frac{1}{\gamma}-\frac{1}{2}\right)\right]^{-1}e^{-\frac{2u(b)}{K}}. \tag{27}$$

Substituting expression (27) into Eq. (26), we finally get

$$\bar{h}(b,p) = \frac{e^{-\frac{2u(b)}{K}}}{2\Gamma\left(\frac{1}{\gamma}-\frac{1}{2}\right)}\frac{\Psi\left(\frac{p}{\epsilon\gamma}+\frac{3}{2}-\frac{1}{\gamma},\frac{3}{2}-\frac{1}{\gamma},\frac{2u(b)}{K}\right)}{\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)}. \tag{28}$$

Consider the function

$$f(p) = p\,\frac{\Psi\left(\frac{p}{\epsilon\gamma}+\frac{3}{2}-\frac{1}{\gamma},\frac{3}{2}-\frac{1}{\gamma},\frac{2u(b)}{K}\right)}{\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)}.$$

When $z\neq 0$, the principal branch of $\Psi(a,c,z)$ is the entire function of a, so $f(p)$ is a meromorphic function. Because $p=0$ is a regular point of $f(p)$, the function can be expanded in rational fractions using the Mittag-Leffler theorem [31]:

$$f(p) = f(0) + \sum_l \operatorname*{Res}_{p=p_l} f(p)\left(\frac{1}{p-p_l}+\frac{1}{p_l}\right).$$

Here p_l are simple poles of $f(p)$ numbered in ascending order of absolute value, i.e., the roots of the equation

$$\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right) = 0. \tag{29}$$

The residue of $f(p)$ at p_l is

$$\operatorname*{Res}_{p=p_l} f(p) = p_l\frac{\Psi\left(\frac{p_l}{\epsilon\gamma}+\frac{3}{2}-\frac{1}{\gamma},\frac{3}{2}-\frac{1}{\gamma},\frac{2u(b)}{K}\right)}{\frac{\partial}{\partial p}\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)\big|_{p=p_l}}.$$

Substituting these expressions into Eq. (26) and with allowance for $f(0)=0$, we obtain

$$\bar{h}(b,p) = \frac{e^{-\frac{2u(b)}{K}}}{2\Gamma\left(\frac{1}{\gamma}-\frac{1}{2}\right)} \times\sum_l\frac{\Psi\left(\frac{p_l}{\epsilon\gamma}+\frac{3}{2}-\frac{1}{\gamma},\frac{3}{2}-\frac{1}{\gamma},\frac{2u(b)}{K}\right)}{\frac{\partial}{\partial p}\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)\big|_{p=p_l}}\frac{1}{p-p_l}. \tag{30}$$

Finally, applying the inverse Laplace transform, we obtain the desired distribution

$$h(b,\tau) = \frac{e^{-\frac{2u(b)}{K}}}{2\Gamma\left(\frac{1}{\gamma}-\frac{1}{2}\right)}\sum_l\frac{\Psi\left(\frac{p_l}{\epsilon\gamma}+\frac{3}{2}-\frac{1}{\gamma},\frac{3}{2}-\frac{1}{\gamma},\frac{2u(b)}{K}\right)}{\frac{\partial}{\partial p}\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)\big|_{p=p_l}}e^{p_l\tau}. \tag{31}$$

The expression for the derivative of the Tricomi function is given in Appendix C.

III. RESULTS AND DISCUSSION

A. Small and medium durations of level crossing

Consider the obtained expressions in detail. First, restriction on duration (14) can be explained by the fact that level crossings, whose duration exceeds a half of the oscillation period T [see Eq. (1)], will appear relatively rarely. Then the mean density of level crossings (16) is proportional to $1/T$, at least in the harmonic case, because on the average there is at most one such crossing during the period of oscillation.

Figure 3 demonstrates a typical dependence of the mean density of crossings on the level and duration (16).

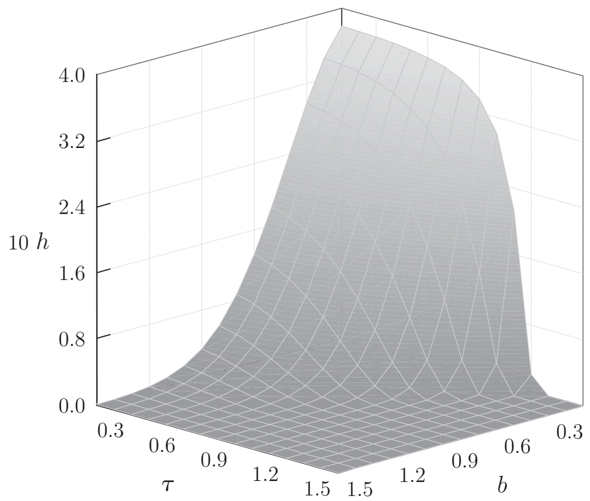

FIG. 3. Plot of the mean density of crossings $h(b,\tau)$ (16) vs the level and duration ($n = 1$, $\alpha = 3$, $\epsilon = 0.1$, $K = 2$).

Here we provide the results of numerical simulation of the stochastic differential equation (3). The continuous-time process is simulated by Euler's discrete-time approximation. The noise impact is a δ-correlated stationary Gaussian process. We use the Mersenne twister [32] as a pseudorandom number generator. The time period of the simulation and the time step are 100 000 and 2×10^{-4}, respectively.

A comparison of analytical and numerical results for the harmonic oscillator ($n = 1$) at various parameters is presented in Figs. 4 and 5. The solid lines correspond to theoretical consideration, whereas the markers denote the results of numerical simulation.

The results for the anharmonic quartic oscillator ($n = 2$) are presented in Figs. 6 and 7.

Note that the theory is in good qualitative and quantitative agreement with numerical experiment. The difference between the analytical consideration and direct numerical simulation appears to be due to discarding the terms of higher order of ϵ in Eq. (10).

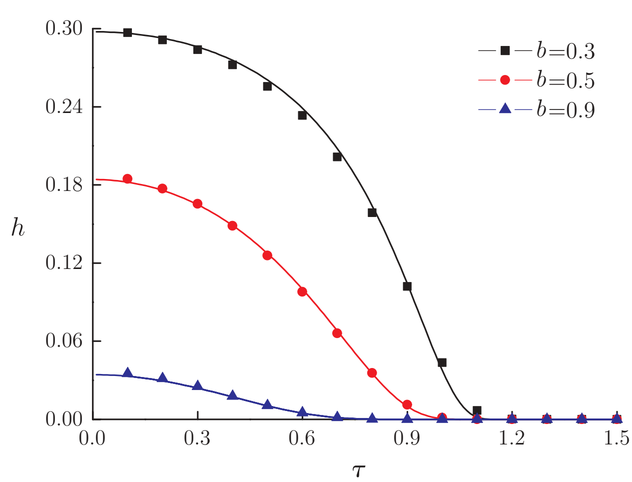

FIG. 4. (Color online) Plot of the mean density of crossings $h(b,\tau)$ vs the duration at different values of level ($n = 1$, $\alpha = 3$, $\epsilon = 0.1$, $K = 2$) from analytical (16) and numerical simulations.

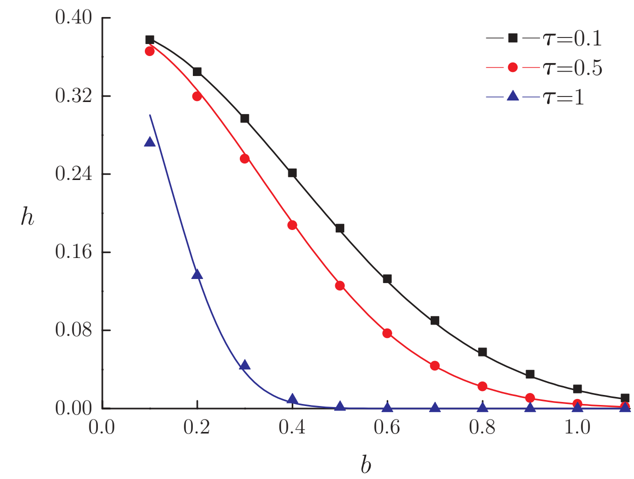

FIG. 5. (Color online) Plot of the mean density of crossings $h(b,\tau)$ vs the level at different values of duration ($n = 1$, $\alpha = 3$, $\epsilon = 0.1$, $K = 2$) from analytical (16) and numerical simulations.

B. Large duration of level crossing

1. Roots of Eq. (29)

It is obvious that the real roots of Eq. (29) are $p_l < 0$ [30]. Wimp [33] demonstrated that the maximum root was

$$p_1 \leqslant -\frac{\epsilon\gamma}{2}\left(\frac{3}{2} - \frac{1}{\gamma}\right).$$

Using the asymptotic expression for $\Psi(a,c,z)$ at $a \to -\infty$ [30], we easily find the large negative roots of Eq. (29):

$$p_l = -\epsilon\gamma\left[l + \frac{2}{\pi}\sqrt{\frac{2u(b)}{K}l} + \frac{4u(b)}{K\pi^2} - \frac{1}{2\gamma} + O\left(\frac{1}{\sqrt{l}}\right)\right].$$

Figure 8 contains a graphical representation of the mean density of crossings on the level and duration (31). Note that this dependence for small and medium durations is for illustrative purposes only.

In the present case of long-lasting level crossings, such that $\tau \gg 1/\epsilon\gamma$, we can restrict sum (31) to the first few terms.

Let us consider the particular cases of distribution (31).

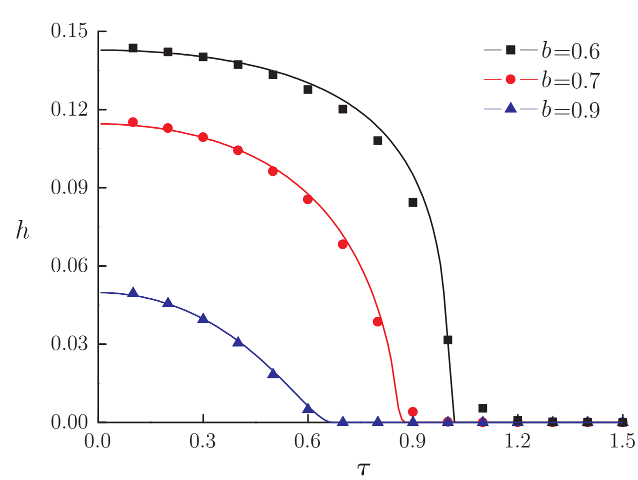

FIG. 6. (Color online) Plot of the mean density of crossings $h(b,\tau)$ vs the duration at different values of level ($n = 2$, $\alpha = 1$, $\epsilon = 0.1$, $K = 1$) from analytical (16) and numerical simulations.

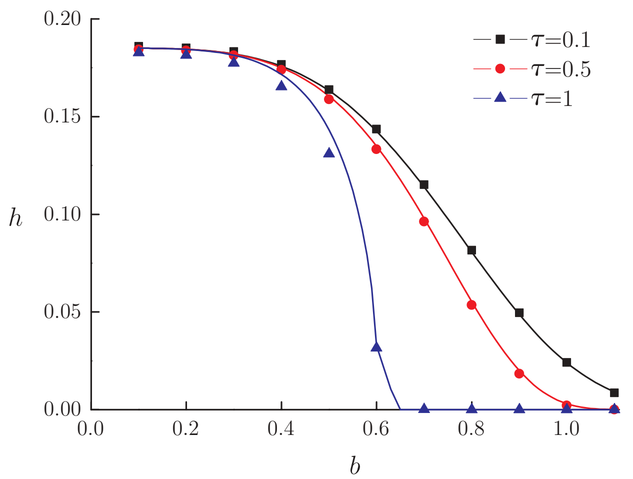

FIG. 7. (Color online) Plot of the mean density of crossings $h(b,\tau)$ vs the level at different values of duration ($n = 2$, $\alpha = 1$, $\epsilon = 0.1$, $K = 1$) from analytical (16) and numerical simulations.

2. Relatively low level

Let $b = 0$. In this case, the solution can be written in terms of elementary functions. We derive the expression for the mean density transform, using the limiting form of the Tricomi function for small values of the argument [30]:

$$\bar{h}(b,p) = \frac{1}{2\Gamma\left(\frac{3}{2}-\frac{1}{\gamma}\right)} \frac{\Gamma\left(\frac{p}{\epsilon\gamma}+\frac{3}{2}-\frac{1}{\gamma}\right)}{\Gamma\left(\frac{p}{\epsilon\gamma}+1\right)}.$$

Using the inverse Laplace transform and the results from Ref. [34], we obtain

$$\begin{aligned} h(b,\tau) &= \frac{\epsilon\gamma}{2\Gamma\left(\frac{3}{2}-\frac{1}{\gamma}\right)\Gamma\left(\frac{1}{\gamma}-\frac{1}{2}\right)}(e^{\epsilon\gamma\tau}-1)^{\frac{1}{\gamma}-\frac{3}{2}} \\ &= \frac{-\epsilon\gamma\ \cos(\pi/\gamma)}{2\pi}(e^{\epsilon\gamma\tau}-1)^{\frac{1}{\gamma}-\frac{3}{2}}. \end{aligned} \quad (32)$$

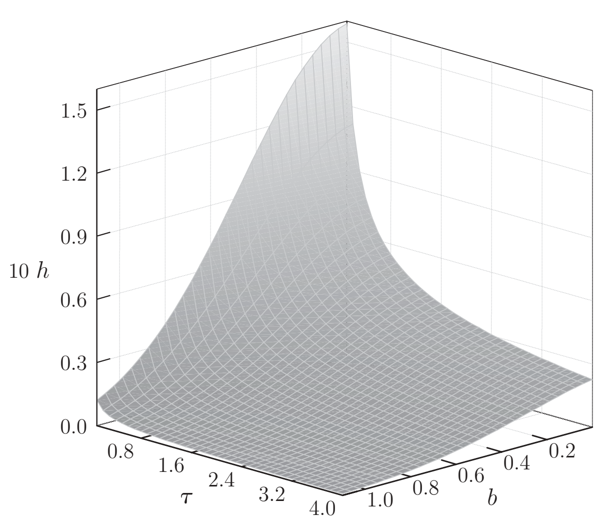

FIG. 8. Plot of the mean density of crossings $h(b,\tau)$ (31) vs the level and duration ($n = 1$, $\alpha = 1$, $\epsilon = 0.1$, $K = 1$).

3. Relatively high level

Let $u(b)/K \gg 1$. Using asymptotic expression of the Tricomi function for large values of the argument and the first parameter $\mathrm{Re}\, p \to \infty$ [30], we derive the following expression:

$$\bar{h}(b,p) = \frac{\sqrt{\epsilon\gamma}\ e^{-\frac{2u(b)}{K}}}{2\Gamma\left(\frac{1}{\gamma}-\frac{1}{2}\right)}\left(\frac{2u(b)}{K}\right)^{1/\gamma-1}\frac{1}{\sqrt{p+\frac{\epsilon\gamma}{2}\frac{u(b)}{K}}}.$$

Applying the inverse Laplace transform [34], we obtain

$$h(b,\tau) = \frac{\sqrt{\epsilon\gamma}\ e^{-\frac{2u(b)}{K}}}{2\Gamma\left(\frac{1}{\gamma}-\frac{1}{2}\right)}\left(\frac{2u(b)}{K}\right)^{1/\gamma-1}\frac{e^{-\frac{u(b)}{2K}\epsilon\gamma\tau}}{\sqrt{\pi\tau}}. \quad (33)$$

4. Harmonic oscillator

Let $n = 1$. In this case, the oscillator moves in a quadratic potential. The expression for the mean density of level crossings (31) can be simplified for the harmonic oscillator using the relationship between confluent hypergeometric functions and parabolic cylinder functions $D_\nu(z)$ [30]. Thus, we obtain

$$h(b,\tau) = \frac{e^{-\frac{2u(b)}{K}}}{\sqrt{2\pi}}\sum_l \frac{D_{-1-2\frac{p_l}{\epsilon\gamma}}\left(2\sqrt{\frac{u(b)}{K}}\right)}{\frac{\partial}{\partial p}D_{-2\frac{p}{\epsilon\gamma}}\left(2\sqrt{\frac{u(b)}{K}}\right)\big|_{p=p_l}}e^{p_l\tau}. \quad (34)$$

IV. SUMMARY

The theory of level crossings of oscillator trajectories can be applied to the description of chemical reactions, in particular, to the processes in the active sites of enzymes, where decreasing the energy barrier of the reaction is determined by the relative position of amino acid residues [35,36]. Due to vibrations, charged groups of enzyme approach the bond so that the reaction barrier is additionally decreased and the reaction rate increases. Note that the above results can also be used in the analysis of stochastic ratchet systems.

Thus, we considered the trajectories of the nonlinear oscillator in the presence of weak noise and derived the mean density of level crossings whose duration exceeds a certain value in the low-friction limit.

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APPENDIX A: CALCULATION OF THE INTEGRALS (23)

We consider the integral

$$\int_{u(x)}^{\infty}\frac{\Psi\left(a,c,-\frac{2E}{K}\right)dE}{\sqrt{E-u(x)}} = u(x)^{1/2}\int_1^{\infty}\frac{\Psi\left(a,c,-\frac{2u(x)}{K}z\right)dz}{\sqrt{z-1}}. \quad (A1)$$

When $z \to \infty$, the integrand is proportional to $z^{-a-1/2}$, so that integral (A1) diverges for $\mathrm{Re}\, a \leqslant 1/2$. We assume that $\mathrm{Re}\, a > 1/2$ and take into account that $1/2 < c \leqslant 1$. Thus, $a \notin \{0 \cup \mathbb{Z}^-\}$ and $c - a \notin \mathbb{Z}^+$, and the Mellin-Barnes integral representation is valid for the Tricomi function [30],

$$\Psi\,(a,c,z) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(a+s)\Gamma(-s)\Gamma(1-c-s)}{\Gamma(a)\Gamma(a-c+1)}z^s ds, \tag{A2}$$

where $|\arg z| < 3\pi/2$, σ is arbitrary, and the contour has loops if necessary so that the poles of $\Gamma(a+s)$ and $\Gamma(-s)\Gamma(1-c-s)$ are on opposite sides of the contour.

$$\begin{aligned}\int_{u(x)}^{\infty}\frac{\Psi\left(a,c,-\frac{2E}{K}\right)dE}{\sqrt{E-u(x)}} &= u(x)^{1/2}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(a+s)\Gamma(-s)\Gamma(1-c-s)}{\Gamma(a)\Gamma(a-c+1)}\left(-\frac{2u(x)}{K}\right)^s\left[\int_1^{\infty}\frac{z^s dz}{\sqrt{z-1}}\right]ds\\ &= u(x)^{1/2}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(a+s)\Gamma(-s)\Gamma(1-c-s)}{\Gamma(a)\Gamma(a-c+1)}\left(-\frac{2u(x)}{K}\right)^s\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-s-\frac{1}{2}\right)}{\Gamma(-s)}ds\\ &= -i\sqrt{\frac{\pi K}{2}}\frac{\Gamma\left(a-\frac{1}{2}\right)}{\Gamma(a)}\Psi\left(a-\frac{1}{2},c-\frac{1}{2},-\frac{2u(x)}{K}\right).\end{aligned} \tag{A3}$$

Here, the inner integral converges for $\operatorname{Re} s < -1/2$, and we choose a vertical line $-\operatorname{Re} a < \sigma < -1/2$ for the contour of the outer integral.

Now, we consider

$$\int_{u(x)}^{\infty}e^{-\frac{2E}{K}}\frac{\Psi\left(a,c,\frac{2E}{K}\right)dE}{\sqrt{E-u(x)}} = u(x)^{1/2}\int_1^{\infty}e^{-\frac{2u(x)}{K}z}\frac{\Psi\left(a,c,\frac{2u(x)}{K}z\right)dz}{\sqrt{z-1}}. \tag{A4}$$

If $|\arg z| < \pi/2$ we have the following integral representation for the Tricomi function [30]:

$$\Psi(a,c,z) = \frac{1}{2\pi i}e^{-i\pi a}\Gamma(1-a)\int_{\infty}^{(0+)}e^{-zt}t^{a-1}(1+t)^{c-a-1}dt, \tag{A5}$$

where the path of integration starts at infinity on the real axis, encircles the origin in the positive direction, and returns to the starting point. Changing the order of integration and using [37], we get

$$\begin{aligned}\int_{u(x)}^{\infty}e^{-\frac{2E}{K}}\frac{\Psi\left(a,c,\frac{2E}{K}\right)dE}{\sqrt{E-u(x)}} &= u(x)^{1/2}\frac{1}{2\pi i}e^{-i\pi a}\Gamma(1-a)\int_{\infty}^{(0+)}\left[\int_1^{\infty}\frac{e^{-\frac{2u(x)}{K}z(1+t)}}{\sqrt{z-1}}dz\right]t^{a-1}(1+t)^{c-a-1}dt\\ &= u(x)^{1/2}\frac{1}{2\pi i}e^{-i\pi a}\Gamma(1-a)\int_{\infty}^{(0+)}\sqrt{\frac{\pi}{\frac{2u(x)}{K}(1+t)}}e^{-\frac{2u(x)}{K}(1+t)}\,t^{a-1}(1+t)^{c-a-1}dt\\ &= \sqrt{\frac{\pi K}{2}}e^{-\frac{2u(x)}{K}}\Psi\left(a,c-\frac{1}{2},\frac{2u(x)}{K}\right).\end{aligned} \tag{A6}$$

APPENDIX B: CALCULATION OF THE INTEGRAL (25)

Consider the integral (25). Using the expression for the derivative of the confluent hypergeometric function [30], we obtain

$$\begin{aligned}\int_b^{\infty}e^{-\frac{2u(x)}{K}}\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(x)}{K}\right)dx &= \left(\frac{K}{2\alpha}\right)^{\frac{1}{\gamma}-\frac{1}{2}}\left(\frac{1}{\gamma}-\frac{1}{2}\right)\int_{\frac{2u(b)}{K}}^{\infty}e^{-z}z^{\frac{1}{\gamma}-\frac{1}{2}-1}\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},z\right)dz\\ &= -\left(\frac{K}{2\alpha}\right)^{\frac{1}{\gamma}-\frac{1}{2}}\left(\frac{1}{\gamma}-\frac{1}{2}\right)\int_{\frac{2u(b)}{K}}^{\infty}\frac{d}{dz}\left[e^{-z}\Psi\left(\frac{p}{\epsilon\gamma}+\frac{3}{2}-\frac{1}{\gamma},\frac{3}{2}-\frac{1}{\gamma},z\right)\right]dz\\ &= \left(\frac{K}{2\alpha}\right)^{\frac{1}{\gamma}-\frac{1}{2}}\left(\frac{1}{\gamma}-\frac{1}{2}\right)e^{-\frac{2u(b)}{K}}\Psi\left(\frac{p}{\epsilon\gamma}+\frac{3}{2}-\frac{1}{\gamma},\frac{3}{2}-\frac{1}{\gamma},\frac{2u(b)}{K}\right).\end{aligned} \tag{B1}$$

APPENDIX C: DERIVATIVE OF THE TRICOMI FUNCTION WITH RESPECT TO THE PARAMETER

We derive an expression for the derivative $\partial\Psi/\partial p$. If $c \notin \mathbb{Z}$ (in our case), then the principal branch of $\Psi(a,c,z)$ can be represented in terms of the Kummer functions [30]:

$$\Psi(a,c,z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a,c,z) + \frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\Phi(a-c+1,2-c,z). \tag{C1}$$

Differentiating expression (C1) with respect to the first parameter and taking into account that the derivative of the confluent hypergeometric function with respect to the parameter can be expressed in terms of two-argument Kampé de Fériet–like hypergeometric functions $\Theta^{(1)}$ [38], we find

$$\begin{aligned}
&\frac{\partial}{\partial p}\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)\bigg|_{p=p_l} \\
&= \frac{1}{\epsilon\gamma}\left\{\frac{\Gamma\left(\frac{3}{2}-\frac{1}{\gamma}\right)}{\Gamma\left(\frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{3}{2}\right)}\left[\frac{\frac{2u(b)}{K}}{\frac{1}{\gamma}-\frac{1}{2}}\Theta^{(1)}\left(\begin{matrix}1,1\big|\frac{p_l}{\epsilon\gamma},1+\frac{p_l}{\epsilon\gamma}\\ 1+\frac{p_l}{\epsilon\gamma}\big|2,\frac{1}{2}+\frac{1}{\gamma}\end{matrix}\,\bigg|\frac{2u(b)}{K},\frac{2u(b)}{K}\right)\right.\right. \\
&\quad\left. -\psi\left(\frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{3}{2}\right)\Phi\left(\frac{p_l}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)\right] \\
&\quad+\frac{\Gamma\left(\frac{1}{\gamma}-\frac{3}{2}\right)}{\Gamma\left(\frac{p_l}{\epsilon\gamma}\right)}\left(\frac{2u(b)}{K}\right)^{\frac{3}{2}-\frac{1}{\gamma}}\left[\frac{\frac{2u(b)}{K}}{\frac{5}{2}-\frac{1}{\gamma}}\Theta^{(1)}\left(\begin{matrix}1,1\big|\frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{3}{2},\frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{5}{2}\\ \frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{5}{2}\big|2,\frac{7}{2}-\frac{1}{\gamma}\end{matrix}\,\bigg|\frac{2u(b)}{K},\frac{2u(b)}{K}\right)\right. \\
&\quad\left.\left. -\psi\left(\frac{p_l}{\epsilon\gamma}\right)\Phi\left(\frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{3}{2},\frac{5}{2}-\frac{1}{\gamma},\frac{2u(b)}{K}\right)\right]\right\}. \qquad \text{(C2)}
\end{aligned}$$

Here $\psi(z)$ is the γ function [30]. With allowance for expression (29), we finally obtain

$$\begin{aligned}
&\frac{\partial}{\partial p}\Psi\left(\frac{p}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)\bigg|_{p=p_l} \\
&= \frac{1}{\epsilon\gamma}\left\{\frac{\Gamma\left(\frac{3}{2}-\frac{1}{\gamma}\right)}{\Gamma\left(\frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{3}{2}\right)}\left[\frac{\frac{2u(b)}{K}}{\frac{1}{\gamma}-\frac{1}{2}}\Theta^{(1)}\left(\begin{matrix}1,1\big|\frac{p_l}{\epsilon\gamma},1+\frac{p_l}{\epsilon\gamma}\\ 1+\frac{p_l}{\epsilon\gamma}\big|2,\frac{1}{2}+\frac{1}{\gamma}\end{matrix}\,\bigg|\frac{2u(b)}{K},\frac{2u(b)}{K}\right)\right.\right. \\
&\quad\left. +\left\{\psi\left(\frac{p_l}{\epsilon\gamma}\right)-\psi\left(\frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{3}{2}\right)\right\}\Phi\left(\frac{p_l}{\epsilon\gamma},\frac{1}{\gamma}-\frac{1}{2},\frac{2u(b)}{K}\right)\right] \\
&\quad\left.+\frac{\Gamma\left(\frac{1}{\gamma}-\frac{3}{2}\right)}{\left(\frac{5}{2}-\frac{1}{\gamma}\right)\Gamma\left(\frac{p_l}{\epsilon\gamma}\right)}\left(\frac{2u(b)}{K}\right)^{\frac{5}{2}-\frac{1}{\gamma}}\Theta^{(1)}\left(\begin{matrix}1,1\big|\frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{3}{2},\frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{5}{2}\\ \frac{p_l}{\epsilon\gamma}-\frac{1}{\gamma}+\frac{5}{2}\big|2,\frac{7}{2}-\frac{1}{\gamma}\end{matrix}\,\bigg|\frac{2u(b)}{K},\frac{2u(b)}{K}\right)\right\}. \qquad \text{(C3)}
\end{aligned}$$

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