Fluctuation theorems on the Nishimori line

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The distribution of the performed work for spin glasses with gauge symmetry is considered. With the aid of gauge symmetry, which leads to exact (rigorous) results in spin glasses, we find a fascinating relation of the performed work to the fluctuation theorem. The integral form of the resultant relation reproduces the Jarzynski-type equation for spin glasses that we have obtained. We show that similar relations can be established not only for the distribution of the performed work but also for that of the free energy of spin glasses with gauge symmetry, which provides another interpretation of the phase transition in spin glasses.

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I. INTRODUCTION

The fluctuation theorem makes current activities to understand the nonequilibrium behavior [1–8]. The theorem consists of a relation between distribution functions in different conditions. The first discovery was in long-time observation for an entropy production rate [2,3]. The rigorous derivation of the fluctuation theorem was given by Gallavotti and Cohen for the thermostated deterministic steady-state ensembles [4,5] and for the stochastic dynamics [6–8]. The fluctuation theorem for the nonequilibrium behavior as well as for the symmetry broken states in equilibrium have been discovered [9]. In the present study, we focus on the fluctuation theorem for the performed work distributions [1]. This type of fluctuation theorem yields a fascinating relation to the expectation of the exponentiated work known as the Jarzynski equality [10,11]. The Jarzynski equality creates a relationship between the equilibrium free energy differences associated with the initial and final conditions and the performed work during a nonequilibrium process.

Recently the author and co-workers investigated the Jarzynski equality for spin glasses with competing interactions between adjacent spins [12,13]. Spin glasses often exhibit extremely long-time relaxation toward equilibrium. The long equilibration time hampers observations of the equilibrium state of spin glasses. However, the previous study pointed out the possibility of investigating the equilibrium property from the observations in a different path through nonequilibrium behavior. Then we use the properties of the Jarzynski equality by evaluating the average of the exponentiated performed work during the nonequilibrium process. In other words, nonequilibrium behavior would not be a nuisance, but would be a benefit to investigating the equilibrium behavior in spin glasses. In the present study, we address the distribution of the performed work in such beneficial nonequilibrium behavior with long-time equilibration in spin glasses.

We revisit the analysis of the nonequilibrium behavior in spin glasses in terms of the distribution function of the performed work by evaluation of the rate function. As a result, we obtain a relation in a similar form to the conventional fluctuation theorem for the performed work for spin glasses. The similar analyses reveal that the gauge symmetry, which leads to simple expressions of several quantities for spin glasses [14,15], is closely related to the existence of the fluctuation-theorem-type relation.

The paper is organized as follows. The second section gives a brief introduction of several notations and tools in spin glasses. In the third section, we review the previous study on the performed work by use of the Jarzynski equality. In the present study we give the fluctuation theorem in spin glasses by employing several techniques developed in spin glasses. We show the detailed analysis in Sec. IV. The analyses with the aid of the specialized tool for spin glasses can yield other types of fluctuation theorems rather than that for the performed work. We show the fluctuation theorems for the free-energy differences and free energy itself in the following sections. In the last section, we conclude our present work.

II. SPIN GLASS AND GAUGE SYMMETRY

We deal with the random-bond Ising model, whose Hamiltonian is defined as

$$H(\mathbf{S}|\{\tau_{ij}\}) = -J \sum_{\langle ij \rangle} \tau_{ij} S_i S_j, \tag{1}$$

where S_i is the Ising spin taking values ± 1, J denotes the strength of the coupling, and τ_{ij} is the sign of the coupling. We use the notation $\mathbf{S} = (S_1, S_2, \ldots, S_N)$ for the spin configuration of total N spins for convenience. We set $J = 1$ without loss of generality. The summation is taken over all bonds, whereas one may suppose usual nearest neighboring bonds on a d-dimensional hypercubic lattice. We make no restrictions on the type or dimension of the lattice in the present study. The distribution function of quenched randomness is specified as

$$P(\tau_{ij}) = p\delta(\tau_{ij} - 1) + (1 - p)\delta(\tau_{ij} + 1) = \frac{e^{\beta_p \tau_{ij}}}{2 \cosh \beta_p}, \tag{2}$$

where β_p is defined as $e^{-2\beta_p} = (1 - p)/p$. The following analyses can readily be applied to other types of interaction and their distribution functions as long as they satisfy a certain type of gauge symmetry [14,15].

We use the gauge transformation, which enables us to perform the exact (rigorous) analyses in spin glasses, especially on a special subspace $\beta = \beta_p$ known as the Nishimori line [14,15]. The gauge transformation is defined as

$$\tau_{ij} \to \tau_{ij}\sigma_i\sigma_j, \quad S_i \to \sigma_i S_i, \tag{3}$$

where σ_i is the gauge variable taking ± 1. The Hamiltonian (1) is gauge invariant, while the distribution function (2) changes as $P(\tau_{ij}) \propto \exp(\beta_p \tau_{ij}\sigma_i\sigma_j)$. If we take the summation over all combinations of $\{\sigma_i\}$, the product of the distribution function for $\{\tau_{ij}\}$ over all bonds can be reduced to the partition function of the random-bond Ising model $Z(\beta_p;\{\tau_{ij}\}) = \sum_\sigma \exp(\beta_p \sum_{\langle ij\rangle} \tau_{ij}\sigma_i\sigma_j)$. We mainly use this property to obtain the results in the present study.

In order to analyze the dynamical property of the Ising spin glass, let us suppose that the system evolves following a stochastic dynamics as governed by the master equation. In the present study, we consider changing the inverse temperature. This case is found in a generic solver of the optimization problem as the simulated annealing [16]. We will formulate our theory for discrete time steps for simplicity, although the continuous case can be treated similarly. We change the coupling β from β_0 at $t=0$ to β_T at $t=T$ in T steps of time evolution $(\beta_0,\beta_1,\ldots,\beta_T)$. Correspondingly, the spin configuration changes as $\mathbf{S}_t$. The path probability for expressing the dynamical behavior of the system is then written by the product of the transition rate from state $\mathbf{S}_t$ to state $\mathbf{S}_{t+1}$ following the master equation $P_t(\mathbf{S}_{t+1}|\mathbf{S}_t,\{\tau_{ij}\})$. The transition rate is also gauge invariant, since it depends on the form of the Hamiltonian [15,17].

III. JARZYNSKI EQUALITY FOR SPIN GLASSES

We analyze the work distribution function for spin glasses with a number of spins in the present study. For convenience, we fix several notations and review the previous study before demonstrating the detailed analysis on our issue.

A. Jarzynski equality

We define the discretized pseudo work given by the difference of the inverse temperature as

$$\delta Y_t(\mathbf{S}_{t+1}|\{\tau_{ij}\}) = \beta_{t+1} H(\mathbf{S}_{t+1}|\{\tau_{ij}\}) - \beta_t H(\mathbf{S}_{t+1}|\{\tau_{ij}\}). \quad (4)$$

If we change the parameters in the Hamiltonian instead of the inverse temperature, then the discretized pseudo work is reduced to the ordinary work as

$$\begin{aligned}\delta Y_t(\mathbf{S}_{t+1}|\{\tau_{ij}\}) &= \beta H_{t+1}(\mathbf{S}_{t+1}|\{\tau_{ij}\}) - \beta H_t(\mathbf{S}_{t+1}|\{\tau_{ij}\}) \\ &\equiv \beta \delta W_t(\mathbf{S}_{t+1}|\{\tau_{ij}\}). \end{aligned} \quad (5)$$

For instance, changing the magnetic field as in Ref. [12] is the case. The performed (pseudo) work consists of a collection of the discretized pseudo work

$$Y(\{\mathbf{S}_t\},\{\tau_{ij}\};0\to T) = \sum_{t=0}^{T-1} \delta Y_t(\{\tau_{ij}\},\mathbf{S}_{t+1}). \quad (6)$$

Notice that the performed work depends on the specific configuration of $\{\tau_{ij}\}$.

We briefly review the previous study for the performed work in spin glasses [12,18]. We applied the Jarzynski equality to the special case for spin glasses in the previous study. The Jarzynski equality states that the expectation of the exponentiated work during the nonequilibrium process is given by the difference of the free energy $[-\beta F = \log Z(\beta;\{\tau_{ij}\})]$ between the initial and final conditions as [10,11]

$$\langle e^{-Y}\rangle_{0\to T} = e^{-\Delta_{0\to T}(\beta F)}, \quad (7)$$

where $\Delta_{0\to T}(\beta F) = \beta_T F(\beta_T;\{\tau_{ij}\}) - \beta_0 F_0(\beta_0;\{\tau_{ij}\})$. The brackets denote the nonequilibrium average over all realizations of the spin configurations in a nonequilibrium process starting from the equilibrium state defined as

$$\langle\cdots\rangle_{0\to T} = \sum_{\{\mathbf{S}_t\}} \prod_{t=0}^{T-1} P_t(\mathbf{S}_{t+1}|\mathbf{S}_t,\{\tau_{ij}\}) P_{\rm eq}^{t=0}(\mathbf{S}_0|\{\tau_{ij}\}), \quad (8)$$

where $P_{\rm eq}^t(\mathbf{S}|\{\tau_{ij}\})$ denotes the equilibrium distribution function

$$P_{\rm eq}^t(\mathbf{S}|\{\tau_{ij}\}) = \frac{1}{Z(\beta_t;\{\tau_{ij}\})} e^{-\beta_t H(\mathbf{S}|\{\tau_{ij}\})}. \quad (9)$$

B. For spin glasses

Since the work is given by the realization of the spin configurations at each time, the path probability can be regarded as the distribution function of the performed work with the specific configuration of $\{\tau_{ij}\}$ as $P(y;\{\tau_{ij}\},0\to T)$. However, in spin glasses, we are often interested in the averaged quantity over all realizations of $\{\tau_{ij}\}$. In the previous study, the author and co-workers evaluated the averaged Jarzynski equality as

$$[\langle e^{-Y}\rangle_{0\to T}]_{\beta_0} = \left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right)^{N_B}, \quad (10)$$

where N_B is the number of bonds, and the square brackets denote the configurational average over all realizations of $\{\tau_{ij}\}$ defined as

$$[\cdots]_{\beta_p} = \sum_{\{\tau_{ij}\}} \prod_{\langle ij\rangle} P(\tau_{ij}) \times \cdots = \sum_{\{\tau_{ij}\}} \prod_{\langle ij\rangle} \frac{e^{\beta_p \tau_{ij}}}{2\cosh\beta_p} \times \cdots. \quad (11)$$

The above equality holds for the special initial condition that the nonequilibrium process starts from the Nishimori line $\beta_p = \beta_0$ [14,15]. Then the averaged free energy on the right-hand side can be reduced to a trivial quantity. However, the Jarzynski equality is for the expectation, which is the average over all realizations. We cannot obtain the detailed structure of the distribution functions only from the expectation. In the present study, we thus revisit the problem on the performed work during a nonequilibrium process in spin glasses by evaluating the distribution function in a different way. That is the motivation of our study.

IV. FLUCTUATION THEOREM FOR SPIN GLASSES

A. Large deviation

Throughout the present study, we assume the large deviation property in the distribution function for the system with a large number of components. For instance, the distribution function $P(y;\{\tau_{ij}\},0\to T)$ of the performed work for the specific configuration of $\{\tau_{ij}\}$ takes an asymptotic form as, for a large N,

$$P(y;\{\tau_{ij}\},0\to T) \sim e^{-NI(y;\{\tau_{ij}\},0\to T)}, \quad (12)$$

where $I(y;\{\tau_{ij}\},0\to T)$ is the rate function and always takes a non-negative value. At the most frequent realization of the performed work (thermodynamic work), the rate function vanishes. Here y stands for the scaled work defined as

$$y = Y(\{\mathbf{S}_t\},\{\tau_{ij}\};0 \to T)/N. \tag{13}$$

In the thermodynamic system, the empirical average as the above scaled work can be evaluated by the zero point y^* of the rate function and coincides with the expectation, since

$$\int dy\, P(y;\{\tau_{ij}\},0 \to T)y = y^*. \tag{14}$$

On the other hand, the rate function can characterize the fluctuation around the zero point. For the rate function, an important relation—the fluctuation theorem—holds. As detailed in Appendix A, the fluctuation theorem states the symmetry of the rate functions as

$$\begin{aligned} I(y;\{\tau_{ij}\},0 \to T) &= I(-y;\{\tau_{ij}\},T \to 0) - y \\ &\quad - \frac{1}{N}[\log Z(\beta_T;\{\tau_{ij}\}) - \log Z(\beta_0;\{\tau_{ij}\})]. \end{aligned} \tag{15}$$

This symmetry of the rate functions yields the well-known fluctuation theorem (Crooks fluctuation theorem) for the distribution functions of the performed work as

$$\frac{P(y;\{\tau_{ij}\},0 \to T)}{P(-y;\{\tau_{ij}\},T \to 0)} = \exp\{N[y - \Delta_{0\to T}(\beta f)]\}. \tag{16}$$

Rather than the Jarzynski equality—namely, the expectation—the fluctuation theorem provides more detailed information on the distribution function. Therefore we consider finding the fluctuation theorem to spin glasses in the present study.

B. Generating function

The above fluctuation theorem holds for the specific configuration of $\{\tau_{ij}\}$, similarly to the Jarzynski equality. However, in spin glasses, we find sample-to-sample fluctuation in observations for different realizations of $\{\tau_{ij}\}$. We thus must evaluate the fluctuation around the most probable realization for the infinite-size system. By the analysis of the generating function of the distribution function of $\{\tau_{ij}\}$ associated with the quantity we are interested in, we can evaluate such sample-to-sample fluctuations by the rate function. Let us define the following generating function of the performed work for spin glasses as

$$\Omega_y(r;\beta_p,0 \to T) \equiv \frac{1}{N}\log[\langle\exp(rNy)\rangle_{0\to T}]_{\beta_p}, \tag{17}$$

and of its inverse process,

$$\Omega_y(r;\beta_p,T \to 0) \equiv \frac{1}{N}\log[\langle\exp(-rNy)\rangle_{T\to 0}]_{\beta_p}. \tag{18}$$

It is convenient to define the generating function for the specific configuration of $\{\tau_{ij}\}$ as

$$\Psi_y(r;\{\tau_{ij}\},0 \to T) \equiv \frac{1}{N}\log\langle\exp(rNy)\rangle_{0\to T}, \tag{19}$$

and for its inverse process

$$\Psi_y(r;\{\tau_{ij}\},T \to 0) \equiv \frac{1}{N}\log\langle\exp[rN(-y)]\rangle_{T\to 0}. \tag{20}$$

Notice that the definition of $\Omega_y(r;\beta_p,0 \to T)$ reads, by Eq. (19),

$$\begin{aligned} \exp[N\Omega_y(r;\beta_p,0 \to T)] &= [\langle\exp(rNy)\rangle_{0\to T}]_{\beta_p} \\ &= [e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)}]_{\beta_p}. \end{aligned} \tag{21}$$

For each realization of $\{\tau_{ij}\}$, the fluctuation theorem for the generating function holds as (see Appendix A)

$$\begin{aligned} \Psi_y(r;\{\tau_{ij}\},0 \to T) &= \Psi_y[-(r+1);\{\tau_{ij}\},T \to 0] \\ &\quad + \frac{1}{N}[\log Z(\beta_T;\{\tau_{ij}\}) - \log Z(\beta_0;\{\tau_{ij}\})]. \end{aligned} \tag{22}$$

We thus obtain

$$e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)} = \frac{Z(\beta_T;\{\tau_{ij}\})}{Z(\beta_0;\{\tau_{ij}\})}e^{N\Psi_y[-(r+1);\{\tau_{ij}\},T\to 0]}. \tag{23}$$

Therefore we evaluate the exponentiated generating function as

$$\begin{aligned} e^{N\Omega_y(r;\beta_p,0\to T)} &= [e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)}]_{\beta_p} \\ &= \left[\frac{Z(\beta_T;\{\tau_{ij}\})}{Z(\beta_0;\{\tau_{ij}\})}e^{N\Psi_y[-(r+1);\{\tau_{ij}\},T\to 0]}\right]_{\beta_p}. \end{aligned} \tag{24}$$

The gauge transformation yields, as detailed in Appendix B,

$$\begin{aligned} 2^N(2\cosh\beta_p)^{N_B}e^{N\Omega_y(r;\beta_p,0\to T)} &= \sum_{\{\tau_{ij}\}} Z(\beta_p;\{\tau_{ij}\})e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)} \\ &= \sum_{\{\tau_{ij}\}} \frac{Z(\beta_p;\{\tau_{ij}\})Z(\beta_T;\{\tau_{ij}\})}{Z(\beta_0;\{\tau_{ij}\})}e^{N\Psi_y[-(r+1);\{\tau_{ij}\},T\to 0]}. \end{aligned} \tag{25}$$

We set $\beta_p = \beta_0$ and then obtain

$$\begin{aligned} 2^N(2\cosh\beta_0)^{N_B}e^{N\Omega_y(r;\beta_0,0\to T)} &= \sum_{\{\tau_{ij}\}} Z(\beta_0;\{\tau_{ij}\})e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)} \\ &= \sum_{\{\tau_{ij}\}} Z(\beta_T;\{\tau_{ij}\})e^{N\Psi_y[-(r+1);\{\tau_{ij}\},T\to 0]}. \end{aligned} \tag{26}$$

On the other hand, let us evaluate the exponentiated generating function of the inverse process

$$\begin{aligned} e^{N\Omega_y(r;\beta_p,T\to 0)} &= [e^{N\Psi_y(r;\{\tau_{ij}\},T\to 0)}]_{\beta_p} \\ &= \left[\frac{Z(\beta_0;\{\tau_{ij}\})}{Z(\beta_T;\{\tau_{ij}\})}e^{N\Psi_y[-(r+1);\{\tau_{ij}\},0\to T]}\right]_{\beta_p}. \end{aligned} \tag{27}$$

When $\beta_p = \beta_T$, the gauge transformation yields

$$\begin{aligned} 2^N(2\cosh\beta_T)^{N_B}e^{N\Omega_y(r;\beta_T,T\to 0)} &= \sum_{\{\tau_{ij}\}} Z(\beta_T;\{\tau_{ij}\})e^{N\Psi_y(r;\{\tau_{ij}\},T\to 0)} \\ &= \sum_{\{\tau_{ij}\}} Z(\beta_0;\{\tau_{ij}\})e^{N\Psi_y[-(r+1);\{\tau_{ij}\},0\to T]}. \end{aligned} \tag{28}$$

Since the second line of Eq. (26) is equal to the third line of Eq. (28) except for the arguments of the generating function Ψ_y, we reach

$$e^{N\Omega_y[-(r+1);\beta_0,0\to T]} = \left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right)^{N_B} e^{N\Omega_y(r;\beta_T,T\to 0)}. \quad (29)$$

C. Rate function and fluctuation theorem

The relation (29) yields the symmetry of the rate function of the performed work for spin glasses. We assume that the existence of the rate function for a large N as

$$P(y;\beta_p,0\to T) \sim e^{-NJ(y;\beta_p,0\to T)}. \quad (30)$$

From the generating function we evaluate the rate function through the Legendre transformation as

$$J(y;\beta_p,0\to T) = \sup_r\{ry - \Omega_y(r;\beta_p,0\to T)\}, \quad (31)$$

and that for its inverse process as

$$J(y;\beta_p,T\to 0) = \sup_r\{ry - \Omega_y(r;\beta_p,T\to 0)\}. \quad (32)$$

Then the relation (29) yields

$$\begin{aligned} &J(y;\beta_0,0\to T)\\ &\quad = \sup_r\{ry - \Omega_y(r;\beta_0,0\to T)\}\\ &\quad = \sup_r\{(r+1)y - \Omega_y[-(r+1);\beta_T,T\to 0]\}\\ &\qquad - y - d\log\left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right)\\ &\quad = J(-y;\beta_T,T\to 0) - y - d\log\left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right), \end{aligned} \quad (33)$$

where $d = N_B/N$. Consequently, we obtain the fluctuation theorem for spin glasses as

$$\frac{P(y;\beta_0,0\to T)}{P(y;\beta_T,T\to 0)} = \left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right)^{N_B} e^{Ny}. \quad (34)$$

The fluctuation theorem for spin glasses immediately reads

$$[\langle\exp(-Ny)\rangle_{0\to T}]_{\beta_0} = \left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right)^{N_B}, \quad (35)$$

which reproduces the Jarzynski equality for spin glasses (29). We here use the fact that we can replace the integration over all realizations of the performed work as the average over all configurations of $\{\tau_{ij}\}$ and $\{\mathbf{S}_t\}$ as

$$[\langle\cdots\rangle_{0\to T}]_{\beta_p} = \int dy\, P(y;\beta_p,0\to T)\times\cdots. \quad (36)$$

By taking the logarithm of the fluctuation theorem (34) and average to make the form of the Kullback-Leibler (KL) divergence $D_{\rm KL}(P_A|P_B) = \int dx\, P_A(x)\log[P_A(x)/P_B(x)]$, we obtain two inequalities:

$$[\langle y\rangle_{0\to T}]_{\beta_0} + d\log\left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right) \geqslant 0 \quad (37)$$

and

$$[\langle y\rangle_{T\to 0}]_{\beta_T} - d\log\left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right) \geqslant 0. \quad (38)$$

Thus we obtain

$$[\langle y\rangle_{0\to T}]_{\beta_0} + [\langle y\rangle_{T\to 0}]_{\beta_T} \geqslant 0. \quad (39)$$

The equality holds, when two of the distribution functions are the same,

$$P(y;\beta_T,T\to 0) = P(y;\beta_0,0\to T), \quad (40)$$

since the left-hand side of Eq. (39) can be evaluated by

$$\begin{aligned} &[\langle y\rangle_{0\to T}]_{\beta_0} + [\langle y\rangle_{T\to 0}]_{\beta_T}\\ &\quad = \int dy\,[P(y;\beta_0;0\to T) - P(-y;\beta_T,T\to 0)]\\ &\qquad \times\log\left(\frac{P(y;\beta_0,0\to T)}{P(-y;\beta_T,T\to 0)}\right). \end{aligned} \quad (41)$$

If we consider the quasistatic process, then $\langle y\rangle_{0\to T} = \Delta(\beta f)_{0\to T}$ following the second law of thermodynamics [i.e., $P(y;\beta_p,0\to T) = P(\Delta(\beta f);\beta_p,0\to T)$]. Therefore, for the difference of the free energy of spin glasses, we expect the existence of a similar relation to the above fluctuation theorem.

V. FLUCTUATION THEOREM FOR FREE ENERGY DIFFERENCE

A. Generating function

Let us consider the sample-to-sample fluctuations for the free-energy difference $\Delta(\beta f)$ by dealing with the rate function. We define the generating function for the free-energy difference as

$$\Psi_{\Delta(\beta f)}(r;\beta_p,0\to T) \equiv \frac{1}{N}\log[\exp[rN\Delta_{0\to T}(\beta f)]]_{\beta_p}. \quad (42)$$

The gauge transformation gives

$$e^{N\Psi_{\Delta(\beta f)}(r;\beta_p,0\to T)} = \sum_{\{\tau_{ij}\}}\frac{Z(\beta_p;\{\tau_{ij}\})}{2^N(2\cosh\beta_p)^{N_B}}\left(\frac{Z(\beta_T;\{\tau_{ij}\})}{Z(\beta_0;\{\tau_{ij}\})}\right)^r. \quad (43)$$

We set $\beta_p = \beta_0$ and obtain

$$e^{N\Psi_{\Delta(\beta f)}(r;\beta_0,0\to T)} = \sum_{\{\tau_{ij}\}}\frac{Z(\beta_T;\{\tau_{ij}\})}{2^N(2\cosh\beta_0)^{N_B}}\left(\frac{Z(\beta_T;\{\tau_{ij}\})}{Z(\beta_0;\{\tau_{ij}\})}\right)^{r-1}. \quad (44)$$

On the other hand, the condition $\beta_p = \beta_T$ leads to

$$e^{N\Psi_{\Delta(\beta f)}(r;\beta_T,0\to T)} = \sum_{\{\tau_{ij}\}}\frac{Z(\beta_T;\{\tau_{ij}\})}{2^N(2\cosh\beta_T)^{N_B}}\left(\frac{Z(\beta_T;\{\tau_{ij}\})}{Z(\beta_0;\{\tau_{ij}\})}\right)^r. \quad (45)$$

Therefore we find a relation of the generating function as

$$\begin{aligned} &\Psi_{\Delta(\beta f)}(r+1;\beta_0,0\to T)\\ &\quad = \Psi_{\Delta(\beta f)}(r;\beta_T,0\to T) + d\log\left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right). \end{aligned} \quad (46)$$

B. Rate function and fluctuation theorem

We assume that the large-deviation property holds for the free energy of the large system $N \to \infty$ as

$$P[\Delta_{0\to T}(\beta f);\beta_p] \sim \exp\{-NK[\Delta_{0\to T}(\beta f);\beta_p]\}. \tag{47}$$

We then obtain the rate function of the free-energy difference through the Legendre transformation as

$$K[\Delta_{0\to T}(\beta f);\beta_p] = \sup_t\{r\Delta(\beta f) - \Psi_{\Delta(\beta f)}(r;\beta_p,0\to T)\}. \tag{48}$$

By use of Eq. (46), we immediately find

$$K[\Delta_{0\to T}(\beta f);\beta_0] = K[\Delta_{0\to T}(\beta f);\beta_T] - \Delta(\beta f) - d\log\left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right). \tag{49}$$

Therefore we find

$$\frac{P\,[\Delta_{0\to T}(\beta f);\beta_0]}{P\,[\Delta_{0\to T}(\beta f);\beta_T]} = \left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right)^{N_B} e^{N\Delta(\beta f)}. \tag{50}$$

Due to the definition of the free-energy difference, $P[\Delta_{0\to T}(\beta f);\beta_p] = P[-\Delta_{T\to 0}(\beta f);\beta_p]$, and we thus obtain the relation in the same form as the fluctuation theorem,

$$\frac{P\,[\Delta_{0\to T}(\beta f);\beta_0]}{P\,[-\Delta_{T\to 0}(\beta f);\beta_T]} = \left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right)^{N_B} e^{N\Delta(\beta f)}. \tag{51}$$

We also establish the Jarzynski-type equality as, by integrating the exponentiated free-energy difference,

$$[\exp\left[-N\Delta_{0\to T}(\beta f)\right]]_{\beta_0} = \left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right)^{N_B}. \tag{52}$$

We here use the fact that we can regard the distribution function of $\{\tau_{ij}\}$ as that of the free-energy difference as

$$[\cdots]_{\beta_p} = \int dx\, P(x:\beta_p)\times\cdots. \tag{53}$$

By making the form of the KL divergence, we obtain the following inequalities:

$$[\Delta_{0\to T}(\beta f)]_{\beta_0} + d\log\left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right) \geqslant 0 \tag{54}$$

and

$$[\Delta_{T\to 0}(\beta f)]_{\beta_T} - d\log\left(\frac{2\cosh\beta_T}{2\cosh\beta_0}\right) \geqslant 0. \tag{55}$$

We thus obtain

$$[\Delta_{T\to 0}(\beta f)]_{\beta_T} + [\Delta_{0\to T}(\beta f)]_{\beta_0} \geqslant 0. \tag{56}$$

The deviation from zero can be written as

$$[\Delta_{T\to 0}(\beta f)]_{\beta_T} + [\Delta_{0\to T}(\beta f)]_{\beta_0} = \int dx\,[P(-x;\beta_T) - P(x;\beta_0)]\log\left(\frac{P(-x;\beta_T)}{P(x;\beta_0)}\right). \tag{57}$$

The equality holds when $P[\Delta_{0\to T}(\beta f);\beta_0] = P[-\Delta_{T\to 0}(\beta f);\beta_T]$. Since two of the distribution functions $P[\Delta_{0\to T}(\beta f);\beta_0]$ and $P[-\Delta_{T\to 0}(\beta f);\beta_T]$ do not coincide with each other in general, the equality in Eq. (56) is not expected to hold; nor is the equality in Eq (39). The magnitude of the violation of the equality can be evaluated by the quantity related to the KL divergence.

As considered above, the gauge symmetry leads to another type of fluctuation theorem not only for the performed work but also the free-energy differences for different configurations of $\{\tau_{ij}\}$. In this sense the obtained relations (34) and (51) are different from the ordinary fluctuation theorem. Our results are related to the sample-to-sample fluctuations of the different realizations of $\{\tau_{ij}\}$. The sample-to-sample fluctuation yields relevant effects even in equilibrium. In theoretical studies in spin glasses, we usually employ the replica method to evaluate the equilibrium property. In the next section we demonstrate how to evaluate the equilibrium property for spin glasses with gauge symmetry from a perspective of the rate function without the replica method. As a result, we find a different way to understand the peculiar behavior in spin glasses.

VI. FLUCTUATION THEOREM FOR FREE ENERGY

A. Free energy statistics

We again assume the large-deviation property for free energy in a large-N system,

$$P(f;\beta_p,\beta) \sim \exp[-NL(f;\beta_p,\beta)]. \tag{58}$$

Here we regard the distribution function of $\{\tau_{ij}\}$ as that of the free energy. We define the generating function of the free energy as

$$\Psi_f(r;\beta_p,\beta) = \frac{1}{N}\log\left[\exp\left[rN(-\beta f)\right]\right]_{\beta_p}. \tag{59}$$

The exponentiated generating function is

$$e^{N\Psi_f(r;\beta_p,\beta)} = [Z^r(\beta;\{\tau_{ij}\})]_{\beta_p}. \tag{60}$$

The analysis by the gauge transformation led us to

$$\Psi_f(r;\beta_p,\beta) = -\log 2 - d\log[2\cosh(\beta_p)] + \frac{1}{N}\log\left(\sum_{\tau_{ij}} Z(\beta_p;\{\tau_{ij}\})Z^r(\beta;\{\tau_{ij}\})\right). \tag{61}$$

On the Nishimori line $\beta_p = \beta$, we find

$$\Psi_f(r;\beta,\beta) = -\log 2 - d\log(2\cosh\beta) + \frac{1}{N}\log\left(\sum_{\tau_{ij}} Z^{r+1}(\beta;\{\tau_{ij}\})\right). \tag{62}$$

We obtain the following similar quantity in the symmetric distribution ($\beta_p = 0$):

$$\Psi_f(r;0,\beta) = -d\log 2 + \frac{1}{N}\log\left(\sum_{\tau_{ij}} Z^r(\beta;\{\tau_{ij}\})\right).$$

B. Fluctuation theorem

We find a relation from the above generating functions:

$$\Psi_f(r;\beta,\beta) + \log 2 + d \log(\cosh \beta) = \Psi_f(r+1;0,\beta), \tag{63}$$

which is essentially the same as the calculation in Ref. [19]. The above relation enables us to analyze the critical behavior of the spin glasses in the symmetry distribution through the free energy on the Nishimori line as shown in Ref. [20]. However, in a modern point of view, this relation can be regarded as the fluctuation theorem for free energy. Indeed we find the symmetry of the rate function of the free energy through the Legendre transformation as

$$L(f;\beta,\beta) = L(f;0,\beta) + \beta f + \log 2 + d \log(\cosh \beta), \tag{64}$$

where we defined the rate function as

$$L(f;\beta_p,\beta) = \sup_r\{r(-\beta f) - \Psi_f(r;\beta_p,\beta)\}. \tag{65}$$

Thus we obtain the fluctuation-theorem-type equality for the free energies on the Nishimori line and in the symmetric distribution as

$$\frac{P(f;\beta,\beta)}{P(f;0,\beta)} = \frac{e^{-N\beta f}}{2^N (\cosh \beta)^{N_B}}. \tag{66}$$

By taking the logarithm and average to make the form of the KL divergence, we obtain two inequalities:

$$-\beta f^*(\beta,\beta) \geqslant \log 2 + d \log(\cosh \beta) \tag{67}$$

and

$$-\beta f^*(0,\beta) \leqslant \log 2 + d \log(\cosh \beta), \tag{68}$$

where we defined the free energy in the thermodynamic limit as

$$f^*(\beta_p,\beta) = \int df\, P(f;\beta_p,\beta) f. \tag{69}$$

C. Phase diagrams in spin glasses

The common quantity on the right-hand sides of Eqs. (67) and (68) is equal to the annealed free energy given in the symmetric distribution as $-\beta f_a(0,\beta) = \log[Z(\beta;\{\tau_{ij}\})]_{\beta_p=0}/N$. Thus the inequalities (67) and (68) read

$$f^*(\beta,\beta) \leqslant f_a(0,\beta) \tag{70}$$

and

$$f^*(0,\beta) \geqslant f_a(0,\beta). \tag{71}$$

From the fluctuation-theorem-type relation (66), the violation of the equality relates to the KL divergence as

$$f^*(\beta,\beta) - f_a(0,\beta) = -\frac{1}{\beta} D_{\rm KL}[P(f;\beta,\beta)|P(f;0,\beta)] \tag{72}$$

and

$$f^*(0,\beta) - f_a(0,\beta) = \frac{1}{\beta} D_{\rm KL}[P(f;0,\beta)|P(f;\beta,\beta)]. \tag{73}$$

When $f^*(0,\beta) = f_a(0,\beta)$, we immediately find $D_{\rm KL}[P(f;\beta,\beta)|P(f;0,\beta)] = D_{\rm KL}[P(f;0,\beta)|P(f;\beta,\beta)] = 0$. Thus we conclude that $f^*(\beta,\beta) = f^*(0,\beta)$. This relation ensures that the critical point on the Nishimori line is located at the same temperature in the symmetry distribution, when $f^*(0,\beta) = f_a(0,\beta)$ as often seen in the paramagnetic solutions for the mean-field spin glass models and the free energy of the Mattis model [21,22] as in the case of (a) in Fig. 1. These models show the parallel phase boundary to the β_p axis from the Nishimori line to the region in the symmetric distribution.

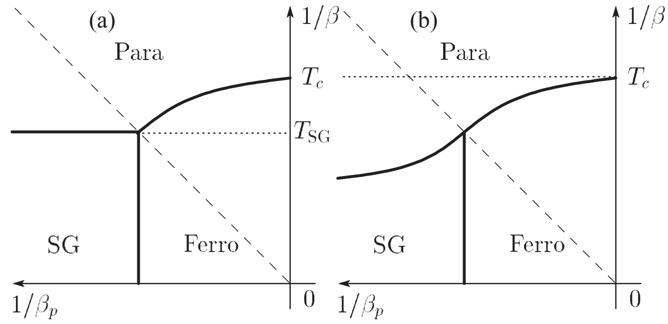

FIG. 1. Phase diagrams of spin glasses. (a) The typical phase diagram of several mean-field models and that of the Mattis model (then $T_c = T_{\rm SG}$). (b) Typical instance of the finite-dimensional $\pm J$ Ising model. The vertical axis denotes the temperature and the horizontal one expresses the density of the quenched randomness. The dashed line depicts the Nishimori line ($\beta_p = \beta$). The solid lines separate the representative phases: the ferromagnetic (Ferro), paramagnetic (Para), and spin glass (SG) ones. In (b), below the dotted line, the Griffiths paramagnetic phase is expected to be laid.

In addition, since $f_a(0,\beta)$ is a trivial function, the nonanalytical point of the free energy f^* is identified as that of the KL divergence. If two of the KL divergences $D_{\rm KL}[P(f;\beta,\beta)|P(f;0,\beta)]$ and $D_{\rm KL}[P(f;0,\beta)|P(f;\beta,\beta)]$ have the nonanalytical points at the same temperature, the phase boundary can be parallel to the β_p axis from the Nishimori line to the region in the symmetric distribution. Note that this does not necessarily mean that two of the KL divergences coincide with each other. This case is expected to be the Griffiths singularity, which is considered to be located at the same temperature T_c for any β_p as the ferromagnetic transition point without the quenched randomness [23–25], as in the case of (b) in Fig. 1. In this sense, the Griffiths singularity might be specified as the appearance of a simultaneous nonanalytical point of two symmetric KL divergences of the free-energy distribution functions. It would increase the understanding of the Griffiths singularity with the aid of the information geometry [26] through the above consideration.

VII. CONCLUSION

We analyzed the distribution function of the performed work for the spin glass with the gauge symmetry. The gauge symmetry revealed the existence of the symmetry in the rate function of the performed work as well as the free energy depending on each realization of the quenched randomness. As a result we obtained several relations in the same form as the fluctuation theorem. In order to analyze a spin glass system, we usually use the replica method, which deals with the highly correlated multiple system of the original model. Although a part of our results overlapped with the known properties via the replica method and gauge transformation as we recovered, our analyses were directly performed on the distribution function without replica method and related the averaged quantities as the performed work and free energy to the KL divergence. In this sense we believe that our analyses should be valuable for providing a different perspective to the understanding of nonequilibrium and critical behaviors for spin glasses.

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APPENDIX A: CROOK'S FLUCTUATION THEOREM

We here demonstrate the Crooks fluctuation theorem [1] of the performed work employing derivation by use of the large-deviation principle as done for the long-time behaviors by Lebowitz and Spohn [7]. Instead of the long-time observation ($T \gg 1$), we consider the large number of components ($N \gg 1$) in the present study.

We assume that the detailed balance condition is satisfied as

$$\frac{P_t(\mathbf{S}_{t+1}|\mathbf{S}_t,\{\tau_{ij}\})}{P_t(\mathbf{S}_t|\mathbf{S}_{t+1},\{\tau_{ij}\})} = \frac{P^t_{\mathrm{eq}}(\mathbf{S}_{t+1}|\{\tau_{ij}\})}{P^t_{\mathrm{eq}}(\mathbf{S}_t|\{\tau_{ij}\})}. \tag{A1}$$

Then the product over all steps can be expressed by

$$\prod_{t=0}^{T-1}\frac{P_t(\mathbf{S}_{t+1}|\mathbf{S}_t,\{\tau_{ij}\})}{P_t(\mathbf{S}_t|\mathbf{S}_{t+1},\{\tau_{ij}\})} = \prod_{t=0}^{T-1}\frac{P^t_{\mathrm{eq}}(\mathbf{S}_{t+1}|\{\tau_{ij}\})}{P^t_{\mathrm{eq}}(\mathbf{S}_t|\{\tau_{ij}\})}$$
$$= \exp\left[-\sum_{t=0}^{T-1}\delta X(\mathbf{S}_{t+1},\mathbf{S}_t|\{\tau_{ij}\})\right], \tag{A2}$$

where we defined the discretized pseudo heat as $\delta X_t(\mathbf{S}_{t+1},\mathbf{S}_t|\{\tau_{ij}\}) = \beta_t H(\mathbf{S}_{t+1}|\{\tau_{ij}\}) - \beta_t H(\mathbf{S}_t|\{\tau_{ij}\})$. The (pseudo) heat is given by

$$X(\{\mathbf{S}_t\},\{\tau_{ij}\};0\to T) = \sum_{t=0}^{T-1}\delta X_t(\mathbf{S}_{t+1},\mathbf{S}_t|\{\tau_{ij}\}). \tag{A3}$$

Then we confirm the first law of thermodynamics as

$$Y(\{\mathbf{S}_t\},\{\tau_{ij}\};0\to T) + X(\{\mathbf{S}_t\},\{\tau_{ij}\};0\to T)$$
$$= \beta_T H(\mathbf{S}_T|\{\tau_{ij}\}) - \beta_0 H(\mathbf{S}_0|\{\tau_{ij}\}). \tag{A4}$$

Therefore we reach a relation between the original process and its inverse one starting from the equilibrium states as

$$\prod_{t=0}^{T-1} P_t(\mathbf{S}_{t+1}|\mathbf{S}_t,\{\tau_{ij}\})e^{-Y(\{\mathbf{S}_t\},\{\tau_{ij}\};0\to T)}$$
$$= \prod_{t=0}^{T-1} P_t(\mathbf{S}_t|\mathbf{S}_{t+1},\{\tau_{ij}\})e^{-\beta_T H(\mathbf{S}_T|\{\tau_{ij}\})+\beta_0 H(\mathbf{S}_0|\{\tau_{ij}\})}. \tag{A5}$$

As a result, we find the following relation:

$$\prod_{t=0}^{T-1} P_t(\mathbf{S}_{t+1}|\mathbf{S}_t,\{\tau_{ij}\})P^0_{\mathrm{eq}}(\mathbf{S}_0|\{\tau_{ij}\})$$
$$= \frac{Z(\beta_T;\{\tau_{ij}\})}{Z(\beta_0;\{\tau_{ij}\})}\prod_{t=0}^{T-1} P_t(\mathbf{S}_t|\mathbf{S}_{t+1},\{\tau_{ij}\})P^T_{\mathrm{eq}}(\mathbf{S}_T|\{\tau_{ij}\})e^{Ny}. \tag{A6}$$

This relation yields Eq. (22).

The rate function is given by the Legendre transformation of the generating function as

$$I(y;\{\tau_{ij}\},0\to T) = \sup_r\{ry - \Psi_y(r;\{\tau_{ij}\},0\to T)\}. \tag{A7}$$

On the other hand, the rate function for the inverse process is also defined as

$$I(y;\{\tau_{ij}\},T\to 0) = \sup_r\{ry - \Psi_y(r;\{\tau_{ij}\},T\to 0)\}. \tag{A8}$$

By use of the relation (22), we find

$$I(y;\{\tau_{ij}\},0\to T)$$
$$= \sup_r\{(r+1)y - \Psi_y[-(r+1);\{\tau_{ij}\},T\to 0]\}$$
$$\quad - y - \frac{1}{N}[\log Z(\beta_T;\{\tau_{ij}\}) - \log Z(\beta_0;\{\tau_{ij}\})]$$
$$= I(-y;\{\tau_{ij}\},T\to 0)$$
$$\quad - y - \frac{1}{N}[\log Z(\beta_T;\{\tau_{ij}\}) - \log Z(\beta_0;\{\tau_{ij}\})]. \tag{A9}$$

This symmetry of the rate functions yields the well-known fluctuation theorem for the distribution functions of the performed work as

$$\frac{P(y;\{\tau_{ij}\},0\to T)}{P(-y;\{\tau_{ij}\},T\to 0)} = \exp\{N[y - \Delta_{0\to T}(\beta f)]\}. \tag{A10}$$

This is the Crooks fluctuation theorem [1].

APPENDIX B: GAUGE TRANSFORMATION

We demonstrate the manipulation of the gauge transformation to obtain Eq. (25) from Eq. (24). The quantity in the second line of Eq. (24) can be written as

$$[e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)}]_{\beta_p} = \sum_{\{\tau_{ij}\}}\frac{\prod_{\langle ij\rangle}e^{\beta_p\tau_{ij}}}{(2\cosh\beta_p)^{N_B}}e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)}. \tag{B1}$$

The gauge transformation yields

$$[e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)}]_{\beta_p} = \sum_{\{\tau_{ij}\}}\frac{\prod_{\langle ij\rangle}e^{\beta_p\tau_{ij}\sigma_i\sigma_j}}{(2\cosh\beta_p)^{N_B}}e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)}. \tag{B2}$$

Thus we sum over all possible configurations of $\{\sigma_i\}$ and obtain

$$2^N[e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)}]_{\beta_p} = \sum_{\{\tau_{ij}\}}\frac{Z(\beta_p;\{\tau_{ij}\})}{(2\cosh\beta_p)^{N_B}}e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)}. \tag{B3}$$

Similarly, we can evaluate the quantity in the third line of Eq. (24) as

$$2^N[e^{N\Psi_y(r;\{\tau_{ij}\},0\to T)}]_{\beta_p} = \sum_{\{\tau_{ij}\}} \frac{Z(\beta_T;\{\tau_{ij}\})}{Z(\beta_0;\{\tau_{ij}\})} \frac{Z(\beta_p;\{\tau_{ij}\})}{(2\cosh\beta_p)^{N_B}} e^{N\Psi_y[-(r+1);\{\tau_{ij}\},T\to 0]}. \tag{B4}$$

Therefore we reproduce Eq. (25).

In addition, the quantity in the second line of Eq. (27) is

$$[e^{N\Psi_y(r;\{\tau_{ij}\},T\to 0)}]_{\beta_p} = \sum_{\{\tau_{ij}\}} \frac{\prod_{\langle ij\rangle} e^{\beta_p\tau_{ij}}}{(2\cosh\beta_p)^{N_B}} e^{N\Psi_y(r;\{\tau_{ij}\},T\to 0)}. \tag{B5}$$

The gauge transformation yields

$$[e^{N\Psi_y(r;\{\tau_{ij}\},T\to 0)}]_{\beta_p} = \sum_{\{\tau_{ij}\}} \frac{\prod_{\langle ij\rangle} e^{\beta_p\tau_{ij}\sigma_i\sigma_j}}{(2\cosh\beta_p)^{N_B}} e^{N\Psi_y(r;\{\tau_{ij}\},T\to 0)}. \tag{B6}$$

The summation over all possible configurations of $\{\sigma_i\}$ yields

$$2^N[e^{N\Psi_y(r;\{\tau_{ij}\},T\to 0)}]_{\beta_p} = \sum_{\{\tau_{ij}\}} \frac{Z(\beta_p;\{\tau_{ij}\})}{(2\cosh\beta_p)^{N_B}} e^{N\Psi_y(r;\{\tau_{ij}\},T\to 0)}. \tag{B7}$$

The same analysis can be applied to the third line of Eq. (27) as

$$2^N[e^{N\Psi_y(r;\{\tau_{ij}\},T\to 0)}]_{\beta_p} = \sum_{\{\tau_{ij}\}} \frac{Z(\beta_0;\{\tau_{ij}\})}{Z(\beta_T;\{\tau_{ij}\})} \frac{Z(\beta_p;\{\tau_{ij}\})}{(2\cosh\beta_p)^{N_B}} e^{N\Psi_y[-(r+1);\{\tau_{ij}\},0\to T]}. \tag{B8}$$

When $\beta_p = \beta_T$, we find Eq. (28).

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